Homework 9 Solutions

CAS CS 132: Geometric Algorithms

Due: **Thursday November 16, 2023 at 11:59PM**

Submission Instructions

- Make the answer in your solution to each problem abundantly clear (e.g., put a box around your answer or used a colored font if there is a lot of text which is not part of the answer).
- Choose the correct pages corresponding to each problem in Gradescope. Note that Gradescope registers your submission as soon as you submit it, so you don't need to rush to choose corresponding pages. **For multipart questions, please make sure each part is accounted for.**

Graders have license to dock points if either of the above instructions are not properly followed.

Note. Solutions written here may be lengthy because they are expository, and may not reflect that amount of detail that you were expected to write in your own solutions.

Practice Problems

The following list of problems comes from *Linear Algebra and its Application 5th Ed* by David C. Lay, Steven R. Lay, and Judi J. McDonald. They may be useful for solidifying your understanding of the material and for studying in general. **They are optional, so please don't submit anything for them**.

- 5.1.1, 5.1.3, 5.1.8, 5.1.13, 5.1.23, 5.1.30
- 5.2.1, 5.2.3, 5.2.7, 5.2.17, 5.2.18

1 Eigenvalues and Eigenvectors

Consider the following matrix.

$$A = \begin{bmatrix} -17 & 28 & 14 \\ -7 & 11 & 7 \\ -7 & 14 & 4 \end{bmatrix}$$

A. (5 points) Determine if the following vectors are eigenvectors of A. For the ones that are, find their associated eigenvalues. Show your work. You may use Python, but if you do, you must include the lines of code you used.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

B. (6 points) Show that -3 is an eigenvalue of A *without doing any row operations*. Use the invertible matrix theorem to justify your answer.

C. (6 points) Find a basis for the eigenspace of A corresponding to the eigenvalue -3.

Solution.

A. $A\mathbf{v}_1$ computes to $[25\ 11\ 11]^T$, so $\mathbf{v}_1$ is not an eigenvector. $A\mathbf{v}_2$ computes to $[22\ 8\ 14]^T$, so $\mathbf{v}_2$ is not an eigenvector. $A\mathbf{v}_3$ computes to $[8\ 4\ 4]^T$, so $\mathbf{v}_3$ is an eigenvector for the eigenvalue 4.

B. We need to determine if $A + 3I$ is invertible. We have

$$A + 3I = \begin{bmatrix} -14 & 28 & 24 \\ -7 & 14 & 7 \\ -7 & 14 & 7 \end{bmatrix}$$

Note that the second and third columns are linearly dependent, so by the invertible matrix theorem, $(A + 3I)\mathbf{x} = \mathbf{0}$ has nontrivial solutions, which implies -3 is an eigenvalue of A.

C. The reduced echelon form of this matrix is

$$\begin{bmatrix} 1 & -2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Writing the solution set as a linear combination of vectors with free variables as weights gives us

$$x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

so

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis for the eigenspace of -3.

2 Determinants

A. (4 points) Compute the determinant of

$$\begin{bmatrix} 3 & -3 & 0 \\ 0 & 3 & -1 \\ 2 & 0 & -1 \end{bmatrix}$$

B. (5 points) Given $\det A = 3.5$ and $\det B = -2$, find the determinant of the matrix $B(AB)^{-1}(AB)^T A$.

C. (6 points) Consider the following reduction sequence

$$\begin{aligned} R_1 &\leftarrow R_1 + R_2 \\ \mathsf{swap}&(R_2, R_3) \\ R_3 &\leftarrow R_3 + 5R_4 \\ R_2 &\leftarrow -3R_2 \\ R_5 &\leftarrow R_5 - 10R_3 \\ R_5 &\leftarrow R_5/11 \\ \mathsf{swap}&(R_5, R_3) \\ \mathsf{swap}&(R_1, R_2) \\ R_4 &\leftarrow R_4 + R_1 \\ R_2 &\leftarrow 5R_2 \\ R_1 &\leftarrow -R_1 \end{aligned}$$

Suppose that $A \in \mathbb{R}^{5\times 5}$ reduces to U by this reduction sequence, where U is in *reduced* echelon form. If $\operatorname{rank} A = 5$, what is $\det A$?

D. (2 points) What is $\det A$ from the previous part if $\operatorname{rank} A = 4$?

Solution.

A. One possible sequence of row operations to convert the given matrix to echelon form is

$$\begin{aligned} R_3 &\leftarrow 3R_3 \\ R_3 &\leftarrow R_3 - 2R_1 \\ R_3 &\leftarrow R_3 - 2R_2 \end{aligned}$$

This required no swaps ($s = 0$) and one scaling ($c = 3$). So

$$\det A = \frac{(-1)^0}{3}(3)(3)(-1) = -3$$

B. Note that we already know that A and B are invertible since they have nonzero determinant. First, we can say that

$$B(AB)^{-1}(AB)^T A = BB^{-1}A^{-1}B^T A^T A$$

Then

$$\det(BB^{-1}A^{-1}B^T A^T A) = \det(B)\det(B^{-1})\det(A^{-1})\det(B^T)\det(A^T)\det(A)$$

Finally since $\det(C^{-1}) = \det(C)$ and $\det(C^T) = \det(C)$, we can simplify this to $\det(B)\det(A)$ which is -7.

C. If U is in reduced echelon from and A has full rank, then $U = I$, and the product of it's diagonals is 1. The above sequence uses three swaps, so $s = 3$, and the scalings -3, $\frac{1}{11}$, 5 and -1, so $c = \frac{15}{11}$. So the determinant is $(-1)^3\frac{11}{15}(1)$, which is $\frac{-11}{15}$.

D. If A does not have full rank, then $\det(A) = 0$.

3 Properties of Determinants

For each of the statements, either argue that it is true for any choice of matrices or give counterexamples in $\mathbb{R}^2$ showing it is false. All matrices are assumed to be square.

A. (3 points) If $A \sim B$ (A is row equivalent to B) then $\det(A) = \det(B)$.

B. (3 points) $\det(5A) = 5 \det A$

C. (3 points) $\det(A^T A) \geq 0$

D. (3 points) $\det(A + B) = \det(A) + \det(B)$

E. (3 points) If A is invertible, then $\det(ABA^{-1}) = \det(B)$.

Solution.

A. False. $I \sim [\mathbf{e}_2\ \mathbf{e}_1]^T$ by a single row swap, but row swapping negates the determinant, i.e., $\det(I) = 1$ and $\det[\mathbf{e}_2\ \mathbf{e}_1] = -1$.

B. False. $\det(5I) = 25$ whereas $5 \det(I) = 5$.

C. True. Since $\det(A^T) = \det(A)$, we just need to show that $\det(A)^2 \geq 0$. This is true because squared numbers must be nonnegative.

D. False. $\det(I) = 1$ and $\det(-I) = 1$, but $\det(I - I) = 0$.

E. True. Since $\det(ABA^{-1}) = \det(A) \det(B) \det(A^{-1})$ and $\det(A^{-1}) = \det(A)^{-1}$, the left side of the equation reduces to $\det(B)$.

4 Characteristic Polynomials

A. (3 points) Find the characteristic polynomial of

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}$$

and use it in order to determine the eigenvalues A.

B. (4 points) Find the characteristic polynomial of

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 5 & 0 & 0 & 0 \\ 2 & 6 & 3 & 0 & 0 \\ 10 & -15 & 3 & 4 & 0 \\ -1 & 5 & 2 & 5 & 5 \end{bmatrix}$$

and use it to determine the eigenvalues of A.

C. (5 points) Find the characteristic polynomial of

$$\begin{bmatrix} 1 & 0 & 2 & 10 & 5 \\ 0 & 0 & 5 & -3 & 15 \\ 0 & 0 & 16 & 6 & -1 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 7 & 4 \end{bmatrix}$$

You do not have to factor the polynomial, but your expression should not contain any fractions.

D. (6 points) Find the characterstic polynomial of

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 5 \\ 0 & 1 & 3 \end{bmatrix}$$

You do not have to factor the polynomial, but your expression should not contain any fractions. *Hint.* Try to row reduce $A - \lambda I$ as usual, scaling rows before zeroing out entries. At the end, the scalings you performed will divide the final polynomial.

Solution.

A. The characteristic polynomial is the determinant of the matrix

$$\begin{bmatrix} 1-\lambda & -1 \\ -1 & 3-\lambda \end{bmatrix}$$

which is $(1-\lambda)(3-\lambda)+1 = \lambda^2 - 4\lambda + 2$. This has roots $2+\sqrt{2}$ and $2-\sqrt{2}$, which are the eigenvalues of hte matrix.

B. The determinant of a triangular matrix is the product of its entries, so the characteristic polynomial of A is $(1-\lambda)(5-\lambda)^2(3-\lambda)(4-\lambda)$. Therefore, the eigenvalues of A are 1, 5, 3, and 4.

C. This matrix is not quite triangular, but it is one row reduction away. We need to find the determinant of

$$\begin{bmatrix} 1-\lambda & 0 & 2 & 10 & 5 \\ 0 & -\lambda & 5 & -3 & 15 \\ 0 & 0 & 16-\lambda & 6 & -1 \\ 0 & 0 & 0 & 1-\lambda & 5 \\ 0 & 0 & 0 & 7 & 4-\lambda \end{bmatrix}$$

We use the row operations $R_5 \leftarrow (1-\lambda)R_5$ and then $R_5 \leftarrow R_5 - 7R_4$ to get the matrix

$$\begin{bmatrix} 1-\lambda & 0 & 2 & 10 & 5 \\ 0 & -\lambda & 5 & -3 & 15 \\ 0 & 0 & 16-\lambda & 6 & -1 \\ 0 & 0 & 0 & 1-\lambda & 5 \\ 0 & 0 & 0 & 0 & (4-\lambda)(1-\lambda)-35 \end{bmatrix}$$

This is in echelon form and we scaled once by $1-\lambda$ so the determinant is

$$\frac{(\lambda-1)\lambda(16-\lambda)(1-\lambda)((4-\lambda)(1-\lambda)-35)}{1-\lambda}$$

which is equal to $(\lambda-1)\lambda(16-\lambda)((4-\lambda)(1-\lambda)-35)$ after cancellation.

D. Using the same process as the previous problem, we have to find the determinant of

$$\begin{bmatrix} 1-\lambda & 0 & 0 \\ 1 & 2-\lambda & 5 \\ 0 & 1 & (3-\lambda) \end{bmatrix}$$

If we follow the process of forward elimination, we can use the row operations $R_2 \leftarrow (1-\lambda)R_2$ and $R_2 \leftarrow R_2 - R_1$ to get

$$\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & (2-\lambda)(1-\lambda) & 5(1-\lambda) \\ 0 & 1 & (3-\lambda) \end{bmatrix}$$

Then we can use the row operations $R_3 \leftarrow (2-\lambda)(1-\lambda)R_2$ and $R_3 \leftarrow R_3 - R_2$ to get

$$\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & (2-\lambda)(1-\lambda) & 5(1-\lambda) \\ 0 & 0 & (3-\lambda)(2-\lambda)(1-\lambda)-5(1-\lambda) \end{bmatrix}$$

We had to do two scalings, so the determinant is

$$\frac{(1-\lambda)^2(2-\lambda)((3-\lambda)(2-\lambda)(1-\lambda)-5(1-\lambda))}{(1-\lambda)^2(2-\lambda)}$$

which reduces to $((3-\lambda)(2-\lambda)-5)(1-\lambda)$.

5 Closed-Form Solution for Fibonacci

Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

A. (3 points) Verify that

$$\mathbf{v}_1 = \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$$

form an eigenbasis of A (recall that this means showing they are eigenvectors of A, and form a basis of $\mathbb{R}^2$). Also determine the eigenvalues for each eigenvector.

B. (3 points) Write the vector $[1\ 0]^T$ in terms of the eigenbasis you found. In other words, determine α_1 and α_2 such that

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2$$

Hint. Don't try to do any row reductions, this will be a messy calculation. Calcuate $\mathbf{v}_1 - \mathbf{v}_2$.

C. (4 points) Write down a closed-form solution for the linear dynamical system determined by A with initial vector $[1\ 0]^T$.

D. (3 points) If you look at the formula given by the second component of your closed-form solution from the previous part, this gives a *non-recursive* definition for Fibonacci numbers. Write down this formula and use it (and Python or a calculator) to calculate F_{20}, the 20th fibonacci number (where $F_0 = 0$ and $F_1 = 1$).[1]

Solution. Let $\varphi = \frac{1+\sqrt{5}}{2}$ and $\varphi' = \frac{1-\sqrt{5}}{2}$.

A.

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \varphi \\ 1 \end{bmatrix} = \begin{bmatrix} \varphi + 1 \\ \varphi \end{bmatrix} = \varphi \begin{bmatrix} \varphi \\ 1 \end{bmatrix}$$

Note that

$$\frac{1+\sqrt{5}}{2}\left(\frac{1+\sqrt{5}}{2}\right) = \frac{1+2\sqrt{5}+5}{4} = \frac{3+\sqrt{5}}{2} = 1 + \frac{1+\sqrt{5}}{2}$$

Likewise,

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \varphi' \\ 1 \end{bmatrix} = \begin{bmatrix} \varphi' + 1 \\ \varphi' \end{bmatrix} = \varphi' \begin{bmatrix} \varphi' \\ 1 \end{bmatrix}$$

[1] You should verify that this is correct by writing a Python function which computes Fibonacci numbers in the usual way, but you do not have to do this.

again noting that

$$\frac{1-\sqrt{5}}{2}\left(\frac{1-\sqrt{5}}{2}\right) = \frac{1-\sqrt{5}+5}{4} = \frac{3-\sqrt{5}}{2} = 1 + \frac{1-\sqrt{5}}{2}$$

B. Note that

$$\mathbf{v}_1 - \mathbf{v}_2 = \begin{bmatrix} \varphi - \varphi' \\ 0 \end{bmatrix}$$

and $\varphi - \varphi' = \sqrt{5}$, so

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{5}}\mathbf{v}_1 - \frac{1}{\sqrt{5}}\mathbf{v}_2$$

C. As we discussed in lecture, the closed-form solution for a linear dynamical system when we can write a vector as a linear combination of the others is

$$\mathbf{u}_k = \frac{\varphi^k}{\sqrt{5}}\mathbf{v}_1 + \frac{\varphi'^k}{\sqrt{5}} + \mathbf{v}_2$$

D. Looking at the formula given by the second component, since both second components of $\mathbf{v}_1$ and $\mathbf{v}_2$ are 1, we get

$$F_k = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right)$$

Using Python to compute the value, we get the value `6765.000000000005`, so after accounting floating-point error, $F_{20} = 6765$.