Week 7 Discussion Solutions

CAS CS 132: Geometric Algorithms

October 16, 2023

During discussion sections, we will go over three problems.

- The first will be a warm-up question, to help you verify your understanding of the material.
- The second will be a solution to a problem on the assignment of the previous week.
- The third will be a problem similar to one on the assignment of the following week.

The remainder of the time will be dedicated to open Q&A.

1 Matrix Inverses (Warm Up)

For each matrix A, determine if it is invertible. If it is, compute its inverse A^{-1} and demonstrate that it is an inverse by computing $A^{-1}A$ and AA^{-1}.

A.
$$A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & -1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

B.
$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix}$$

Solution.

A. We can determine if a matrix is invertible by trying to compute its inverse. Starting with the matrix

$$\begin{bmatrix} 2 & 3 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 2 & 1 & 1 \end{bmatrix}$$

After one row operation, we quickly see that the last row becomes a scalar multiple of the second, so there will be no way for the final matrix to contain the identity matrix in its first three columns. So this matrix is not invertible. In particular, there will be no solution to the equation $Ax = \mathbf{e}_3$.

B. We can reduce the following matrix to reduced echelon form.

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 2 & 2 & 0 & 0 & 0 & 1 \end{bmatrix}$$

This will give us the matrix

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & -0.5 \\ 0 & 0 & 1 & 1 & 1 & -0.5 \end{bmatrix}$$

The rest is a matter of working out the matrix multiplications.

2 Span and Linear Independence (Midterm)

Consider the following vectors in $\mathbb{R}^4$.

$$\mathbf{v}_1 = \begin{bmatrix} -3 \\ 4 \\ 3 \\ 7 \end{bmatrix} \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 2 \end{bmatrix} \mathbf{v}_3 = \begin{bmatrix} 0 \\ -2 \\ -1 \\ -1 \end{bmatrix} \mathbf{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -2 \end{bmatrix}$$

A. Determine if $\mathbf{v}_1$ is in $\mathsf{span}\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$. Justify your answer. In particular, if $\mathbf{v}_1$ is in $\mathsf{span}\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$, then write $\mathbf{v}_1$ as a linear combination of $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$.

B. Determine if the vectors $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ are linearly independent. Justify your answer. In particular, if they are linearly dependent, then write a dependence relation for them (that is, write the zero vector $\mathbf{0}$ as a linear combination of the vectors $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$).

C. Determine if the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent. Justify your answer. In particular, if they are linearly dependent, then write a dependence relation for them.

Solution. Before going into the solution, we emphasize that part of the point of this question was to recognize that you only need to solve one system of equations to answer all three of these question.

A. The matrix $[\mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4\ \mathbf{v}_1]$ has the reduced echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which means that $\mathbf{v}_1 = 5\mathbf{v}_2 - \mathbf{v}_3 + 2\mathbf{v}_4$. In detail,

$$\begin{aligned}
\begin{bmatrix} -1 & 0 & 1 & -3 \\ 0 & -2 & 1 & 4 \\ 0 & -1 & 1 & 3 \\ 2 & -1 & -2 & 7 \end{bmatrix} &\sim \begin{bmatrix} -1 & 0 & 1 & -3 \\ 0 & -2 & 1 & 4 \\ 0 & -1 & 1 & 3 \\ 0 & -1 & 0 & 1 \end{bmatrix} \\
&\sim \begin{bmatrix} -1 & 0 & 1 & -3 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 3 \\ 0 & -2 & 1 & 4 \end{bmatrix} \\
&\sim \begin{bmatrix} -1 & 0 & 1 & -3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & -2 & 1 & 4 \end{bmatrix} \\
&\sim \begin{bmatrix} -1 & 0 & 1 & -3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}$$

B. The above calculation also tells us that the matrix $[\mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4]$ has a pivot in every column, so these vectors are linearly independent.

C. The above calculation also tells us that the matrix $[\mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_1]$ has a pivot in every column, so these vectors are linearly independent. Another way of thinking about this, the above calculation tells us there is a unique solution to the equation $x_1\mathbf{v}_2 + x_2\mathbf{v}_3 + x_3\mathbf{v}_4 = \mathbf{v}_1$. It also tells us that the vectors $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly independent. Therefore, if $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ were linearly dependent, it would be possible to write $\mathbf{v}_1$ as a linear combination of $\mathbf{v}_2$ and $\mathbf{v}_3$, and this would give us another solution to the above equation.

3 3D Rotation Matrices

The matrices used for rotating counter-clockwise by θ around the x-, y-, and z-axes are

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \quad R_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \quad R_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that difference in sign for $R_{y,\theta}$. This is due to a convention called the *right-hand rule* which says that the orientation of the positive directions of each axis is given by thinking of the index finger, middle finger, and thumb of our right hand as the x-, y-, and $z-$axes (and that when we talk about counter-clockwise around an axis, we mean counter-clockwise when the positive axis is pointing towards us).

A. Compute $\mathbf{v} = R_{z,45^\circ}[1\ (-1)\ \sqrt{2}]^T$. Recall that $\cos 45^\circ = \sin 45^\circ = \frac{\sqrt{2}}{2}$.

B. Computer $R_{y,45^\circ}\mathbf{v}$, where $\mathbf{v}$ is the vector from the previous part.

Try to draw, to the best of your ability, what is happening to this vector after each transformation.

Solution.

A.

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 0 \\ \sqrt{2} \end{bmatrix}$$

B.

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ 0 \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$