Week 5 Discussion

CAS CS 132: Geometric Algorithms

October 2, 2023

During discussion sections, we will go over three problems.

- The first will be a warm-up question, to help you verify your understanding of the material.
- The second will be a solution to a problem on the assignment of the previous week.
- The third will be a problem similar to one on the assignment of the following week.

The remainder of the time will be dedicated to open Q&A.

1 Definitions

Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n \end{bmatrix}$ be a $m \times n$ matrix (in particular, each column $\mathbf{a}_i$ is a vector in $\mathbb{R}^m$), let $\mathbf{b}$ be an arbitrary nonzero vector in $\mathbb{R}^m$ and let T be the linear transformation implemented by A. Consider the following list of statements.

1. The equation $A\mathbf{x} = \mathbf{b}$ is consistent.
2. The equation $A\mathbf{x} = \mathbf{c}$ is consistent for any choice of $\mathbf{c}$.
3. $A\mathbf{x} = \mathbf{b}$ has a unique solution.
4. $A\mathbf{x} = \mathbf{0}$ has a unique solution.
5. $A\mathbf{x} = \mathbf{c}$ has a unique solution for any choice of $\mathbf{c}$.
6. $\mathbf{b} \in \mathsf{span}\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$.
7. $\mathbf{c} \in \mathsf{span}\{\mathsf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$ for any choice of $\mathbf{c}$.
8. $\mathbf{a}_n \in \mathsf{span}\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{n-1}\}$.
9. A has a pivot in every column.
10. A has a pivot in every row.
11. $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}$ has a pivot in every row.
12. $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}$ has a pivot in its last (rightmost) column.
13. $\mathbf{b}$ is an image under T.
14. $\mathbf{b} \in \mathsf{ran}(T)$.
15. $\mathbf{c} \in \mathsf{ran}(T)$ for any choice of $\mathbf{c}$.
16. $\mathsf{ran}(T) = \mathsf{cod}(T)$.
17. $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ span $\mathbb{R}^m$.
18. $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ are linearly independent.
19. $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n, \mathbf{b}$ are linearly independent.
20. $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n, \mathbf{c}$ are linearly dependent for any choice of $\mathbf{c}$.

For any pair (i) and (j) you should be able to determine which of the following hold.

- if (i) is true than (j) must be true.
- if (i) is true then (j) must be false.
- if (i) is true then (j) may or may not be true.

For example, if (1) is true then (6) must be true. Also, if (8) is true then (18) must be false. But if (3) is true, then (7) may or may not be true.

Suppose (5) is true. For each statement, determine if it must be true, must be false, or may or may not be true. (The remaining cases you will have to do on your own, this may be a good way to study the material.)

Solution.

2 Linearly Dependent Sets

(15 points) Write down three **nonzero** vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ in $\mathbb{R}^3$ such that

- $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent and
- $\mathbf{v}_1$ **cannot** be written as a linear combination of $\mathbf{v}_2$ and $\mathbf{v}_3$.

Your solution should be given in the following form:

$$\mathbf{v}_1 = \begin{bmatrix} * \\ * \\ * \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} * \\ * \\ * \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} * \\ * \\ * \end{bmatrix}$$

where $*$ represents an arbitrary entry.

Solution.

3 3D Matrix Transformations

A. Find the 3×3 matrix which implements the linear transformation which rotates vectors around the x_3-axis.

B. Find the 3×3 matrix which implements the linear transformation which rotates vectors around the line generated by the span of the vector $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ by 180 degrees.

Solution.