Practice Final Solutions

CAS CS 132: Geometric Algorithms

December 14, 2023

Name:

BUID:

Location:

- You will have approximately 120 minutes to complete this exam.
- Make sure to read every question, some are easier than others.
- Please write your name and BUID **on every page**.

(Extra page)

1 Orthogonal Projections and Linear Equations

Consider the linear equation

$$x_1 - x_2 + x_3 = 0$$

and the vector

$$\mathbf{v} = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$$

A. (3 points) Write down a vector $\mathbf{z}$ which is orthogonal to the plane given by the above linear equation (that is, the vector which is orthogonal to every solution in its solution set.)

B. (5 points) Find a basis $\{\mathbf{b}_1, \mathbf{b}_2\}$ for the plane given by the above linear equation.

C. (5 points) Find a solution to the vector equation $y_1\mathbf{z} + y_2\mathbf{b}_1 + y_3\mathbf{b}_2 = \mathbf{v}$.

D. (5 points) Find the orthogonal projection of $\mathbf{v}$ onto the plane given by the above linear equation. (*Hint.* Use the previous parts.)

Solution.

A. The expression $x_1 - x_2 + x_3$ is the equal to

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

so we can take

$$\mathbf{z} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

B. A single equation is still a system of linear equations. The general form solution to this system is

$$\begin{aligned} &x_1 = x_2 - x_3 \\ &x_2 \text{ is free} \\ &x_3 \text{ is free} \end{aligned}$$

which can be written is as a linear combination of vectors with free variables as weights:

$$x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

so

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis for the plane.

C. This requires converting the augmented matrix

$$\begin{bmatrix} 1 & 1 & -1 & 4 \\ -1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

The unique solution to this system is $[1, 2, -1]^T$.

D. We have expressed $\mathbf{v}$ the sum of a vector orthogonal to the plane and vectors within the plane. The part of that sum which is in the plane must be the orthogonal projection of $\mathbf{v}$. Therefore

$$2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

is the orthogonal projection of $\mathbf{v}$ onto the plane.

Solution. (Continued)

Solution. (Continued)

Solution. (Continued)

2 True/False Questions

A. (2 points) For any matrix A, if A is square and $\det(A) = 0$, then the columns of A are linearly dependent.

B. (2 points) For any stochastic matrix A, if A has a unique stationary state, then it is regular.

C. (2 points) For any matrix A, the dimension of the null space of A is at most the rank of A.

D. (2 points) For any matrix A, if A has n distinct eigenvalues, then it is invertible.

E. (2 points) Every orthogonal set is linearly independent.

F. (2 points) For any two matrices A and B, if A is invertible and A is row equivalent to B then B is invertible.

G. (2 points) For any two matrices A and B, if AB is defined then $AB \neq BA$.

H. (2 points) For any matrix A and quadratic form $Q(\mathbf{x})$, if $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$, then A is symmetric.

Solution.

A. True

B. False

C. False

D. False

E. True

F. True

G. False

H. False

Solution. (Continued)

Solution. (Continued)

Solution. (Continued)

3 Elementary Matrices

A. (5 points) Find the 3×3 matrix E which implements the following row operations:

$$\begin{aligned} &\mathsf{swap}(R_1, R_2) \\ &R_1 \leftarrow 3R_1 \\ &R_3 \leftarrow R_3 + 2R_2 \end{aligned}$$

B. (6 points) Find values for i through m such that E^T implements the following row operations:

$$\begin{aligned} &\mathsf{swap}(R_i, R_j) \\ &R_k \leftarrow 3R_k \\ &R_l \leftarrow R_l + 2R_m \end{aligned}$$

C. (6 points) Compute AE where

$$A = \begin{bmatrix} 11 & 22 & 33 \\ 11 & 22 & 33 \\ 11 & 22 & 33 \end{bmatrix}$$

(*Hint.* Use the previous part and the fact that $(B^T)^T = B$.)

Solution.

A.

$$\begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

B. The matrix E^T is

$$\begin{bmatrix} 0 & 1 & 2 \\ 3 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which implements the row operations

$$\begin{aligned} &\mathsf{swap}(R_1, R_2) \\ &R_2 \leftarrow 3R_2 \\ &R_1 \leftarrow R_1 + 2R_3 \end{aligned}$$

C. Note that $AE = ((AE)^T)^T = (E^T A^T)^T$. This means multiplying A by E performs *column* operations on A. After doing the operations from the previous part (treated as column operations) we get

$$\begin{bmatrix} 88 & 33 & 33 \\ 88 & 33 & 33 \\ 88 & 33 & 33 \end{bmatrix}$$

Solution. (Continued)

Solution. (Continued)

Solution. (Continued)

4 Diagonalizability

$$A = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & -1 \\ 0 & 1 & 3 \end{bmatrix}$$

A. (7 points) Find the characteristic polynomial of A.

B. (8 points) Find bases for every eigenspace of A. That is for each eigenvalue λ of A, find a basis for $\text{Nul}(A - \lambda I)$.

C. (3 points) Determine if A is diagonalizable. If it is, provide a diagonalization. Otherwise, justify your answer.

Solution.

A. We need to determine the determinant of

$$\begin{bmatrix} 1-\lambda & 1 & 4 \\ 0 & 1-\lambda & -1 \\ 0 & 1 & 3-\lambda \end{bmatrix}$$

We first have to perform the row operations $R_3 \leftarrow (1-\lambda)R_3$ and $R_3 \leftarrow R_3 - R_2$ to get the matrix

$$\begin{bmatrix} 1-\lambda & 1 & 4 \\ 0 & 1-\lambda & -1 \\ 0 & 0 & (3-\lambda)(1-\lambda)+1 \end{bmatrix}$$

This included one scaling operation by $(1-\lambda)$, one replacement, and no swaps, so the determinant is

$$\begin{aligned} \frac{(-1)^0}{(1-\lambda)}(1-\lambda)(1-\lambda)((3-\lambda)(1-\lambda)+1) &= (1-\lambda)(\lambda^2 - 4\lambda + 4) \\ &= (1-\lambda)(\lambda-2)^2 \end{aligned}$$

B. To find a basis for the eigenspace of 1, we have to find a solution to the equation $(A - I)\mathbf{x} = \mathbf{0}$. Rather than solving a system of linear equations, we may notice that $(A - I)$ has $\mathbf{0}$ as its first column. In this case, $\{[1\ 0\ 0]^T\}$ is a basis for the eigenspace of 1.

We follow the same process for the eigenvalue 2. Starting with the matrix $(A - 2I)$:

$$\begin{bmatrix} -1 & 1 & 4 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

which means that $\{[3\ (-1)\ 1]^T\}$ is a basis for the eigenspace of 2.

C. A is not diagonalizable. There is no eigenbasis for $\mathbb{R}^3$ of A.

Solution. (Continued)

Solution. (Continued)

Solution. (Continued)

5 Interpreting Matrices

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 & 8 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 0 & 7 \\ 0 & 0 & -4 & 1 \\ 3 & -3 & 2 & 0 \\ 0 & 2 & -1 & 1 \end{bmatrix}$$

A. (2 points) Is A in echelon form?

B. (5 points) Find a basis of $\operatorname{Col} A$ with vectors that are columns of A.

C. (5 points) Find a basis of $\operatorname{Nul} A$.

D. (5 points) Compute $\det B$.

E. (2 points) Is B invertible?

Solution.

A. A is not in echelon form. The leading entry of the first row appears to the right of the leading entry of the second row.

B. It only takes one swap to put A is echelon form:

$$\begin{bmatrix} 0 & 1 & 2 & 1 & 8 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

We look to the pivot positions for vectors which form a basis of the columns space, so

$$\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

is a basis for $\operatorname{Col} A$. It would have also been possible to use the vector $[2\ 8]^T$.

C. We first have to put A into reduced echelon form:

$$\begin{bmatrix} 0 & 1 & 2 & 0 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

which has the general form solution

$$\begin{aligned} &x_1 \text{ is free} \\ &x_2 = -2x_3 + -6x_5 \\ &x_3 \text{ is free} \\ &x_4 = -2x_5 \\ &x_5 \text{ is free} \end{aligned}$$

which can be rewritten as

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ -6 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

These vectors form a basis for $\operatorname{Nul} A$.

D. B is almost a triangular matrix, we can get to

$$\begin{bmatrix} 3 & -3 & 2 & 0 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

by three swaps, no replacements and no scalings. This means the determinant is

$$(-1)^3(3)(2)(-4)(7) = 168$$

E. Yes, B has nonzero determinant.

Solution. (Continued)

Solution. (Continued)

Solution. (Continued)

6 Linear Models

Suppose we are given the data

$$(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$$

A. (5 points) Construct the design matrix for the given data which can be used to find the best-fit curve of the form

$$f_{\beta_1,\beta_2}(\theta) = \beta_1 \cos\theta + \beta_2 \sin\theta$$

where β_1 and β_2 are parameters.

B. (7 points) Consider trying to fit the data with a curve of the form

$$g_\alpha(\theta) = \cos(\theta + \alpha)$$

where α is a parameter. Note that g_α is not linear in its parameters. Given $\hat{\alpha}$ and $\hat{\beta_1}$ and $\hat{\beta_2}$, the parameters for the best-fit curves, show that

$$\sum_{i=1}^{4} \|\hat{\beta_1}\cos(x_i) + \hat{\beta_2}\sin(x_i) - y_i\|^2 \leq \sum_{i=1}^{4} \|\cos(x_i + \hat{\alpha}) - y_i\|^2$$

using the trigonometric identity

$$\cos(a + b) = \cos(a)\sin(b) + \sin(a)\cos(b)$$

In other words, show that the best-fit curve from part A has error at least as small as the error of the best-fit curve from part B.

C. (4 points, **Extra Credit**) Compute $\hat{\alpha}$ from $\hat{\beta_1}$ and $\hat{\beta_2}$. This implies that, in fact, the errors are equal.

Solution.

A.

$$\begin{bmatrix} \cos x_1 & \sin x_1 \\ \cos x_2 & \sin x_2 \\ \cos x_3 & \sin x_3 \\ \cos x_4 & \sin x_4 \end{bmatrix}$$

B. Since

$$\cos(\theta + \hat{\alpha}) = \sin(\hat{\alpha})\cos(\theta) + \cos(\hat{\alpha})\sin(\theta)$$

we know that $\beta_1 = \sin(\hat{\alpha})$ and $\beta_2 = \cos(\hat{\alpha})$ are possible coefficients for models in part A. Since $\hat{\beta_1}$ and $\hat{\beta_2}$ have the smallest error for any choice of coefficients, we know that

$$\begin{aligned} \sum_{i=1}^{4} \|\hat{\beta_1}\cos(x_i) + \hat{\beta_2}\sin(x_i) - y_i\|^2 &\leq \sum_{i=1}^{4} \|\sin(\hat{\alpha})\cos(x_i) + \cos(\hat{\alpha})\sin(x_i) - y_i\|^2 \\ &= \sum_{i=1}^{4} \|\cos(x_i + \hat{\alpha}) - y_i\|^2 \end{aligned}$$

C. If we take α such that

$$\hat{\beta}_1 = \cos\alpha \qquad \hat{\beta}_2 = \sin\alpha$$

we can note that

$$\frac{\sin\alpha}{\cos\alpha} = \frac{\hat{\beta}_2}{\hat{\beta}_1}$$

In other words

$$\alpha = \tan^{-1}\frac{\hat{\beta}_2}{\hat{\beta}_1}$$

So the best model of the form given in part A also provides a model of the form for part B. Since $\hat{\alpha}$ gives the best such model, together with part B this implies their errors are equal, and in fact they are the same model so,

$$\hat{\alpha} = \tan^{-1}\frac{\hat{\beta}_2}{\hat{\beta}_1}$$

This means we can train a model of the form given in part A, and then derive a model of the form given in part B.

Solution. (Continued)

Solution. (Continued)

Solution. (Continued)