Week 5 Discussion Solutions

CAS CS 132: Geometric Algorithms

October 2, 2023

During discussion sections, we will go over three problems.

- The first will be a warm-up question, to help you verify your understanding of the material.
- The second will be a solution to a problem on the assignment of the previous week.
- The third will be a problem similar to one on the assignment of the following week.

The remainder of the time will be dedicated to open Q&A.

1 Definitions

Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n \end{bmatrix}$ be a $m \times n$ matrix (in particular, each column $\mathbf{a}_i$ is a vector in $\mathbb{R}^m$), let $\mathbf{b}$ be an arbitrary nonzero vector in $\mathbb{R}^m$ and let T be the linear transformation implemented by A. Consider the following list of statements.

1. The equation $A\mathbf{x} = \mathbf{b}$ is consistent.
2. The equation $A\mathbf{x} = \mathbf{c}$ is consistent for any choice of $\mathbf{c}$.
3. $A\mathbf{x} = \mathbf{b}$ has a unique solution.
4. $A\mathbf{x} = \mathbf{0}$ has a unique solution.
5. $A\mathbf{x} = \mathbf{c}$ has a unique solution for any choice of $\mathbf{c}$.
6. $\mathbf{b} \in \mathsf{span}\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$.
7. $\mathbf{c} \in \mathsf{span}\{\mathsf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$ for any choice of $\mathbf{c}$.
8. $\mathbf{a}_n \in \mathsf{span}\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{n-1}\}$.
9. A has a pivot in every column.
10. A has a pivot in every row.
11. $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}$ has a pivot in every row.
12. $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}$ has a pivot in its last (rightmost) column.
13. $\mathbf{b}$ is an image under T.
14. $\mathbf{b} \in \mathsf{ran}(T)$.
15. $\mathbf{c} \in \mathsf{ran}(T)$ for any choice of $\mathbf{c}$.
16. $\mathsf{ran}(T) = \mathsf{cod}(T)$.
17. $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ span $\mathbb{R}^m$.
18. $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ are linearly independent.
19. $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n, \mathbf{b}$ are linearly independent.
20. $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n, \mathbf{c}$ are linearly dependent for any choice of $\mathbf{c}$.

For any pair (i) and (j) you should be able to determine which of the following hold.

- if (i) is true than (j) must be true.
- if (i) is true then (j) must be false.
- if (i) is true then (j) may or may not be true.

For example, if (1) is true then (6) must be true. Also, if (8) is true then (18) must be false. But if (3) is true, then (7) may or may not be true.

Suppose (5) is true. For each statement, determine if it must be true, must be false, or may or may not be true. (The remaining cases you will have to do on your own, this may be a good way to study the material.)

Solution. If (5) is true then

1. must be true
2. must be true
3. must be true
4. must be true
5. identical
6. must be true: this is a rephrasing of (1) in terms of span
7. must be true: this is a rephrasing of (2) in terms of span
8. must be false: this would imply the columns are linearly dependent, which would allow for nontrival solutions to $\mathbf{A}\mathbf{x} = \mathbf{0}$.
9. must be true: equivalent restatement of linear independence, which is equivalent to (4).
10. must be true: equivalent restatement of spanning $\mathbb{R}^m$.
11. must be true: equivalent to (10).
12. must be false: this would imply the system has an row representing an inconsistent equation, which contradicts (1).
13. must be true: equilalent to (1).
14. must be true: equivalent to (13).
15. must be true: equivalent to (2).
16. must be true: equivalent to spanning $\mathbb{R}^m$.
17. must be true: equivalent to (16).

18. must be true: equivalent to (9).
19. must be false: if (1) is true, then there is a nontrivial linear combination.
20. must be true: implied by (7).

2 Linearly Dependent Sets

(15 points) Write down three **nonzero** vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ in $\mathbb{R}^3$ such that

- $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent and
- $\mathbf{v}_1$ **cannot** be written as a linear combination of $\mathbf{v}_2$ and $\mathbf{v}_3$.

Your solution should be given in the following form:

$$\mathbf{v}_1 = \begin{bmatrix} * \\ * \\ * \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} * \\ * \\ * \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} * \\ * \\ * \end{bmatrix}$$

where $*$ represents an arbitrary entry.

Solution. If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent then *at least one* of the vector can be written as a linear combination of the others. The second condition indicates that this cannot be $\mathbf{v}_1$, so it has to be $\mathbf{v}_2$ or $\mathbf{v}_3$. We start by taking $\mathbf{v}_2$ to be something simple, like

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

We now need to take $\mathbf{v}_1$ and $\mathbf{v}_3$ such that $\mathbf{v}_2$ can be written as one of their linear combinations. Furthermore, since $\mathbf{v}_1$ cannot be written a linear combinator of $\mathbf{v}_2$ and $\mathbf{v}_3$, then it *must* be the case that the coefficient α_1 for $\mathbf{v}_1$ in

$$\mathbf{v}_2 = \alpha_1 \mathbf{v}_1 + \alpha_3 \mathbf{v}_3$$

is 0. If $\alpha_1 \neq 0$, then $\mathbf{v}_1 = (-1/\alpha_1)\mathbf{v}_2 + (\alpha_3/\alpha_1)\mathbf{v}_3$. In other words, it must be that $\mathbf{v}_3$ is a scalar multiple of $\mathbf{v}_2$. Let's take

$$\mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

Finally, we just have to take $\mathbf{v}_1$ to be any vector which is *not* as scalar multiple of $\mathbf{v}_2$ (by definition of $\mathbf{v}_3$, any linear combination of $\mathbf{v}_2$ and $\mathbf{v}_3$ is a scalar multiple of $\mathbf{v}_2$). So we get a final answer:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

3 3D Matrix Transformations

A. Find the 3×3 matrix which implements the linear transformation which rotates vectors around the x_3-axis.

B. Find the 3×3 matrix which implements the linear transformation which rotates vectors around the line generated by the span of the vector $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ by 180 degrees.

Solution.

A. The standard basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are transformed as if the x_1x_2-plane was $\mathbb{R}^2$. The vector $\mathbf{e}_3$ is not transformed at all because it is in the direction of the axis around which we are rotating. This gives us the matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If we multiply this matrix with a vector, the x_1-part and x_2-part are rotated while the x_3 part remains the same.

B. The line generated by this vector is the same as $y = x$, again thinking of the x_1x_2-plane as $\mathbb{R}^2$. If we rotate by 180 degrees about this line the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ trade places. The vector $\mathbf{e}_3$ is flipped in direction. Note that we know that the lengths of these vectors aren't going to change. This gives us the matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$