Week 10 Discussion Solutions

CAS CS 132: Geometric Algorithms

November 6, 2023

During discussion sections, we will go over three problems.

- The first will be a warm-up question, to help you verify your understanding of the material.
- The second will be a solution to a problem on the assignment of the previous week.
- The third will be a problem similar to one on the assignment of the following week.

The remainder of the time will be dedicated to open Q&A.

1 Column Space and Null Space (Warm Up)

Consider the following matrix

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -1 \\ -3 & 1 & 5 \end{bmatrix}$$

A. Find the reduced echelon form of A.

B. Write down a general form solution for the equation $A\mathbf{x} = \mathbf{0}$.

C. Find a basis for $\operatorname{Col} A$.

D. Find a basis for $\operatorname{Nul} A$.

E. Find a linear equation whose solution set (i.e., those points in the plane represented by the linear equation) is $\operatorname{Col} A$.

Solution.

A.

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

B.

$$\begin{aligned} x_1 &= 2x_3 \\ x_2 &= x_3 \\ x_3 &\text{ is free} \end{aligned}$$

C. The pivot columns of A are its first and second columns. These columns form a basis for the column space of A:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

D. The above general form solution can be represented as a linear combination of column vectors with free variables as weights:

$$x_3 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

So this single vector forms a basis for the null space of A.

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

E. In order to find a plane which is spanned by two given vectors, we have to reduce a matrix of the form

$$\begin{bmatrix} 1 & -2 & x_1 \\ 0 & 1 & x_2 \\ -3 & 1 & x_3 \end{bmatrix}$$

to echelon form. Do so gives us

$$\begin{bmatrix} 1 & -2 & x_1 \\ 0 & 1 & x_2 \\ 0 & 0 & x_3 + 3x_1 + 5x_2 \end{bmatrix}$$

In order for the system of linear equations represented by the above matrix to be consisent, it must be that $3x_1 + 5x_2 + x_3 = 0$. In other words, in order for a vector $[x_1 \; x_2 \; x_3]^T$ to lie in the span of $\operatorname{Col} A$, it must be a solution to the equation $3x_1 + 5x_2 + x_3 = 0$.

2 Order of Matrix Multiplcation

The matrices for the 2D transformations (on homogeneous coordinates) of translation and counterclockwise rotation about the origin are

$$T_{x,y} = \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \qquad R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A. Show that $T_{1,1}R_{\pi/4} \neq R_{\pi/4}T_{1,1}$. Recall that $\pi/4$ in radians is 45° and $\cos(\pi/4) = \sin(\pi/4) = \sqrt{2}/2$.

B. If we want to do rotation *and then* translation, which of the two matrices in the previous part do we want to use?

C. Draw the effects $T_{1,1}R_{\pi/4}$ and $R_{\pi/4}T_{1,1}$ on the unit square on two separate graphs.

Solution.

A.

$$T_{1,1}R_{\pi/4} = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 1 \\ \sqrt{2}/2 & \sqrt{2}/2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad R_{\pi/4}T_{1,1} = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & \sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$$

B. Application order is the reverse of multiplication order. If we want to apply $R_{\pi/4}$ and then $T_{1,1}$, then we need to apply $T_{1,1}R_{\pi/4}$.

C. Please consult the solution given during your discussion section.

3 Intersections and Unions of Subspaces

Let H_1 and H_2 be arbitrary subspaces of $\mathbb{R}^n$.

A. The *intersection* of H_1 and H_2, written $H_1 \cap H_2$, is the set of vectors which appear in *both* H_1 and H_2:

$$H_1 \cap H_2 = \{\mathbf{v} : \mathbf{v} \in H_1 \text{ and } \mathbf{v} \in H_2\}$$

Show that $H_1 \cap H_2$ is a subspace. (In other words, the intersection of two spans is a span).

B. The *union* of H_1 and H_2, written $H_1 \cup H_2$, is the set of vectors which appear in *at least one of* H_1 and H_2:

$$H_1 \cup H_2 = \{\mathbf{v} : \mathbf{v} \in H_1 \text{ or } \mathbf{v} \in H_2\}$$

Give an explicit example of two subspaces $\mathbb{R}^2$ whose union is *not* a subspace of $\mathbb{R}^2$. *Hint.* Pick pretty much any two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$ and take the two subspaces to be $\text{span}\{\mathbf{u}\}$ and $\text{span}\{\mathbf{v}\}$.

Solution.

A. We have to verify two things: closure under addition and closure under scaling. First, closure under addition. Suppose we have two arbitrary vectors $\mathbf{u}$ and $\mathbf{v}$ of $H_1 \cap H_2$. We need to show that $\mathbf{u} + \mathbf{v}$ is in both H_1 and H_2. First note that $\mathbf{u}$ and $\mathbf{v}$ are both in H_1 and in H_2. Since H_1 is a subspace, it is closed under addition, so $\mathbf{u} + \mathbf{v} \in H_1$. Likewise, since H_2 is a subspace, it is closed under addition so $\mathbf{u} + \mathbf{v} \in H_2$. Therefore, $\mathbf{u} + \mathbf{v}$ is in $H_1 \cap H_2$.

Now for closure under scaling. Suppose we have an arbitrary vector $\mathbf{v}$ in $H_1 \cap H_2$ and an arbitrary scalar c. Then $\mathbf{v}$ is in both H_1 and H_2. Since H_1 is closed under scaling, we have $c\mathbf{v}$ is in H_1. And since H_2 is closed under scaling, we also have $c\mathbf{v}$ is in H_2. Therefore, $c\mathbf{v} \in H_1 \cap H_2$.

B. It is very unlikely that the union of spans of two distinct vectors will form a subspace. Consider $\text{span}\{\mathbf{e}_1\}$ and $\text{span}\{\mathbf{e}_2\}$. Then

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \text{span}\{\mathbf{e}_1\} \qquad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \text{span}\{\mathbf{e}_2\}$$

but $[1\ 1]^T$ does not appear in either spans. Try to draw a picture to make this more clear.