Week 8 Discussion

CAS CS 132: Geometric Algorithms

October 23, 2023

During discussion sections, we will go over three problems.

- The first will be a warm-up question, to help you verify your understanding of the material.
- The second will be a solution to a problem on the assignment of the previous week.
- The third will be a problem similar to one on the assignment of the following week.

The remainder of the time will be dedicated to open Q&A.

1 Warm up

Consider the following matrices in echelon form and reduced echelon form.

$$A = \begin{bmatrix} 2 & -3 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 10 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 3 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 10 & 1 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 6 \end{bmatrix}$$

A. If $D \sim A$ (that is D is row equivalent to A) and D is the augmented matrix of a system of linear equations, what can we say about the solutions of the system? (e.g., general form, number of solutions)

B. If $D \sim C$, then what can we say about the columns of A? (e.g., linearly independence, span)

C. If $D \sim C$, then what can we say about the solutions to the equation $D\mathbf{x} = \mathbf{0}$?

D. If $D \sim B$, then what can we say about the columns of D?

E. If $D \sim B$ and D represents the augmented matrix of a system of linear equations of B, what can we say about the columns of the coefficient matrix for the system?

F. If $D \sim A$ and D is the augmented matrix of a system of linear equations, what can we say about the solutions of the system for which $x_1 = 0$.

Solution.

2 Composing Rotations

Consider the following $\mathbb{R}^3$ rotation matrices.

$$A = \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ & 0 \\ \sin 45^\circ & \cos 45^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 180^\circ & -\sin 180^\circ \\ 0 & \sin 180^\circ & \cos 180^\circ \end{bmatrix}$$

The matrix A rotates vectors around the x_3-axis by 45 degrees, and B rotates vectors around the x_1-axis by 180 degrees. Also remember that

$$\begin{aligned} \cos 45^\circ &= \frac{\sqrt{2}}{2} \\ \sin 45^\circ &= \frac{\sqrt{2}}{2} \\ \cos 180^\circ &= -1 \\ \sin 180^\circ &= 0 \\ \cos(-\theta) &= \cos(\theta) \text{ for any } \theta \\ \sin(-\theta) &= -\sin(\theta) \text{ for any } \theta \end{aligned}$$

A. (4 points) Determine A^{-1}. Every entry should be a scalar multiple of 1, $\cos 45^\circ$ or $\sin 45^\circ$. In particular, don't include any trigonometric functions applied to negative angles. *Hint.* You don't have to do any calculations for this. Think about what it means to "undo" a rotation.

B. (6 points) Calculate ABA^{-1}. Your solutions should be as reduced as possible, and should not contain any trigonometric functions.

C. (4 points) Describe what the transformation implemented by ABA^{-1} does geometrically. Your answer should be of the form "rotation by [NUMBER] degrees around the span of [VECTOR]." *Hint.* We've seen this transformation before.

Solution.

3 Regular Stochastic Matrices

For each of the following stochastic matrices:

- Determine if it is regular.
- Write down a solution to the homogeneous equation $(A - I)\mathbf{x} = \mathbf{0}$.
- If it is possible, find a steady-state vector from this solution.

A.

$$\begin{bmatrix} 1 & 0.5 \\ 0 & 0.5 \end{bmatrix}$$

B.

$$\begin{bmatrix} 0.2 & 1 \\ 0.8 & 0 \end{bmatrix}$$

Solution.