# Week 8 Discussion Solutions

CAS CS 132: Geometric Algorithms

October 23, 2023

During discussion sections, we will go over three problems.

- The first will be a warm-up question, to help you verify your understanding of the material.
- The second will be a solution to a problem on the assignment of the previous week.
- The third will be a problem similar to one on the assignment of the following week.

The remainder of the time will be dedicated to open Q&A.

## 1 Warm up

Consider the following matrices in echelon form and reduced echelon form.

$$A = \begin{bmatrix} 2 & -3 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 10 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 3 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 10 & 1 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 6 \end{bmatrix}$$

A. If $D \sim A$ (that is $D$ is row equivalent to $A$) and $D$ is the augmented matrix of a system of linear equations, what can we say about the solutions of the system? (e.g., general form, number of solutions)

B. If $D \sim C$, then what can we say about the columns of $A$? (e.g., linearly independence, span)

C. If $D \sim C$, then what can we say about the solutions to the equation $D\mathbf{x} = \mathbf{0}$?

D. If $D \sim B$, then what can we say about the columns of $D$?

E. If $D \sim B$ and $D$ represents the augmented matrix of a system of linear equations of $B$, what can we say about the columns of the coefficient matrix for the system?

F. If $D \sim A$ and $D$ is the augmented matrix of a system of linear equations, what can we say about the solutions of the system for which $x_1 = 0$.

*Solution.* I'm going to leave this solution open-ended. Feel free to ask on Piazza.

## 2 Composing Rotations

Consider the following $\mathbb{R}^3$ rotation matrices.

$$A = \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ & 0 \\ \sin 45^\circ & \cos 45^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 180^\circ & -\sin 180^\circ \\ 0 & \sin 180^\circ & \cos 180^\circ \end{bmatrix}$$

The matrix $A$ rotates vectors around the $x_3$-axis by 45 degrees, and $B$ rotates vectors around the $x_1$-axis by 180 degrees. Also remember that

$$\begin{aligned} \cos 45^\circ &= \frac{\sqrt{2}}{2} \\ \sin 45^\circ &= \frac{\sqrt{2}}{2} \\ \cos 180^\circ &= -1 \\ \sin 180^\circ &= 0 \\ \cos(-\theta) &= \cos(\theta) \text{ for any } \theta \\ \sin(-\theta) &= -\sin(\theta) \text{ for any } \theta \end{aligned}$$

A. (4 points) Determine $A^{-1}$. Every entry should be a scalar multiple of 1, $\cos 45^\circ$ or $\sin 45^\circ$. In particular, don't include any trigonometric functions applied to negative angles. *Hint.* You don't have to do any calculations for this. Think about what it means to "undo" a rotation.

B. (6 points) Calculate $ABA^{-1}$. Your solutions should be as reduced as possible, and should not contain any trigonometric functions.

C. (4 points) Describe what the transformation implemented by $ABA^{-1}$ does geometrically. Your answer should be of the form "rotation by [NUMBER] degrees around the span of [VECTOR]." *Hint.* We've seen this transformation before.

*Solution.*

A. Rotation by $-45^\circ$:

$$\begin{bmatrix} \cos 45^\circ & \sin 45^\circ & 0 \\ -\sin 45^\circ & \cos 45^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

B. I'll leave the calculations to you, but you should get

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

C. This is rotation by 180 degrees around the span of $[1\ 1\ 0]^T$.

## 3 Regular Stochastic Matrices

For each of the following stochastic matrices:

- Determine if it is regular.
- Write down a solution to the homogeneous equation $(A - I)\mathbf{x} = \mathbf{0}$.
- If it is possible, find a steady-state vector from this solution.

A.

$$\begin{bmatrix} 1 & 0.5 \\ 0 & 0.5 \end{bmatrix}$$

B.

$$\begin{bmatrix} 0.2 & 1 \\ 0.8 & 0 \end{bmatrix}$$

*Solution.*

A. This matrix is not regular. Note that $(A^2)_{21} = 0(1) + 0.5(0) = 0$. More generally, For any $2 \times 2$ matrices $A$ and $B$,

$$(AB)_{21} = A_{21}B_{11} + A_{22}B_{21}$$

so if $A_{21} = B_{21} = 0$, then $(AB)_{21} = 0$. So any power of the above matrix has 0 in row 2 and column 1.

It is still possible to solve the equation $(A - I)\mathbf{x} = 0$.

$$(A - I)\begin{bmatrix} 0 & 0.5 \\ 0 & -0.5 \end{bmatrix}$$

which has the reduced echelon form

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

so $(A - I)\mathbf{x} = \mathbf{0}$ has the general form solution

$$\begin{aligned} &x_1 \text{ is free} \\ &x_2 = 0 \end{aligned}$$

The only steady-state vector which can be derived from this general form solution (remember that a steady-stead vector must be a probability vector is $[1\ 0]^T$.

B. The matrix is regular. Note that

$$A^2 = \begin{bmatrix} 0.86 & 0.2 \\ 0.16 & 0.8 \end{bmatrix}$$

As above,

$$\begin{bmatrix} -0.8 & 1 \\ 0.8 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1.25 \\ 0 & 0 \end{bmatrix}$$

so the system $(A - I)\mathbf{x} = \mathbf{0}$ has the general form solution

$$\begin{aligned} x_1 &= 1.25x_2 \\ x_2 &\text{ is free} \end{aligned}$$

In order to get a solution which is a probability vector, it must be that $x_1 + x_2 = 1$ (and they are both non-negative). Due to the first equation this means $(2.25)x_2 = 1$, so $x_2 = 4/9$ and $x_1 = 5/9$.