Week 4 Discussion Solutions

CAS CS 132: Geometric Algorithms

September 25, 2023

During discussion sections, we will go over three problems.

- The first will be a warm-up question, to help you verify your understanding of the material.
- The second will be a solution to a problem on the assignment of the previous week.
- The third will be a problem similar to one on the assignment of the following week.

The remainder of the time will be dedicated to open Q&A.

1 Unique Solutions (Warm-up)

Determine if the following equations of the form $A\mathbf{x} = \mathbf{b}$ have unique solutions. If they do, write down what the solution is.

A. $\begin{bmatrix} 1 & -1 & 0.5 & 3 \\ 1 & 2 & 15 & 6 \\ 2 & 46 & -24 & 3 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 3 \\ 0 & 2 & 4 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Solution.

A. There are number of ways to approach these problems. You can, of course, convert A to reduced echelon form and then determine what the solution set looks like. But you can also reason about the shape of A without doing any calculations.

First, does this system have a solution? Yes, because $\mathbf{b}$ is a multiple of the first column of A, so

$$\mathbf{s} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

is a solution. However, this solution cannot be unique. Because A has more columns than rows (more variables than equations) the columns of A are linearly dependent. This means there is a nontrivial solution to equation $A\mathbf{x} = \mathbf{0}$. In particular, it means that A does not have a pivot in every column, so the general solution of $A\mathbf{x} = \mathbf{b}$ has a free variable, which implies there is more than one solution.

We can also reason about the algebraic properties of matrix-vector multiplication. Let

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

be a nontrivial solution to the equation $A\mathbf{x} = \mathbf{0}$. By the algebraic properties of matrix-vector multiplication, we know that

$$A(\mathbf{s} + \mathbf{v}) = A\mathbf{s} + A\mathbf{v} = \mathbf{b} + \mathbf{0} = \mathbf{b}$$

And since $\mathbf{v}$ is nontrivial, it must be that $\mathbf{s} + \mathbf{v} \neq \mathbf{s}$. So $\mathbf{s} + \mathbf{v}$ is a second solution to the equation $A\mathbf{x} = \mathbf{b}$.

B. By just a few row operations, we can convert A to an echelon form with a pivot in each column *and* each row. Since there is a pivot in each row, this means the columns of A span all of $\mathbb{R}^3$, so there must be a solution. Since there is a pivot in each column, this means that the general form solution of $A\mathbf{x} = \mathbf{b}$ has no free variables so the solution must be unique.

We can, again, reason about the algebraic properties of matrix-vector multiplication. Since there is a pivot in every column of A, the columns of A are linearly independent. Suppose there were two solutions $\mathbf{u}$ and $\mathbf{v}$ to the equation $A\mathbf{x} = \mathbf{b}$. Then

$$A(\mathbf{u} - \mathbf{v}) = A\mathbf{u} - A\mathbf{v} = \mathbf{b} - \mathbf{b} = \mathbf{0}$$

so $\mathbf{u} - \mathbf{v}$ is solution to the equation $A\mathbf{x} = \mathbf{0}$. Since the columns of A are linearly independent, it must be that $\mathbf{u} - \mathbf{v} = \mathbf{0}$, which means $\mathbf{u} = \mathbf{v}$.

2 Switching Free and Bound Variable (From the Previous Assignment)

Before presenting the solution to this problem, a note about general form solutions.

A **general form solution** is a description of the solution set of a system of linear equations (or vector equation or matrix equation) such that every variable either

- is designated as **free**,
- or is written as *a linear equation of the free variables*, and is called **basic**.

For example, the equation

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

has the general form solution

$$\begin{aligned} x_1 &= 1 - x_3 \\ x_2 &= 2 \\ x_3 &\text{ is free} \end{aligned}$$

but it also has the general form solution

$$\begin{aligned} x_1 &\text{ is free} \\ x_2 &= 2 \\ x_3 &= 1 - x_1 \end{aligned}$$

Both are valid general form solutions to the above equation. The first, we get from our procedure for reading a general form solution off of a reduced echelon form. But again, the second is also a general form solution, since *they both describe the same solution set.*

General form solutions are also sometimes called **parametric solutions** because the free variables are parameters in which the basic variables are written. Since reduced echelon forms are unique, there is a single general form solution that you will get by reading off the reduced echelon form of a matrix. But this does not mean that a system has a unique general form solution, it may have many general form solutions which describe the same solution set.

Consider the following augmented matrix in reduced echelon form.

$$\begin{bmatrix} 1 & 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 1 & 4 \end{bmatrix}$$

A. Write down the solution to this system in general form.

B. Rewrite the solution in general form so that x_1 and x_3 are free (and x_2 and x_4 are written in terms of x_1 and x_3).

C. Write down the solution from part B as an augmented matrix (that is, your general form solution from part B should have two linear equations, you should rearrange these equations and write them as an augmented matrix). Write down the row operation of the form $R_i \leftarrow R_i + cR_j$ which converts the matrix from this part to the one above.

Solution.

A. We can follow the procedure that we covered in lecture.

$$\begin{aligned} &x_1 = 3 - x_2 - 2x_4 \\ &x_2 \text{ is free} \\ &x_3 = 4 - x_4 \\ &x_4 \text{ is free} \end{aligned}$$

B. The equations in the previous part for the values of x_1 and x_3 can be rearranged to isolate x_2 and x_4. It is then possible to substitute values so that x_2 and x_4 are written entirely in terms of x_1 and x_3. This gives the following general form solution.

$$\begin{aligned} &x_1 \text{ is free} \\ &x_2 = (-5) - x_1 + 2x_3 \\ &x_3 \text{ is free} \\ &x_4 = 4 - x_3 \end{aligned}$$

C. As noted in the hint, the equations in the previous part for x_2 and x_4 can also be viewed as linear equations. If we rearrange the equations and write them down as an augmented matrix, we get the following.

$$\begin{bmatrix} 1 & 1 & -2 & 0 & -5 \\ 0 & 0 & 1 & 1 & 4 \end{bmatrix}$$

We can get to the matrix above from this one by the row operation $R_1 \leftarrow R_1 + 2R_2$.

3 Matrix-Vector Multiplications (Similar to the Current Assignment)

Write down a matrix A such that the following equations hold. For each part, write down A in the specified shape, where ■ represents a nonzero entry.

A. $$\begin{bmatrix} \blacksquare & 0 & \blacksquare \\ 0 & \blacksquare & 0 \\ \blacksquare & 0 & \blacksquare \\ 0 & \blacksquare & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

B. $$\begin{bmatrix} \blacksquare & \blacksquare & \blacksquare \\ 0 & \blacksquare & \blacksquare \\ 0 & 0 & \blacksquare \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Solution.

A. We can look at each row, one by one, to determine the restriction on these values. For the first row we need to choose values so that $1\blacksquare + 0(2) + 1\blacksquare = 1$. Since they both need to be nonzero, we can take each to be 0.5. Repeating this for every row, we can get this solution

$$\begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \\ 1.5 & 0 & 1.5 \\ 0 & 2 & 0 \end{bmatrix}$$

B. The process is the same, we can get the solution

$$\begin{bmatrix} 1/3 & 1/6 & 1/6 \\ 0 & 1/2 & 1/2 \\ 0 & 0 & 3/2 \end{bmatrix}$$