Homework 1 Solutions

CAS CS 132: Geometric Algorithms

Due: **September 17 at 5PM EST**

Submission Instructions

- Make the answer in your solution to each problem abundantly clear (e.g., put a box around your answer or used a colored font if there is a lot of text which is not part of the answer).
- Choose the correct pages corresponding to each problem in Gradescope. Note that Gradescope registers your submission as soon as you submit it, so you don't need to rush to choose corresponding pages.

Graders have license to dock points if either of the above instructions are not properly followed.

Note. Solutions written here may be lengthy because they are expository, and may not reflect that amount of detail that you were expected to write in your own solutions.

Practice Problems

The following list of problems comes from *Linear Algebra and its Application 5th Ed* by David C. Lay, Steven R. Lay, and Judi J. McDonald. They may be useful for solidifying your understanding of the material and for studying in general. **They are optional, so please don't submit anything for them**.

- (page 10) 1.1.11, 1.1.17, 1.1.18, 1.1.19
- (page 11) 1.1.25, 1.1.26, 1.1.30

1 Solving Systems of Linear Equations

Consider the following system of linear equations.

$$\begin{aligned} 3x - 4y + 3z &= -9 \\ 6x + 7y - 3z &= 0 \\ x + 10z &= -21 \end{aligned}$$

A. (5 pts) Write down the augmented matrix for this system.

B. (10 pts) Write down a solution to this system. Solve the system using elementary row operations. Show your work by writing down the row operations you used as well as the intermediate matrices.

Solution.

A.
$$\begin{bmatrix} 3 & -4 & 3 & -9 \\ 6 & 7 & -3 & 0 \\ 1 & 0 & 10 & -21 \end{bmatrix}$$

B.

$$R_2 \leftarrow R_1 - 2R_1 \quad \begin{bmatrix} 3 & -4 & 3 & -9 \\ 0 & 15 & -9 & 18 \\ 1 & 0 & 10 & -21 \end{bmatrix}$$

$$R_3 \leftarrow 3R_3 \quad \begin{bmatrix} 3 & -4 & 3 & -9 \\ 0 & 15 & -9 & 18 \\ 3 & 0 & 30 & -63 \end{bmatrix}$$

$$R_3 \leftarrow R_3 - R_1 \quad \begin{bmatrix} 3 & -4 & 3 & -9 \\ 0 & 15 & -9 & 18 \\ 0 & 4 & 27 & -54 \end{bmatrix}$$

$$R_3 \leftarrow R_3 + 3R_2 \quad \begin{bmatrix} 3 & -4 & 3 & -9 \\ 0 & 15 & -9 & 18 \\ 0 & 49 & 0 & 0 \end{bmatrix}$$

$$R_3 \leftarrow R_3/49 \quad \begin{bmatrix} 3 & -4 & 3 & -9 \\ 0 & 15 & -9 & 18 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$R_2 \leftarrow R_2 - 15R_3 \quad \begin{bmatrix} 3 & -4 & 3 & -9 \\ 0 & 0 & -9 & 18 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$R_1 \leftarrow R_1 + 4R_3 \quad \begin{bmatrix} 3 & 0 & 3 & -9 \\ 0 & 0 & -9 & 18 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$R_2 \leftarrow R_2/(-9) \quad \begin{bmatrix} 3 & 0 & 3 & -9 \\ 0 & 0 & 1 & -2 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$R_1 \leftarrow R_1 - 3R_2 \quad \begin{bmatrix} 3 & 0 & 0 & -3 \\ 0 & 0 & 1 & -2 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$R_1 \leftarrow R_1/3 \quad \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

The solution is $x = -1, y = 0, z = -2$.

2 Plane Intersection

(10 pts) Write the slope-intercept form of the line equation which defines the intersection of the plane

$$2x + 3y + 3z = 6$$

with the xy-plane.

Solution. The linear equation whose solutions form the xy-plane is $z = 0$, so the line is given by $2x + 3y = 6$. This equation has slope-intercept form $y = (-2/3)x + 2$.

3 Homogeneous Systems

A system of linear equations is *homogeneous* if it is of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

That is, all of its equations have 0 on the right-hand side. Consider the following pair of systems

$$\begin{aligned} ax + by &= c \\ dx + ey &= f \end{aligned}$$

$$\begin{aligned} ax + by - cz &= 0 \\ dx + ey - fz &= 0 \end{aligned}$$

A. (5 pts) Show that if the first system has a solution, then so does the second one.

B. (5 pts) Give **nonzero** values to a through f such that the second system has a solution, but the first does not. Present your solution as an augmented matrix for the first system, i.e., of the form

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

Solution.

A. Suppose (s_1, s_2) is a solution to the first system. Then, by definition,

$$\begin{aligned} as_1 + bs_2 &= c \\ ds_1 + es_2 &= f \end{aligned}$$

hold. Our goal now is to exhibit a solution (t_1, t_2, t_3) of the second system. Technically, the second system has the solution $(0, 0, 0)$ for any choice of a through f (this would be a valid answer). But if we want to give a solution which uses the values s_1 and s_2, we can note that if we take $t_1 = s_1$ and $t_2 = s_2$, the requirement on t_3 is that

$$\begin{aligned} as_1 + bs_2 - ct_3 &= 0 \\ ds_1 + bs_2 - et_3 &= 0 \end{aligned}$$

But by the equalities we get from the fact that (s_1, s_2) is a solution to the first system, we get that

$$\begin{aligned} c - ct_3 &= 0 \\ d - dt_3 &= 0 \end{aligned}$$

so we can take $t_3 = 1$. All togther, we get: if (s_1, s_2) is a solution to the first system, then $(s_1, s_2, 1)$ is a solution to the second system.

B. Again, the second system has the solution $(0, 0, 0)$ for any choices of a through f and so will always be consistent. It then suffices to choose a through f such that the first system is inconsistent. The easiest way to do this is to choose to equations with the same coefficients, but set them equal to different values:

$$\begin{aligned} x + y &= 1 \\ x + y &= 2 \end{aligned}$$

which has the augmented matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

4 Composing Systems of Linear Equations

Consider the following pair of systems of linear equations.

$$\begin{aligned} x_1 - 2x_2 &= 3 \\ 4x_1 + x_2 &= 21 \end{aligned}$$

$$\begin{aligned} 10x_3 + 2x_4 &= x_1 \\ (-8)x_3 + 9x_4 &= x_2 \end{aligned}$$

A. (5 pts) Solve the first system of linear equations (in x_1 and x_2) and write down the augmented matrix of the second system with the solutions of x_1 and x_2 substituted in.

B. (5 pts) Write the augmented matrix of a *single* system of linear equations *with all four equations* in the variables x_1, x_2, x_3, x_4. Discuss the relationship between this matrix and the one in the previous part. (Hint. Move around x_1 and x_2 in the second system.)

Solution.

A. Solving the first system for x_1 and x_2, we get the solution $x_1 = 5$ and $x_2 = 1$. Substituting these values into the second system and writing down its augmented matrix, we get:

$$\begin{bmatrix} 10 & 2 & 5 \\ -8 & 9 & 1 \end{bmatrix}$$

B. We first have to move x_1 and x_2 in the second system to the LHS of the equations. We then have four linear equations we can write down as an augmented matrix:

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 4 & 1 & 0 & 0 & 21 \\ -1 & 0 & 10 & 2 & 0 \\ 0 & -1 & -8 & 9 & 0 \end{bmatrix}$$

If we eliminate and back substitute for the first two rows, then the first two rows form a solution to the first system, and the last two rows define the same system as the one from the previous part. Note, in particular, that the -1 entries in the last two rows have the effect of setting the last column of the last two equations to the derived values of x_1 and x_2 when they are eliminated.

5 Pairwise Consistency

Given an example of a system of linear equations in 3 variables such that every pair of equations is consistent, but the system as a whole is inconsistent. That is, give values for a through l in the system

$$\begin{aligned} ax + by + cz &= d \\ ex + fy + gz &= h \\ ix + jy + kz &= l \end{aligned}$$

such that the system is inconsistent, but each pair of equations forms a consistent system. Present your solution to each part as the an augmented matrix, i.e., of the form

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{bmatrix}$$

(Hint. Make the third equation inconsistent with the sum of the first and second equation)

A. (7 pts) Achieve this with no more than 5 nonzero values for a through l.

B. (8 pts) Achieve this with all nonzero values for a through l.

Solution. Let's break down what this question is asking. To say that every pair of equations forms a consistent system, we require that each augmented matrix

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix} \qquad \begin{bmatrix} e & f & g & h \\ i & j & k & l \end{bmatrix} \qquad \begin{bmatrix} a & b & c & d \\ i & j & k & l \end{bmatrix}$$

represents a consistent system. It then must also be the case that the augmented matrix

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{bmatrix}$$

represents an inconsistent system.

Now let's break down what this question means geometrically. If each pair of equations is consistent, then the planes they represent intersect. But if the entire is inconsistent, there is no point where all three planes intersect.

This means that we cannot choose the planes so that two of them are parallel, since every plane needs to intersect every other plane at least once. What would such an orientation of planes look like?

Finally, let's break down what this question means algebraically. Per the hint, we can choose a pair of consistent equations, and then find a third equation which is inconsistent with their sum. We also need to verify that this equation is consistent with the first two we wrote down.

For part (a), since we're trying to minimize the number of nonzero values we use, we can start with the consistent system.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Geometrically, this is the yz-plane and the xz-plane (why?). If we take their sum, we get the equation: $x+y=0$. The easiest way to write down an equation which is inconsistent with another is to write down an equation with the same coefficients but equal to a different value, e.g., $x+y=1$ (it cannot possibly be the case that $x+y=0$ and $x+y=1$ simultaneously). This given us the entire system

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

By a cursory check, we see that each pair of systems is consistent. One tip: the only way that a pair of linear equations can be inconsistent is if they have the same coefficients up to scaling but are equal to a different value after scaling, e.g.,

$$\begin{aligned} x + 2y - z &= 1 \\ 2x + 4y - 2z &= 3 \end{aligned}$$

This system has the augmented matrix

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -2 & 3 \end{bmatrix}$$

and the row-operation $R_2 \leftarrow R_2 - 2R_1$ produces a matrix with an inconsistent row.

This is not the case for any pair of rows in the solution above so each pair must be consistent.

For the second part, we do the same thing, but we start with a pair of consistent equations without any zero values, e.g.,

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \end{bmatrix}$$

The sum of the these two equations is: $2x + 3y + 2z = 2$. And an equation which is inconsistent with this is: $2x + 3y + 2z = 3$. This gives us the system

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 2 & 3 & 2 & 3 \end{bmatrix}$$

To verify, no pair of equations has the same coefficients up to scaling, so each pair is consistent. Also, the row-operations $R_3 = R_3 - R_2$ and $R_3 = R_3 - R_1$ produces a matrix with the row $[0\ 0\ \ldots\ 0\ 1]$.

6 More than 2 Solutions

(10 pts) Consider the system of linear equations given by

$$\begin{aligned} ax + by + cz &= d \\ ex + fy + gz &= h \\ ix + jy + kz &= l \end{aligned}$$

where the values a through l are fixed real numbers. Show that if (s_x, s_y, s_z) and (t_x, t_y, t_z) are solutions to the above system, then

$$\left(\frac{s_x + t_x}{2}, \frac{s_y + t_y}{2}, \frac{s_z + t_z}{2}\right)$$

is also a solution. Discuss why if $(s_x, s_y, s_z) \neq (t_x, t_y, t_z)$, this implies the system has infinitely many solutions.

Solution. To show that

$$\left(\frac{s_x + t_x}{2}, \frac{s_y + t_y}{2}, \frac{s_z + t_z}{2}\right)$$

is a solution, we have to show that it simultaneously satisfies all the equations in the system. Let's consider the first equation. Since (s_x, s_y, s_z) is a solution to the system, it must be that

$$as_x + bs_y + cs_z = d$$

and since (t_x, t_y, t_z) is a solution, it must be that

$$at_x + bt_y + ct_z = d$$

We need to show that

$$a\left(\frac{s_x + t_x}{2}\right) + b\left(\frac{s_y + t_x}{2}\right) + c\left(\frac{s_z + t_z}{2}\right) = d$$

We do this by expanding the term on the LHS of the equation and substituting according the above two equations:

$$\begin{aligned} &a\left(\frac{s_x + t_x}{2}\right) + b\left(\frac{s_y + t_y}{2}\right) + c\left(\frac{s_z + t_z}{2}\right) \\ &\quad = \frac{as_x + at_x}{2} + \frac{bs_y + bt_y}{2} + \frac{cs_z + ct_z}{2} \\ &\quad = \frac{1}{2}\left(as_x + at_x + bs_y + bt_y + cs_z + ct_z\right) \\ &\quad = \frac{1}{2}\left((as_x + bs_y + cs_z) + (at_x + bt_y + ct_z)\right) \\ &\quad = \frac{1}{2}(2d) \\ &\quad = d \end{aligned}$$

The same calculation can be done for each equation. If (s_x, s_y, s_z) and (t_x, t_y, t_z) are distinct, we can then generate infinitely many solutions from them by finding a solution between them, and then finding a solution between that and the first solution, and so on.

7 Lines in 3 Dimensions (Extra Credit)

(5 pts) As we discussed in lecture, a 3 dimensional linear equation does not describe a line, but a plane. One way to describe a line in 3 dimensions is to use a parameter t. That is, after fixing values a_x, b_x, a_y, b_y, a_z, and b_z, a line can be described as all points of the form

$$(a_x t + b_x, a_y t + b_y, a_z t + b_z)$$

for any real value of t (make sure to convince yourself of this!). We also saw the intersection of two planes can describe a line.

Give a pair of 3 dimensional linear equations (each of which represents a plane) whose intersection is exactly the line whose points are defined by the above parametric form. (Hint. The line above can be thought of as a system in the variables x, y, z, and t, e.g., one of its equations is $x - a_x t = b_x$. Write t in terms of x and substitute this value for t into the other equations.)

Solution. According to the hint, the first observation that we can make is that the set of points can be described by linear equations if we have an additional variable t:

$$\begin{aligned} x - a_x t &= b_x \\ y - a_y t &= b_y \\ z - a_z t &= b_z \end{aligned}$$

And per the hint, we can solve for t:

$$t = \frac{x - b_x}{a_x}$$

By substituting this value of t into the second equation, we get

$$y - a_y \left(\frac{x - b_x}{a_x} \right) = b_y$$

which can be scaled and rewritten as

$$a_x y - a_y x = a_x b_y - a_y b_x$$

Likewise, for the second equation

$$a_x z - a_z x = a_x b_z - a_z b_x$$

These two equations are exactly linear equations, which intersect at the above line. You can verify this by showing that

$$(a_x t + b_x, a_y t + b_y, a_z t + b_z)$$

is a solution to the system

$$\begin{aligned} (-a_y)x + a_x y &= a_x b_y - a_y b_x \\ (-a_z)x + a_x z &= a_x b_z - a_z b_x \end{aligned}$$

for any value of t.

8 (Programming) Floating-Point Accuracy

In this problem you will be verifying that you've set everything up Python correctly and that you are comfortable submitting coding assignments on Gradescope. We'll also be reasoning a bit about floating point error.

NOTE. The coding submission on gradescope is just a practice run this week. It's worth 0 points, and this question is graded in the analytic part. But you will NOT receive credit if you do not submit your code via Gradescope as well.

Consider the function

```
def sub_error(n):
  return abs(n / 7 / 10 * 7 - n / 10)
```

We will be looking at the way this error varies as x increases.

A. (5 pts) Fill in the function

```
def next_error(start):
  # TODO
```

which returns the returns `sub_error(x')` where `x'` is the smallest value greater than or equal to `start`, taken from the set of possibilities `start, start + 1, start + 2,...`, such that `sub_error(x')` is nonzero. **Please reproduce your function in your submission to the analytic part of the assignment.**

B. (3 pts) Plot `next_error` on a log scale graph using `matplotlib` (you don't have to write any code for this part, its included in the starter code). And include a copy of the image produced in your solution.

C. (2 pts) Discuss the trend in the plot. Give an informal justification for its shape.

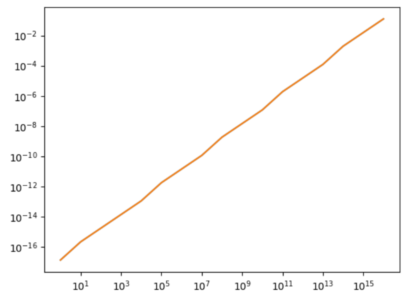
Solution.

A.

```
def next_error(start):
i = start
while True:
next = error(i)
if next > 0:
return next
i += 1

```

B.



C. As the input to `next_error` grows, so does the error. Around 10^{16}, the error is within 1 decimal point. Roughly speaking, this is because the absolute error of the representation is growing in the input, and so the error blows up after subtraction.