# Midterm Solutions (Version 2)

CAS CS 132: Geometric Algorithms

October 12, 2023

Name:

BUID:

Location:

- You will have approximately 70 minutes to complete this exam.
- Make sure to read every question, some are easier than others.
- Please write your name and BUID **on every page**.
- Your work will only be seens on the **front facing pages**. You may write on the backs of pages if you are using a pencil, but you will not be evaluated on what is written there.

# 1 Span and Linear Independence

Consider the following vectors in $\mathbb{R}^4$.

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ -6 \\ 0 \\ -7 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \\ -3 \\ -1 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ -2 \\ 0 \end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \end{bmatrix}$$

A. (6 points) Determine if $\mathbf{v}_1$ is in $\mathsf{span}\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$. Justify your answer. In particular, if $\mathbf{v}_1$ is in $\mathsf{span}\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$, then write $\mathbf{v}_1$ as a linear combination of $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$.

B. (6 points) Determine if the vectors $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ are linearly independent. Justify your answer. In particular, if they are linearly dependent, then write a dependence relation for them (that is, write the zero vector $\mathbf{0}$ as a linear combination of the vectors $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$).

C. (6 points) Determine if the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent. Justify your answer. In particular, if they are linearly dependent, then write a dependence relation for them.

*Solution.*

A. The matrix $[\mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4\ \mathbf{v}_1]$ has the reduced echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which means that $\mathbf{v}_1 = 2\mathbf{v}_2 - 3\mathbf{v}_3 + 5\mathbf{v}_4$.

B. The above calculation also tells us that the matrix $[\mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4]$ has a pivot in every column, so these vectors are linearly independent.

C. There are a couple ways to reason about this. First, the above calculation again tells us that the matrix $[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3]$ has a pivot in every column, so these vectors are linearly independent. Alternatively, the above calculation tells us there is a unique solution to the equation $x_1\mathbf{v}_2 + x_2\mathbf{v}_3 + x_3\mathbf{v}_4 = \mathbf{v}_1$. It also tells us that the vectors $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly independent. Therefore, if $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ were linearly dependent, it would be possible to write $\mathbf{v}_1$ as a linear combination of $\mathbf{v}_2$ and $\mathbf{v}_3$. This would give us another solution to the above equation.

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## 2 True/False Questions

For each of the following statements, determine if they are true or false. You do not have to show your work or justify your answer. You just have to write "true" or "false."

A. (2 points) For any $20 \times 24$ matrix $A$, the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

B. (2 points) For any $m \times n$ matrix $A$, if $m > n$ then the columns of $A$ must be linearly independent.

C. (2 points) Every matrix has a unique echelon form.

D. (2 points) For any nonzero real values $a$ and $b$, the matrix $\begin{bmatrix} a & 2a \\ b & 3b \end{bmatrix}$ has a pivot in every column and every row.

E. (2 points) If $A$ is the augmented matrix of an inconsistent system, then $A$ has a pivot in its last (rightmost) column.

F. (2 points) To show that a transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is linear, it is enough to show that $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ for any two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^m$.

G. (2 points) For any vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ in $\mathbb{R}^n$, if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, then so is $\{\mathbf{v}_1, \mathbf{v}_2\}$.

H. (2 points) Any set of distinct standard basis vectors in $\mathbb{R}^n$ is linearly independent.

*Solution.*

A. True

B. False

C. False

D. True

E. True

F. False

G. True

H. True

*Extra Page.*

## 3 Inner Products and Matrix Equations

Consider the following matrix and vector.[1]

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -2 & 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

A. (3 points) Compute the following matrix-vector multiplications.

$$\begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \left( A \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right)$$

B. (6 points) Write down a general form solution for the solution set of the matrix equation $A\mathbf{x} = \mathbf{b}$.

C. (6 points) Use your solution to the previous part to find a **nonzero** vector $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ such that

$$\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \left( A \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \right) = 0$$

*Solution.*

A.

$$A \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -4 \\ 4 \end{bmatrix} = 3 + 4 + 4 = 11$$

B. The matrix $[A\ \mathbf{b}]$ has the reduced echelon form

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

from which we get the general form solution

$$\begin{aligned} x_1 &= 2 - x_3 \\ x_2 &= 1 \\ x_3 &\text{ is free} \end{aligned}$$

[1]Credit to Vishesh Jain for suggesting a version of this problem.

C. We need to find a vector in the solution set above such that

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0$$

which means that $x_1 = 2 - x_3 = 0$. Since $x_3$ is free we can choose it two be 2, so that $x_1 = 0$. So the choice of vector is

$$\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

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## 4 Linear Transformations

(5 points) Consider the following linear transformation $T$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} x_1 + 2x_2 - x_3 \\ x_3 \\ 2x_3 \end{bmatrix}$$

Find a set of linearly independent vectors which span the range of $T$. *Hint.* First find the matrix implementing $T$.

*Solution.* The matrix implementing $T$ is

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

The range of $T$ is exactly the span of the columns of $A$, since the span of the columns of a matrix $A$ is exactly the set of images of the matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$. The first two columns above are linearly dependent, so we're left with the set

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \right\}$$

## 5 Matrix Equations

Consider the following matrices.

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \quad \mathbf{e}_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

A. (3 points) Explain why the columns of $B$ span $\mathbb{R}^6$.

B. (7 points) Find a solution to the equation $B\mathbf{x} = \mathbf{e}_6$.

C. **(3 points, Extra Credit)** Find a solution to the equation $C\mathbf{x} = \mathbf{e}_6$.

*Solution.*

A. $A$ is in echelon form and it has a pivot in every row.

B. The matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

is almost in reduced echelon form, if we back substitute from here, we get the solution

$$\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

*Extra Page.*