Homework 3 Solutions

CAS CS 132: Geometric Algorithms

Due: **Thursday September 28, 2023 at 11:59PM**

Submission Instructions

- Make the answer in your solution to each problem abundantly clear (e.g., put a box around your answer or used a colored font if there is a lot of text which is not part of the answer).
- Choose the correct pages corresponding to each problem in Gradescope. Note that Gradescope registers your submission as soon as you submit it, so you don't need to rush to choose corresponding pages. **For multipart questions, please make sure each part is accounted for.**

Graders have license to dock points if either of the above instructions are not properly followed.

Note. Solutions written here may be lengthy because they are expository, and may not reflect that amount of detail that you were expected to write in your own solutions.

Practice Problems

The following list of problems comes from *Linear Algebra and its Application 5th Ed* by David C. Lay, Steven R. Lay, and Judi J. McDonald. They may be useful for solidifying your understanding of the material and for studying in general. **They are optional, so please don't submit anything for them**.

- (page 40) 1.4.1, 1.4.2
- (page 41) 1.4.10, 1.4.15, 1.4.16, 1.4.29, 1.4.30
- (page 61) 1.7.1, 1.7.5
- (page 62) 1.7.15, 1.7.17, 1.7.35, 1.7.37

1 Spanning Columns

(10 points) Do the columns of the following matrix span all of $\mathbb{R}^3$?

$$\begin{bmatrix} 1 & -5 & 4 \\ -1 & 6 & -3 \\ -2 & 13 & -7 \end{bmatrix}$$

Justify your answer. (This means explaining why your answer is correct. You **do not** have to write down any row operations you use.)

Solution. The columns of a (3×3) matrix span $\mathbb{R}^3$ if any vector in $\mathbb{R}^3$ can be written as a linear combinations of them. This means, for any vector $\mathbf{b}$ in $\mathbb{R}^3$, the vector equation

$$x_1 \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} -5 \\ 6 \\ 13 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ -3 \\ -7 \end{bmatrix} = \mathbf{b}$$

has a solution. This is equivalent to saying that the above matrix has a pivot position in every row.

The reduced echelon form of the above matrix is the identity matrix, which has a pivot position in every row, so its columns span $\mathbb{R}^3$.

2 Matrix-Vector Multiplications

Write down a matrix A in $\mathbb{R}^{4\times 4}$ such that

$$A \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

For each part, write down A in the specified shape, where $\blacksquare$ represents a nonzero entry.

A. (5 points)

$$\begin{bmatrix} \blacksquare & 0 & 0 & 0 \\ 0 & \blacksquare & 0 & 0 \\ 0 & 0 & \blacksquare & 0 \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}$$

B. (5 points)

$$\begin{bmatrix} \blacksquare & 0 & 0 & 0 \\ \blacksquare & \blacksquare & 0 & 0 \\ \blacksquare & \blacksquare & \blacksquare & 0 \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare \end{bmatrix}$$

C. (5 points)

$$\begin{bmatrix} \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare & 0 \\ \blacksquare & \blacksquare & 0 & 0 \\ \blacksquare & 0 & 0 & 0 \end{bmatrix}$$

Solution.

A.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

B.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

C.

$$\begin{bmatrix} 4 & -1 & -1 & -1 \\ 4 & -1 & -1 & 0 \\ 4 & -1 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix}$$

For B and C there are many correct answers.

3 Linear Independence

(10 points) Are the following vectors linearly independent?

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} -3 \\ 9 \\ -8 \end{bmatrix}$$

If so, justify your answer. If not, write one of the vectors as a linear combination of the others.

Solution. The reduced echelon form of the matrix

$$\begin{bmatrix} 1 & -1 & -3 \\ -1 & 4 & -3 \\ -3 & 9 & -8 \end{bmatrix}$$

is

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix does not have a pivot in every column, so these vectors are not linearly independent. Reading a solution from this matrix, we see that we can write $\mathbf{v}_3$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$(-1)\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} -3 \\ 9 \\ -8 \end{bmatrix}$$

4 Linearly Dependent Sets

(15 points) Write down three **nonzero** vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ in $\mathbb{R}^3$ such that

- $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent and
- $\mathbf{v}_1$ **cannot** be written as a linear combination of $\mathbf{v}_2$ and $\mathbf{v}_3$.

Your solution should be given in the following form:

$$\mathbf{v}_1 = \begin{bmatrix} * \\ * \\ * \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} * \\ * \\ * \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} * \\ * \\ * \end{bmatrix}$$

where $*$ represents an arbitrary entry.

Solution. If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent then *at least one* of the vector can be written as a linear combination of the others. The second condition indicates that this cannot be $\mathbf{v}_1$, so it has to be $\mathbf{v}_2$ or $\mathbf{v}_3$. We start by taking $\mathbf{v}_2$ to be something simple, like

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

We now need to take $\mathbf{v}_1$ and $\mathbf{v}_3$ such that $\mathbf{v}_2$ can be written as one of their linear combinations. Furthermore, since $\mathbf{v}_1$ cannot be written a linear combinator of $\mathbf{v}_2$ and $\mathbf{v}_3$, then it *must* be the case that the coefficient α_1 for $\mathbf{v}_1$ in

$$\mathbf{v}_2 = \alpha_1 \mathbf{v}_1 + \alpha_3 \mathbf{v}_3$$

is 0. If $\alpha_1 \neq 0$, then $\mathbf{v}_1 = (-1/\alpha_1)\mathbf{v}_2 + (\alpha_3/\alpha_1)\mathbf{v}_3$. In other words, it must be that $\mathbf{v}_3$ is a scalar multiple of $\mathbf{v}_2$. Let's take

$$\mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

Finally, we just have to take $\mathbf{v}_1$ to be any vector which is *not* as scalar multiple of $\mathbf{v}_2$ (by definition of $\mathbf{v}_3$, any linear combination of $\mathbf{v}_2$ and $\mathbf{v}_3$ is a scalar multiple of $\mathbf{v}_2$). So we get a final answer:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

5 Linear Independence, Computationally

Consider the following matrix, presented as a numpy array.

```
a = np.array(
    [[ 13., -19,  19,  16,   5,   1,  10,   5,  15],
     [-11,   -7, -10,   7,   2,  -8, -10, -19,   6],
     [ -5,   10,  -7,   2,  -8,   2, -15, -16, -11],
     [ 17,  -13,   9,  13,  19,   8,  -3,  -9,   0],
     [  9,  -18,   5,   1,   4,  14,   9,   8,  -4],
     [  8,   14,  17,   5,  -6,   7, -13,   2,  12],
     [ 18,   12,  -7,   2, -10,  15, -12,   1, -12],
     [ 19,  -12,   1, -16,   2,  -6,  -4,  17,  15],
     [-19,   -6, -16, -20, -20,  -3,   7,   3,  14],
     [  6,    8, -15,   5,  -5,   8, -14,   5, -19]])
```

A. (5 points) Do the columns of this matrix span all of $\mathbb{R}^{10}$? Justify your answer.

B. (5 points) Are the columns of this matrix linearly independent? Justify your answer.

You may use python to solve this problem (I recommend it, you can copy-paste the above lines). You must describe what you did in python to come to your solution.

Solution.

```
[[13.  -19.    19.     16.      5.      1.      10.      5.      15.  ]
 [ 0.  -23.08   6.08   20.54    6.23   -7.15    -1.54  -14.77    18.69]
 [ 0.    0.     1.02   10.55   -5.35    1.55   -11.33  -15.8     -3.05]
 [ 0.    0.     0.    134.69  -51.31   22.42  -158.74 -220.9    -48.2 ]
 [ 0.    0.     0.      0.    -18.6    15.29    -4.35   -2.01   -16.73]
 [ 0.    0.     0.      0.      0.     14.75   -17.5    -8.05     2.94]
 [ 0.    0.     0.      0.      0.      0.     -16.6    13.11    54.56]
 [ 0.    0.     0.      0.      0.      0.       0.    -84.51  -171.92]
 [ 0.    0.     0.      0.      0.      0.       0.      0.      35.66]
 [ 0.    0.     0.      0.      0.      0.       0.     26.73    12.88]]
```

A. This matrix has more rows than columns, so its columns cannot possibly span all of $\mathbb{R}^{10}$.

B. Calling `gaussian_elimination` on this matrix gives the one above, which has a pivot in every column.

6 A Small Interface

(20 points) This week you will be filling in a small interface for answering some of the questions we have considered so far in the course. **Read through the docstring of each function carefully.** You will see what you need to implement described there.

You are given starter code in the file `hw03prog.py`. You are also provided with a file `gauss.py` which implements gaussian elimination. You are only required to submit `hw03prog.py`, a copy of `gauss.py` will be available to the autograder. **Don't change the name of this file when you submit.** Also don't change any of the names of the functions included in the starter code. **The only changes you should make are to fill in the TODO items in the starter code**.

Each function can be written in a single line (you are not required to do so). This is to say that the code is not lengthy, it just requires that you understand what each function is supposed to do.

Some guidelines for this assignment (and others as well):

- Work incrementally. Don't try to implement the entire program in one go and then debug.
- Write some test cases, or at least check your implemenetations in the command line. We are not providing any tests this week.

You will upload the single python file `hw03prog.py` to Gradescope with your implementations of the required functions. We will be running autograder tests on your submission to determine its correctness. **You will not have access to the autograder tests.**