Week 12 Discussion Solutions

CAS CS 132: Geometric Algorithms

November 20, 2023

During discussion sections, we will go over three problems.

- The first will be a warm-up question, to help you verify your understanding of the material.
- The second will be a solution to a problem on the assignment of the previous week.
- The third will be a problem similar to one on the assignment of the following week.

The remainder of the time will be dedicated to open Q&A.

1 Diagonalization by Hand

Diagonalize the following matrix.

$$A = \begin{bmatrix} -3 & -4 \\ 2 & 3 \end{bmatrix}$$

Solution. First, we have to find the eigenvalues of A. This is a matter of finding the roots of its characteristic polynomial.

$$\det(A - \lambda) = (\lambda - 3)(\lambda + 3) + 8 = \lambda^2 - 1 = (\lambda + 1)(\lambda - 1)$$

This implies the eigenvalues of this matrix are 1 and -1.

Next we have to find bases for the eigenspaces of each of these eigenvalues. First, we will find a basis for $\text{Nul}(A - I)$. Since

$$A - I = \begin{bmatrix} -4 & -4 \\ 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

the solution set of the equation $(A - I)\mathbf{x} = \mathbf{0}$ can be written as

$$x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Therefore, $[-1\ 1]^T$ forms a basis of the eigenspace of A for the eigenvalue 1.

For the eigenvalue -1, we can do the same:

$$A + I = \begin{bmatrix} -2 & -4 \\ 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

which implies (by the same process as above) that $[-2\ 1]^T$ forms a basis of the eigenspace of A for -1. Therefore, we can take

$$P = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Finally, we have to find the inverse of P. One way to do this is to use the equation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

In this case,

$$P^{-1} = \frac{1}{(-1)(1) - (-2)(1)} \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$$

2 Determinant

Consider the following reduction sequence

$$\begin{aligned}
R_1 &\leftarrow R_1 + R_2 \\
\mathsf{swap}&(R_2, R_3) \\
R_3 &\leftarrow R_3 + 5R_4 \\
R_2 &\leftarrow -3R_2 \\
R_5 &\leftarrow R_5 - 10R_3 \\
R_5 &\leftarrow R_5/11 \\
\mathsf{swap}&(R_5, R_3) \\
\mathsf{swap}&(R_1, R_2) \\
R_4 &\leftarrow R_4 + R_1 \\
R_2 &\leftarrow 5R_2 \\
R_1 &\leftarrow -R_1
\end{aligned}$$

Suppose that $A \in \mathbb{R}^{5\times 5}$ reduces to U by this reduction sequence, where U is in *reduced* echelon form. If rank $A = 5$, what is $\det A$?

Solution. If U is in reduced echelon from and A has full rank, then $U = I$, and the product of it's diagonals is 1. The above sequence uses three swaps, so $s = 3$, and the scalings -3, $\frac{1}{11}$, 5 and -1, so $c = \frac{15}{11}$. So the determinant is $(-1)^3 \frac{11}{15}(1)$, which is $\frac{-11}{15}$.

3 2×2 Triangular Matrices and Diagonalization

Consider an arbitrary 2×2 matrix

$$A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$$

A. Find an expression for the characteristic polynomial of A.

B. If $a \neq d$, then A has 2 distinct eigenvalues, and so there must be an eigenbasis of $\mathbb{R}^2$ for A. If $a = d$, for what values of b is there an eigenbasis of $\mathbb{R}^2$ for A? Justify your answer.

Solution.

A.
$$(\lambda - a)(\lambda - d) = \lambda^2 - (a + d)\lambda + ad$$

B. If $a = d$, then A has a single eigenvalue, so we need to determine when $\dim(\text{Nul}(A - dI) = 2$. Since

$$A - dI = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$$

We need to count the number of pivot columns in this matrix. This matrix has 0 pivot positions exactly when $b = 0$. This means there is an eigenbasis for A exactly when $b = 0$.

This implies that the only triangular diagonalizable 2×2 matrices with a single eigenvalue are of the from cI for some constant c.