Week 4 Discussion

CAS CS 132: Geometric Algorithms

September 25, 2023

During discussion sections, we will go over three problems.

- The first will be a warm-up question, to help you verify your understanding of the material.
- The second will be a solution to a problem on the assignment of the previous week.
- The third will be a problem similar to one on the assignment of the following week.

The remainder of the time will be dedicated to open Q&A.

1 Unique Solutions (Warm-up)

Determine if the following equations of the form $A\mathbf{x} = \mathbf{b}$ have unique solutions. If they do, write down what the solution is.

A. $\begin{bmatrix} 1 & -1 & 0.5 & 3 \\ 1 & 2 & 15 & 6 \\ 2 & 46 & -24 & 3 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 3 \\ 0 & 2 & 4 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Solution.

2 Switching Free and Bound Variable (From the Previous Assignment)

Before presenting the solution to this problem, a note about general form solutions.

A **general form solution** is a description of the solution set of a system of linear equations (or vector equation or matrix equation) such that every variable either

- is designated as **free**,
- or is written as *a linear equation of the free variables*, and is called **basic**.

For example, the equation

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

has the general form solution

$$\begin{aligned} x_1 &= 1 - x_3 \\ x_2 &= 2 \\ x_3 &\text{ is free} \end{aligned}$$

but it also has the general form solution

$$\begin{aligned} x_1 &\text{ is free} \\ x_2 &= 2 \\ x_3 &= 1 - x_1 \end{aligned}$$

Both are valid general form solutions to the above equation. The first, we get from our procedure for reading a general form solution off of a reduced echelon form. But again, the second is also a general form solution, since *they both describe the same solution set.*

General form solutions are also sometimes called **parametric solutions** because the free variables are parameters in which the basic variables are written. Since reduced echelon forms are unique, there is a single general form solution that you will get by reading off the reduced echelon form of a matrix. But this does not mean that a system has a unique general form solution, it may have many general form solutions which describe the same solution set.

Consider the following augmented matrix in reduced echelon form.

$$\begin{bmatrix} 1 & 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 1 & 4 \end{bmatrix}$$

A. Write down the solution to this system in general form.

B. Rewrite the solution in general form so that x_1 and x_3 are free (and x_2 and x_4 are written in terms of x_1 and x_3).

C. Write down the solution from part B as an augmented matrix (that is, your general form solution from part B should have two linear equations, you should rearrange these equations and write them as an augmented matrix). Write down the row operation of the form $R_i \leftarrow R_i + cR_j$ which converts the matrix from this part to the one above.

Solution.

3 Matrix-Vector Multiplications (Similar to the Current Assignment)

Write down a matrix A such that the following equations hold. For each part, write down A in the specified shape, where $\blacksquare$ represents a nonzero entry.

A. $\begin{bmatrix} \blacksquare & 0 & \blacksquare \\ 0 & \blacksquare & 0 \\ \blacksquare & 0 & \blacksquare \\ 0 & \blacksquare & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

B. $\begin{bmatrix} \blacksquare & \blacksquare & \blacksquare \\ 0 & \blacksquare & \blacksquare \\ 0 & 0 & \blacksquare \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Solution.