# Week 14 Discussion

CAS CS 132: Geometric Algorithms

December 4, 2023

During discussion sections, we will go over three problems.

- The first will be a warm-up question, to help you verify your understanding of the material.
- The second will be a solution to a problem on the assignment of the previous week.
- The third will be a problem similar to one on the assignment of the following week.

The remainder of the time will be dedicated to open Q&A.

# 1 Basis of the column space (Warm up)

Consider the following matrices. Note that $A'$ is an echelon from of $A$.

$$A = \begin{bmatrix} 1 & 1 & 2 & 0 & 2 \\ 3 & 4 & 9 & -2 & 5 \\ -2 & -3 & -7 & 2 & -2 \\ 2 & 2 & 4 & 0 & 5 \end{bmatrix} \qquad A' = \begin{bmatrix} 1 & 0 & -1 & 2 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A. Use the echelon form above to find a basis of $\text{Col}\, A$ made up of columns of $A$.

B. Write down a NumPy expression in terms of `A` (a 2D NumPy array representing the matrix $A$ above) for the matrix whose columns are the basis vectors you found in the previous part.

C. Let $A_i$ be the matrix whose columns are the first $i$ columns of $A$. For example,

$$A_3 = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 4 & 9 \\ -2 & -3 & -7 \\ 2 & 2 & 4 \end{bmatrix}$$

Find $\text{rank}(A_i)$ for each $i$ using the echelon form above.

D. Write down a NumPy expression for $\text{rank}(A_i)$ in terms of `A` and `i` and the NumPy function `numpy.linalg.matrix_rank`, which returns the rank of its argument.

E. Let $B$ be an arbitrary $m \times 5$ matrix and let $B_i$ be the matrix whose columns are the first $i$ columns of $B$. Further suppose that $\text{rank}(B_1) = 1$, $\text{rank}(B_2) = 1$, $\text{rank}(B_3) = 2$, $\text{rank}(B_4) = 3$, and $\text{rank}(B_5) = 3$. Which columns of $B$ form a basis of $\text{Col}\, B$?

F. Use the previous parts to describe in an informal procedure you can use to find a basis for the column space of a small matrix using Python.

*Solution.*

## 2 Boundary Reflection without a Matrix

Suppose that $A$ is a $n \times n$ matrix and $\mathbf{z}$ is a vector in $\mathbb{R}^n$ whose $i$th component is 1 if the $i$th column of $A$ is $\mathbf{0}$, and 0 otherwise, e.g.,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & 2 & 0 \end{bmatrix} \qquad \mathbf{z} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

For an arbitrary vector $\mathbf{v}$ in $\mathbb{R}^n$, write down an expression in terms of $A$, $\mathbf{z}$ and $\mathbf{v}$ for the vector

$$A'\mathbf{v}$$

where $A'$ is the same as $A$, but every all-zeros column of $A$ is replaced with the vector $c\mathbf{1}$ for some scalar $c$, e.g., as it pertains to the example above,

$$A' = \begin{bmatrix} c & 1 & c \\ c & -3 & c \\ c & 2 & c \end{bmatrix}$$

Furthermore, write down a NumPy expression which computes this *without* using the function `numpy.ones`.

*Solution.*

## 3 Multiple Least Squares Solutions

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 2 \\ 5 \\ 1 \end{bmatrix}$$

A. Find the orthogonal projection $\hat{\mathbf{b}}$ onto $\operatorname{Col} A$. (*Hint.* Note that the columns of $A$ are linearly dependent. It will be easier to do the computation if you take the last three columns of $A$ to find the projection.)

B. Find a general form solution for the homogeneous equation $A^T A\mathbf{x} = \mathbf{0}$. Then write this general form solution as a linear combination of vectors with free variables as weights.

C. Find the normal equations for the system $A\mathbf{x} = \mathbf{b}$.

D. Using the normal equations find a general form solution for the set of least squares solutions of $A\mathbf{x} = \mathbf{b}$. Then write this general form solution as a linear combination of vectors with free variables **and the scalar 1** as weights.

*Solution.*