Homework 8 Solutions

CAS CS 132: Geometric Algorithms

Due: **Thursday November 9, 2023 at 11:59PM**

Submission Instructions

- Make the answer in your solution to each problem abundantly clear (e.g., put a box around your answer or used a colored font if there is a lot of text which is not part of the answer).

- Choose the correct pages corresponding to each problem in Gradescope. Note that Gradescope registers your submission as soon as you submit it, so you don't need to rush to choose corresponding pages. **For multipart questions, please make sure each part is accounted for.**

Graders have license to dock points if either of the above instructions are not properly followed.

Note. Solutions written here may be lengthy because they are expository, and may not reflect that amount of detail that you were expected to write in your own solutions.

Practice Problems

The following list of problems comes from *Linear Algebra and its Application 5th Ed* by David C. Lay, Steven R. Lay, and Judi J. McDonald. They may be useful for solidifying your understanding of the material and for studying in general. **They are optional, so please don't submit anything for them**.

- 2.8.1-5, 2.8.11-12, 2.8.15-16, 2.8.21-22, 2.8.25, 2.8.31-32, 2.8.35

- 2.9.3-6, 2.9.12, 2.9.17-21

1 Subspaces and Matrices (Basics)

For each of the following parts, justify your answer.

A. (2 points) If $A \in \mathbb{R}^{3 \times 6}$, then for what value of n is $\operatorname{Nul} A$ is a subspace of $\mathbb{R}^n$?

B. (4 points) If $A \in \mathbb{R}^{10 \times 13}$ matrix then what is the minimum dimension of $\operatorname{Nul} A$?

C. (3 points) If $A \in \mathbb{R}^{7 \times 5}$ matrix and $\dim(\operatorname{Col} A) = 4$, what is $\dim(\operatorname{Nul} A)$?

D. (3 points) Determine if $\mathbf{v}$ is in $\operatorname{Nul} A$ where

$$\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \qquad A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 6 & 0 \\ 5 & 7 & 2 \end{bmatrix}$$

E. (4 points) Determine if $\mathbf{v}$ is in $\operatorname{Col} A$, where $\mathbf{v}$ and A are as in the previous part.

F. (4 points) Determine $\operatorname{rank} A$ where

$$A = \begin{bmatrix} 2 & 1 & -8 & 3 \\ -1 & 3 & 4 & 2 \\ 3 & 2 & -12 & 5 \\ 1 & -2 & -4 & -1 \end{bmatrix}$$

Hint. Attempt to do this without performing any calculations.

Solution.

A. The null space consists of solutions to the equation $A\mathbf{x} = \mathbf{0}$, and solutions lie in $\mathbb{R}^6$ (Remember that $\mathbf{x}$ defines weights a linear combination of the columns of A, and A has 6 columns).

B. The dimension of the null space is determined by the number of free variables in a general form solution to the equation $\mathbf{Ax} = \mathbf{0}$. There can be at most 10 pivots in A, which leaves at least 3 free variables. Another way: by the rank theorem, we know that $\operatorname{rank} A + \dim(\operatorname{Nul} A) = 13$, and $\operatorname{rank} A \leq 10$ since $\operatorname{Col} A \subset \mathbb{R}^{10}$, so $\dim(\operatorname{Nul} A) \geq 3$.

C. By the rank theorem, $\operatorname{rank} A + \dim(\operatorname{Nul} A) = 5$, so $\dim(\operatorname{Nul} A) = 1$.

D. $\mathbf{v}$ is in $\operatorname{Nul} A$ if $A\mathbf{v} = \mathbf{0}$. But $A\mathbf{v} = [\, 0 \;\; 0 \;\; 1 \,]^T$, so $\mathbf{v} \notin \operatorname{Nul} A$.

E. To say that $\mathbf{v}$ is in $\operatorname{Col} A$ is to say that $\mathbf{v}$ lies in the span of the columns of A, something we know how to do: find a solution to the system with the augmented matrix $[A \mid \mathbf{v}]$. However, it is also somewhat clear that the columns of A are linearly independent. Without doing any row operations: the previous part tells us that $\mathbf{e}_3 \in \operatorname{Col} A$, but then $\mathbf{e}_1 = \mathbf{a}_3 - 2\mathbf{e}_3$ and $6\mathbf{e}_2 = (\mathbf{a}_2 - 7\mathbf{e}_3)$.

F. Per the hint, we might notice that $\mathbf{a}_3 = 4\mathbf{a}_1$ and $\mathbf{a}_4 = \mathbf{a}_1 + \mathbf{a}_2$. So there are two linearly independent vectors which span Col A, making the rank 2.

2 Bases for Null Spaces and Column Spaces

Consider the following matrix.

$$\begin{bmatrix} 1 & 2 & 4 & 3 & -4 & 1 \\ -3 & -4 & -10 & -8 & 13 & -3 \\ 5 & 6 & 16 & 10 & -13 & 9 \\ -7 & -8 & -22 & -12 & 13 & -9 \\ 13 & 18 & 44 & 32 & -47 & 11 \end{bmatrix}$$

A. (2 points) Find the reduced echelon form of A. You may (should) use Python. Describe the process you used. For your convenience:

```
1 np.array([[1., 2, 4, 3, -4, 1], [-3, -4, -10, -8, 13, -3], [5,
    6, 16, 10, -13, 9], [-7, -8, -22, -12, 13, -9], [13, 18,
    44, 32, -47, 11]])
```

B. (6 points) Find a basis for Col A.

C. (6 points) Find a basis for Nul A.

Solution.

A. The reduced echelon form of A can be obtained using code we've written in this course, like `cs132lib.reduced_echelon_form(a)`:

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

B. We can build a basis for Col A by looking at the pivot columns of A:

$$\left\{ \begin{bmatrix} 1 \\ -3 \\ 5 \\ -7 \\ 13 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 6 \\ -8 \\ 18 \end{bmatrix}, \begin{bmatrix} 3 \\ -8 \\ 10 \\ -12 \\ 32 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 9 \\ -9 \\ 11 \end{bmatrix} \right\}$$

C. First we have to find a general form solution for the equation $A\mathbf{x} = \mathbf{0}$.

$$x_1 = -2x_3 - x_5$$
$$x_2 = -x_3 - 2x_5$$
$$x_3 \text{ is free}$$
$$x_4 = 3x_5$$
$$x_5 \text{ is free}$$
$$x_6 = 0$$

We can think of this a linear transformation

$$\begin{bmatrix} s \\ t \end{bmatrix} \mapsto \begin{bmatrix} -2s - t \\ -s - 2t \\ s \\ 3t \\ t \\ 0 \end{bmatrix}$$

The columns of the matrix implementing this tranformation form a basis for Nul A.

$$\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix} \right\}$$

3   Affine Spaces

Let A be an $m \times n$ matrix and let $\mathbf{v}$ be a vector in $\mathbb{R}^n$.

   A. (6 points) Show that if $A\mathbf{v} = \mathbf{b}$ and $\mathbf{w} \in \operatorname{Nul} A$, then $\mathbf{v} + \mathbf{w}$ is a solution to the equation $A\mathbf{x} = \mathbf{b}$.

   B. (5 points) Show that if $\mathbf{b} \neq 0$, then the solution set of $A\mathbf{x} = \mathbf{b}$ is *not* a subspace of $\mathbb{R}^n$.

   C. (5 points) **(Extra Credit)** A set H is an **affine** subspace of $\mathbb{R}^n$ if there is a subspace U of $\mathbb{R}^n$ and a vector $\mathbf{o}$ such that

$$H = \{\mathbf{u} + \mathbf{o} \mid \mathbf{u} \text{ is in } U\}$$

   Show that **if the solution set of $A\mathbf{x} = \mathbf{b}$ is nonempty** then it is an affine subspace. This means choosing a vector $\mathbf{o}$ and subspace U.

Solution.

   A. We need to show that $A(\mathbf{v} + \mathbf{w}) = \mathbf{b}$:

$$A(\mathbf{v} + \mathbf{w}) = A\mathbf{v} + A\mathbf{w} = \mathbf{b} + \mathbf{0} = \mathbf{b}$$

   B. The vector $\mathbf{0}$ is not a solution to $A\mathbf{x} = \mathbf{b}$, so this solution set is not a subspace of $\mathbb{R}^n$.

   C. Let $\mathbf{v}$ be a solution to the equation $A\mathbf{x} = \mathbf{b}$. Then we can take $\mathbf{o}$ to be $\mathbf{v}$ and U to be $\operatorname{Nul} A$. Part A shows that every element in the set $H = \{\mathbf{u} + \mathbf{v} \mid \mathbf{u} \in \operatorname{Nul} A\}$ is a solution to $A\mathbf{x} = \mathbf{b}$. In order to show that every solution can be written in this form, we have to recognize that for any solution $\mathbf{w}$, we have $\mathbf{w} - \mathbf{v} \in \operatorname{Nul} A$. This is because

$$A(\mathbf{w} - \mathbf{v}) = A\mathbf{w} - A\mathbf{v} = \mathbf{b} - \mathbf{b} = \mathbf{0}$$

Then $\mathbf{w} = \mathbf{v} + (\mathbf{w} - \mathbf{v})$, so it is H.

4 Complement of the Column Space

A. (2 points) Find a vector which is *not* in $\operatorname{Col} A$, where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

B. (5 points) Find a vector which is *not* in $\operatorname{Col} A$, where

$$A = \begin{bmatrix} 1 & 1 & 5 \\ -1 & 0 & -1 \\ 1 & 2 & 9 \end{bmatrix}$$

C. (8 points) Let A be a $5 \times n$ matrix such that $\operatorname{rank} A = 4$, which has an LU decomposition where

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ 0 & 3 & -3 & 0 & 1 \end{bmatrix}$$

Determine if $\mathbf{v}$ in $\operatorname{Col} A$, where

$$\mathbf{v} = \begin{bmatrix} 2 \\ -5 \\ -11 \\ 5 \\ -12 \end{bmatrix}$$

Solution.

A. A vector $\mathbf{v}$ is not in the column space if $A\mathbf{x} = \mathbf{v}$ has no solution. A simple choice of vector for which this equation does not have a solution is $\mathbf{e}_4$.

B. We can determine this by reducing a matrix of the form

$$\begin{bmatrix} 1 & 1 & 5 & b_1 \\ -1 & 0 & -1 & b_2 \\ 1 & 2 & 9 & b_3 \end{bmatrix}$$

to reduced echelon form:

$$\begin{bmatrix} 1 & 1 & 5 & b_1 \\ 0 & 1 & 4 & b_1 + b_2 \\ 0 & 0 & 0 & b_3 - b_1 - (b_1 + b_2) \end{bmatrix}$$

We can then take $b_3 - 2b_1 - b_2 = 1$, which mean $\mathbf{e}_3$ is such a vector.

C. Since rank $A = 4$, every echelon form of A has exactly one row of all zeros. We have to verify that the matrix $[A \ \mathbf{v}]$ does not have an inconsistent last row. We can read from L a sequence of row operations which takes A to echelon form.

$$R_2 \leftarrow R_2 + R_1$$
$$R_4 \leftarrow R_4 - 2R_1$$
$$R_3 \leftarrow R_3 - 4R_2$$
$$R_5 \leftarrow R_5 - 3R_2$$
$$R_5 \leftarrow R_5 + 3R_3$$

If we apply these operations to $\mathbf{v}$, we get the vector $[2 \ -3 \ 1 \ 1 \ 0]^T$, which means the equation $A\mathbf{x} = \mathbf{v}$ has a solution (since the last row doesn't represent an inconsistent equation).

5 Problem (Programming)

(15 points) A lot of what doing computational linear algebra entails is building and manipulating matrices. NumPy provides a powerful interface for doing this, but it takes some time to become familiar with. In this problem, you will be constructing NumPy arrays using functions from the NumPy standard library. You are required to complete each part **with a single line of code**.

You are given starter code in the file `hw08prog.py`. **Don't change the name of this file when you submit.** Also don't change the names of any functions or variables provided in the starter code. **The only changes you should make are to fill in the provided TODO items.** You will upload a single file `hw08prog.py` to Gradescope.

For each of the following parts, fill in (at the corresponding TODO item in `hwprog08.py`) a **single line (fewer than 100 characters))** of Python code which builds the 2D NumPy arrays representing each matrix. In particular, you can't hardcode the arrays. This may require reading through some of the NumPy documentation.

A.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

B.

$$\begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

C. *Hint.* Look at some of the examples in the documentation for `np.eye`.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

D. *Hint.* Look at `np.arange` and `np.diag`.

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 7 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 8
\end{bmatrix}$$

E. *Hint.* Use matrix addition.

$$\begin{bmatrix}
2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 4 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 3 & 6 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 & 8 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 3 & 10 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & 12 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 3 & 14 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 3 & 16
\end{bmatrix}$$

F. *Hint.* Look at some of the examples in the documentation for `np.arange`.

$$\begin{bmatrix}
8 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 7 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 6 & 3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 5 & 4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 4 & 5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 3 & 6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

G.

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

H. *Hint.* Look at `np.ones`.

$$\begin{bmatrix} 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{bmatrix}$$

I. *Hint.* Look at `np.hstack`.

$$\begin{bmatrix} 3 & 3 & 3 & 3 & 1 & 0 & 0 & 0 \\ 3 & 3 & 3 & 3 & 0 & 1 & 0 & 0 \\ 3 & 3 & 3 & 3 & 0 & 0 & 1 & 0 \\ 3 & 3 & 3 & 3 & 0 & 0 & 0 & 1 \end{bmatrix}$$

J. *Hint.* Look also at `np.vstack`.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -2 & -2 & -2 & -2 \\ 0 & 1 & 0 & 0 & -2 & -2 & -2 & -2 \\ 0 & 0 & 1 & 0 & -2 & -2 & -2 & -2 \\ 0 & 0 & 0 & 1 & -2 & -2 & -2 & -2 \end{bmatrix}$$

K. *Hint.* Remember you can index matrices on lists.

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

L.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

M. *Hint.* Look at `np.tril`.

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 2 & 2 & 0 & 0 & 0 \\ 2 & 2 & 2 & 2 & 2 & 2 & 0 & 0 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{bmatrix}$$