Week 9 Discussion Solutions

CAS CS 132: Geometric Algorithms

October 30, 2023

During discussion sections, we will go over three problems.

- The first will be a warm-up question, to help you verify your understanding of the material.
- The second will be a solution to a problem on the assignment of the previous week.
- The third will be a problem similar to one on the assignment of the following week.

The remainder of the time will be dedicated to open Q&A.

1 LU Factorization by Hand (Warm Up)

Compute the LU factorization of the following matrix. Then verify that your factorization is correct by carrying out the matrix multiplication LU.

$$\begin{bmatrix} 1 & 2 & -3 \\ -1 & -1 & 4 \\ 2 & 3 & -6 \end{bmatrix}$$

Solution. This matrix can be reduced to an echelon form by the following row operations.

$$\begin{aligned} R_2 &\leftarrow R_2 + R_1 \\ R_3 &\leftarrow R_3 + (-2)R_1 \\ R_3 &\leftarrow R_3 + R_2 \end{aligned}$$

which yields the matrix

$$U = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

We can then build the matrix L using the procedure we discussed in lecture: we start with $L \leftarrow I$ and then perform the following sequence of operations on L according to the row operations above.

$$\begin{aligned} L_{21} &\leftarrow -1 \\ L_{31} &\leftarrow 2 \\ L_{32} &\leftarrow -1 \end{aligned}$$

so we get the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$$

2 Matrix Algebra

A. Suppose A and B are invertible matrices and $AB^TXA^{-1}B = I$. Solve for X in terms of A and B.

B. Show that $A + A^T$ is symmetric for any square matrix A. That is, show that $(A + A^T)^T = A + A^T$.

C. Show that if A and B are symmetric and $AB = BA$ then AB is symmetric.

Solution.

A. Solving for X is a matter of multiplying both sides of the equation by matrices until X is isolated:

$$\begin{aligned}
AB^TXA^{-1}B &= I \\
A^{-1}AB^TXA^{-1}B &= A^{-1}I \\
B^TXA^{-1}B &= A^{-1} \\
(B^{-1})^TB^TXA^{-1}B &= (B^{-1})^TA^{-1} \\
(B^T)^{-1}B^TXA^{-1}B &= (B^{-1})^TA^{-1} \\
XA^{-1}B &= (B^{-1})^TA^{-1} \\
XA^{-1}BB^{-1} &= (B^{-1})^TA^{-1}B^{-1} \\
XA^{-1} &= (B^{-1})^TA^{-1}B^{-1} \\
XA^{-1}A &= (B^{-1})^TA^{-1}B^{-1}A \\
X &= (B^{-1})^TA^{-1}B^{-1}A
\end{aligned}$$

It is not necessary for you have explicitly included the 5th line.

B. As noted in the problem statement, we need to show that $(A + A^T)^T = A + A^T$. We have to use the equalities we know about transposes to do this.

$$(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$$

C. Same idea as the previous part:

$$(AB)^T = B^TA^T = BA = AB$$

We can replace B^T with B and A^T with A because they are both symmetric.

3 NumPy and SciPy

A. Construct a random 10×10 matrix and a random vector in $\mathbb{R}^{10}$ using

```
import numpy
import scipy

a = numpy.random.rand(10, 10)
b = numpy.random.rand(10)
```

Then construct its inverse and its LU factoration using `numpy.linalg.inv` and `scipy.linalg.lu_factor`, respectively. Finally, solve equation `ax = b` in three ways:

```
s1 = numpy.linalg.solve(a, b)
s2 = a_inv @ b
s3 = scipy.lu_solve(lu) # where lu is the result of lu_factor
```

Verify that these three solutions are the same. Note, this process is not guaranteed to succeed since not all square matrices are invertible, but it is incredibly unlikely that a random 10×10 matrix is not invertible.

B. Construct a random integer matrix with entries between 1 and 10 as follows:

```
a = numpy.random.randint(1, 11, (10, 10))
```

Write a function called `norm_col` which divides a NumPy vector by the sum of its entries (you don't need to consider the case in which the input is the zero vector). Then read about the function `numpy.apply_along_axis` in the NumPy documentation. Check the value of

```
numpy.apply_along_axis(norm_col, 0, a)
```