Matrix-Vector Equations

Geometric Algorithms
Lecture 5

CAS CS 132

Recap Problem (+ Drill)

Is the vector $\begin{bmatrix} 9 \\ 3 \\ -14 \end{bmatrix}$ *in* $\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix} \right\}$*?*

Answer

solve the system of linear equations with the augmented matrix

$$\begin{bmatrix} 1 & 1 & 3 & 9 \\ 0 & 1 & 2 & 3 \\ -2 & -1 & -4 & -14 \end{bmatrix}$$

Answer

solve the system of linear equations with the augmented matrix

$$\begin{bmatrix} 1 & 1 & 3 & 9 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 4 \end{bmatrix}$$

$$R_3 \leftarrow R_3 + 2R_1$$

Answer

solve the system of linear equations with the augmented matrix

$$\begin{bmatrix} 1 & 1 & 3 & 9 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_3 \leftarrow R_3 - R_1$$

Answer

$$\begin{bmatrix} 1 & 1 & 3 & 9 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

`no solution ≡ not in the span`

Objectives

1. motivation
2. define matrix-vector multiplication
3. Revisit span
4. take stock of our perspectives on systems of linear equations

Keywords

matrix-vector multiplication

the matrix equation

inner-product

row-column rule

Motivation

Recall: Vector "Interface"

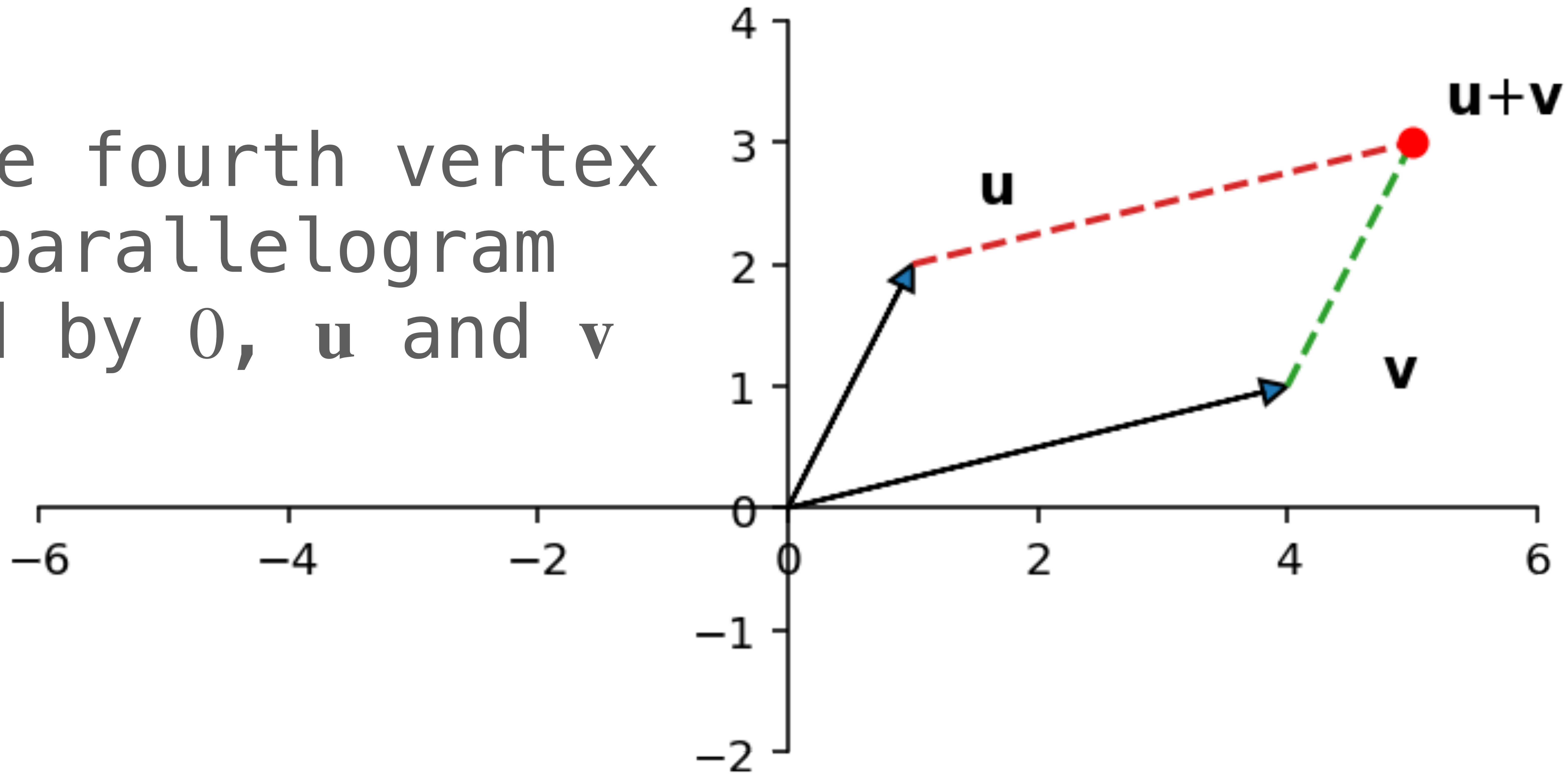
addition — what does $\mathbf{u} + \mathbf{v}$ (adding two vectors mean?

scaling — what does $a\mathbf{v}$ (multiplying a vector by a real number) mean?

Recall: Vector Addition (Geometrically)

in $\mathbb{R}^2$ it's called the *parallelogram rule*

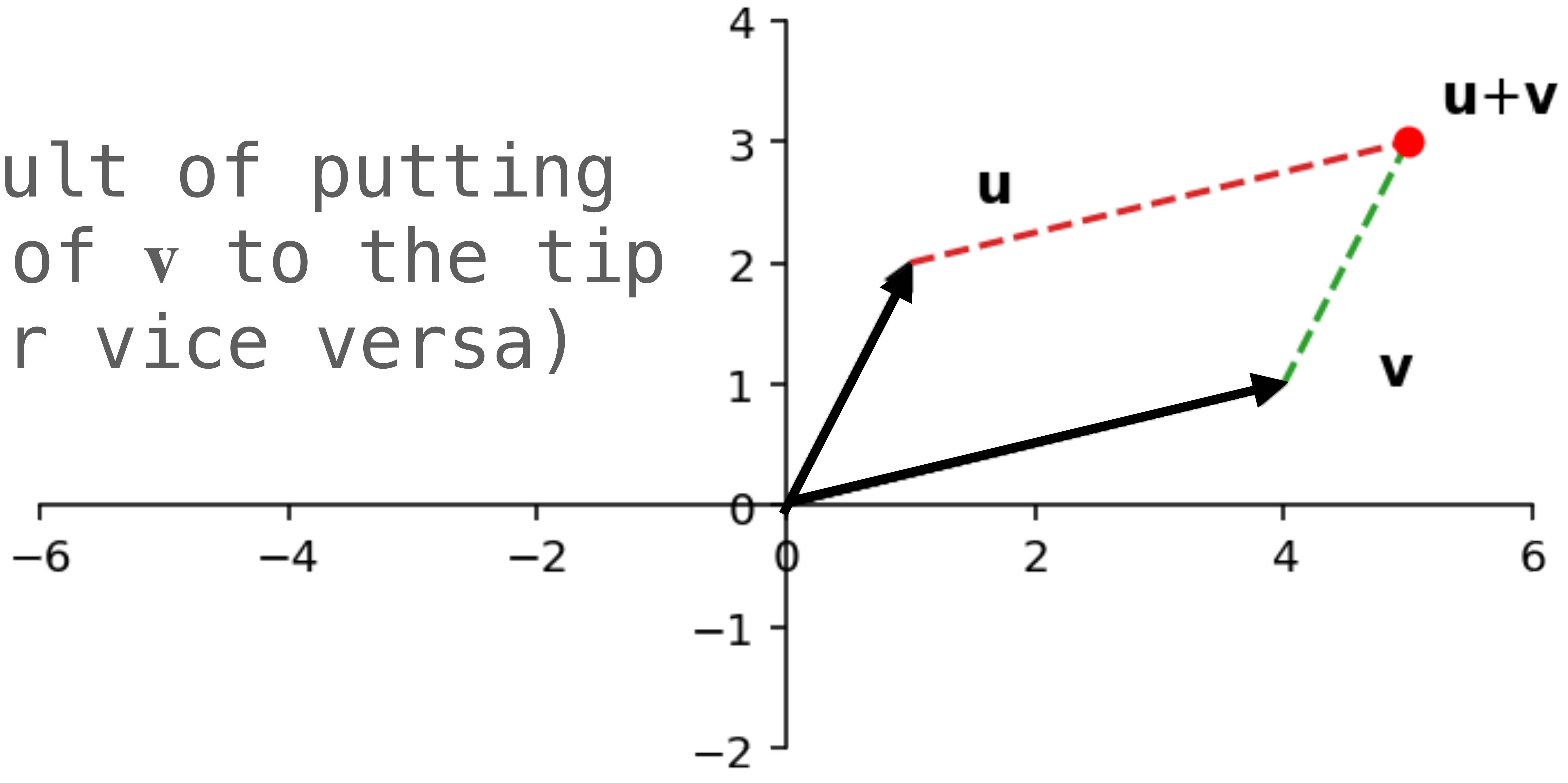
$\mathbf{u} + \mathbf{v}$ is the fourth vertex of the parallelogram generated by 0, $\mathbf{u}$ and $\mathbf{v}$



Vector Addition (Geometrically)

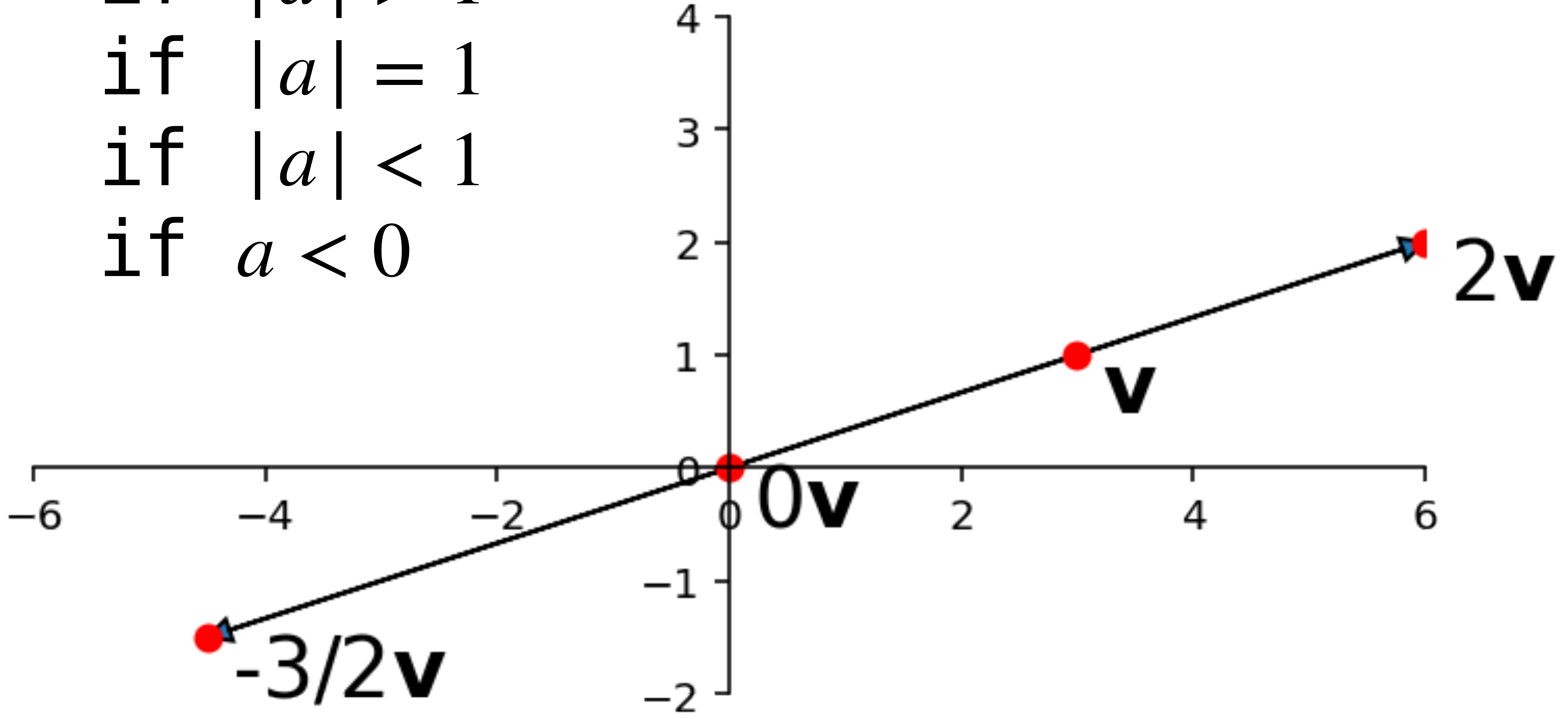
or the *tip-to-tail rule*

$\mathbf{u} + \mathbf{v}$ result of putting the tail of $\mathbf{v}$ to the tip of $\mathbf{u}$ (or vice versa)



Recall Vector Scaling (Geometrically)

longer	if $\|a\| > 1$
the same length	if $\|a\| = 1$
shorter	if $\|a\| < 1$
reversed	if $a < 0$



Recall Vector Scaling (Geometrically)

longer	if $\|a\| > 1$
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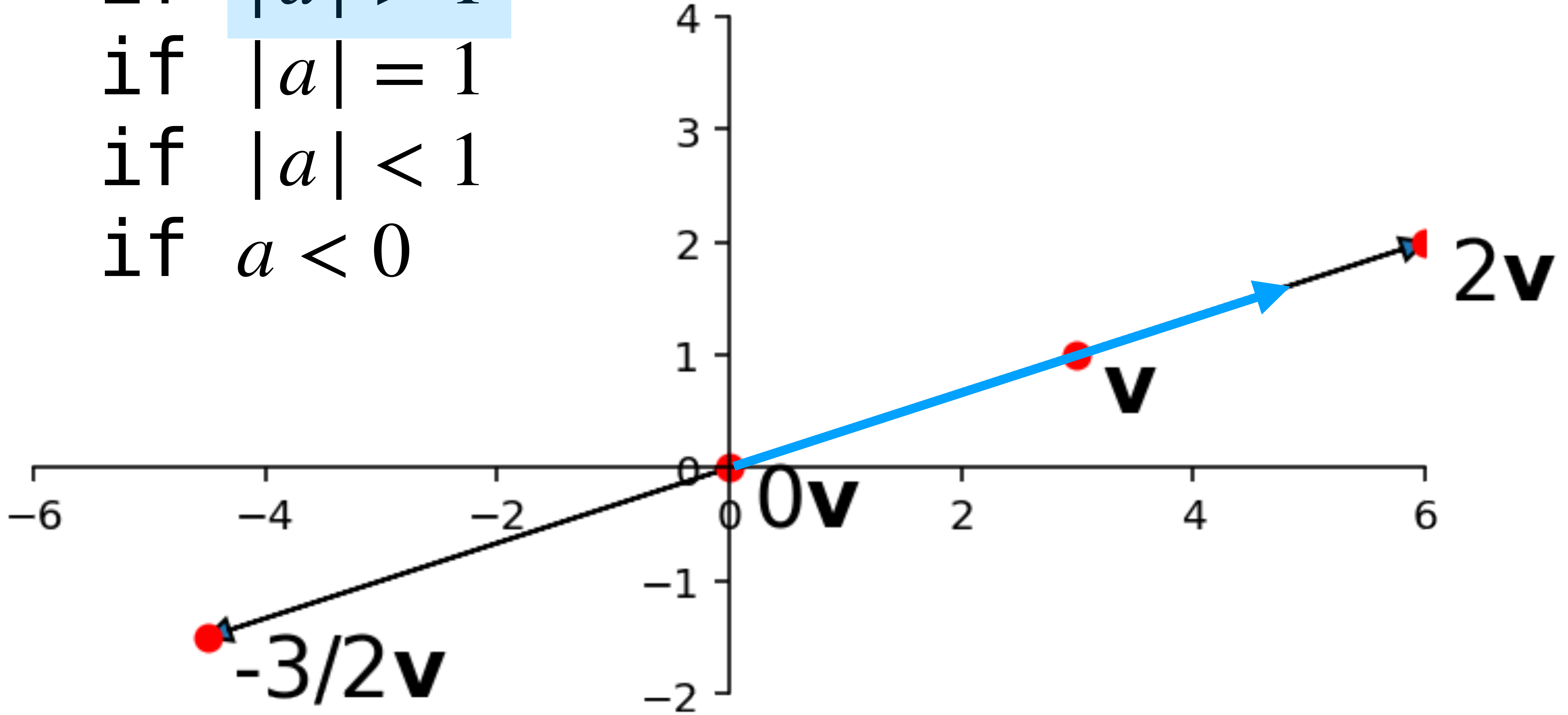
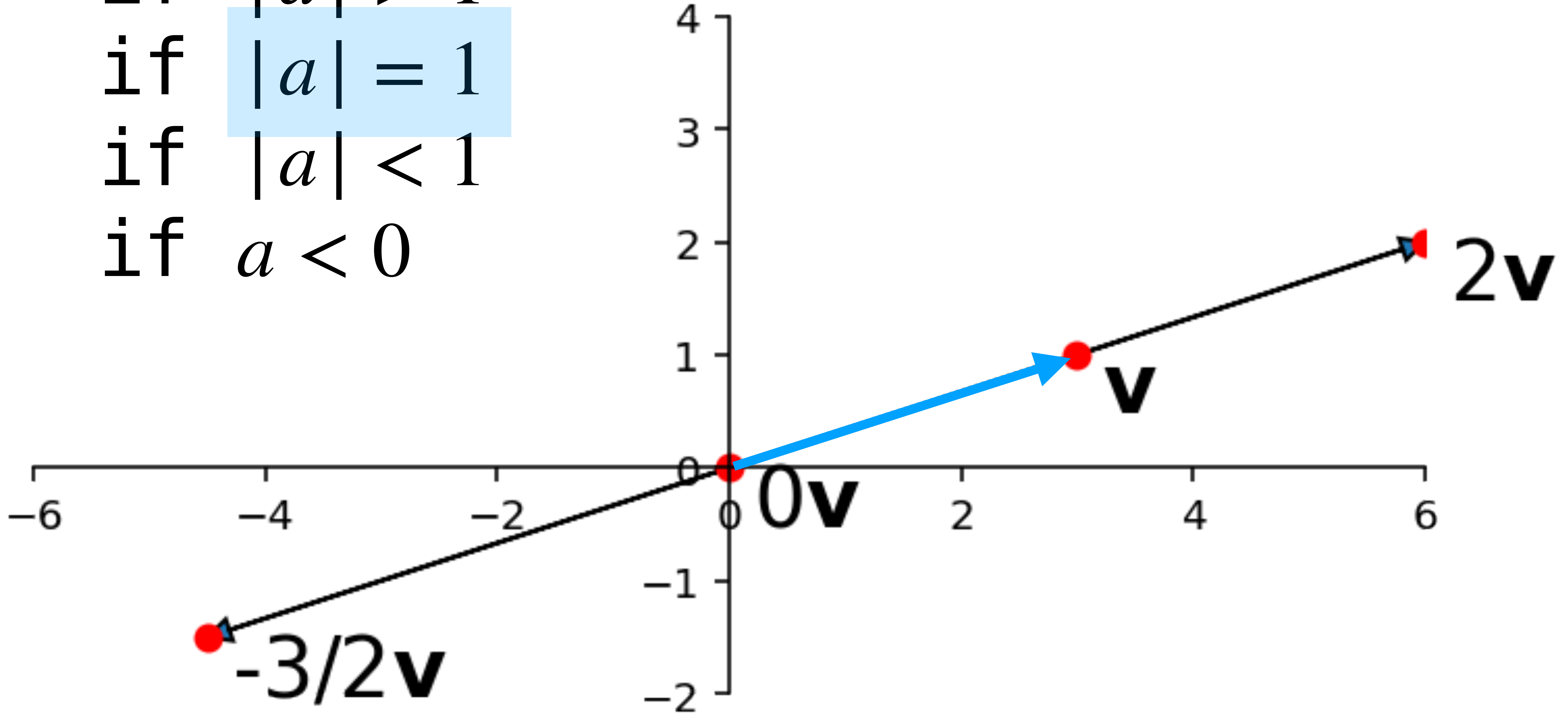


Recall Vector Scaling (Geometrically)

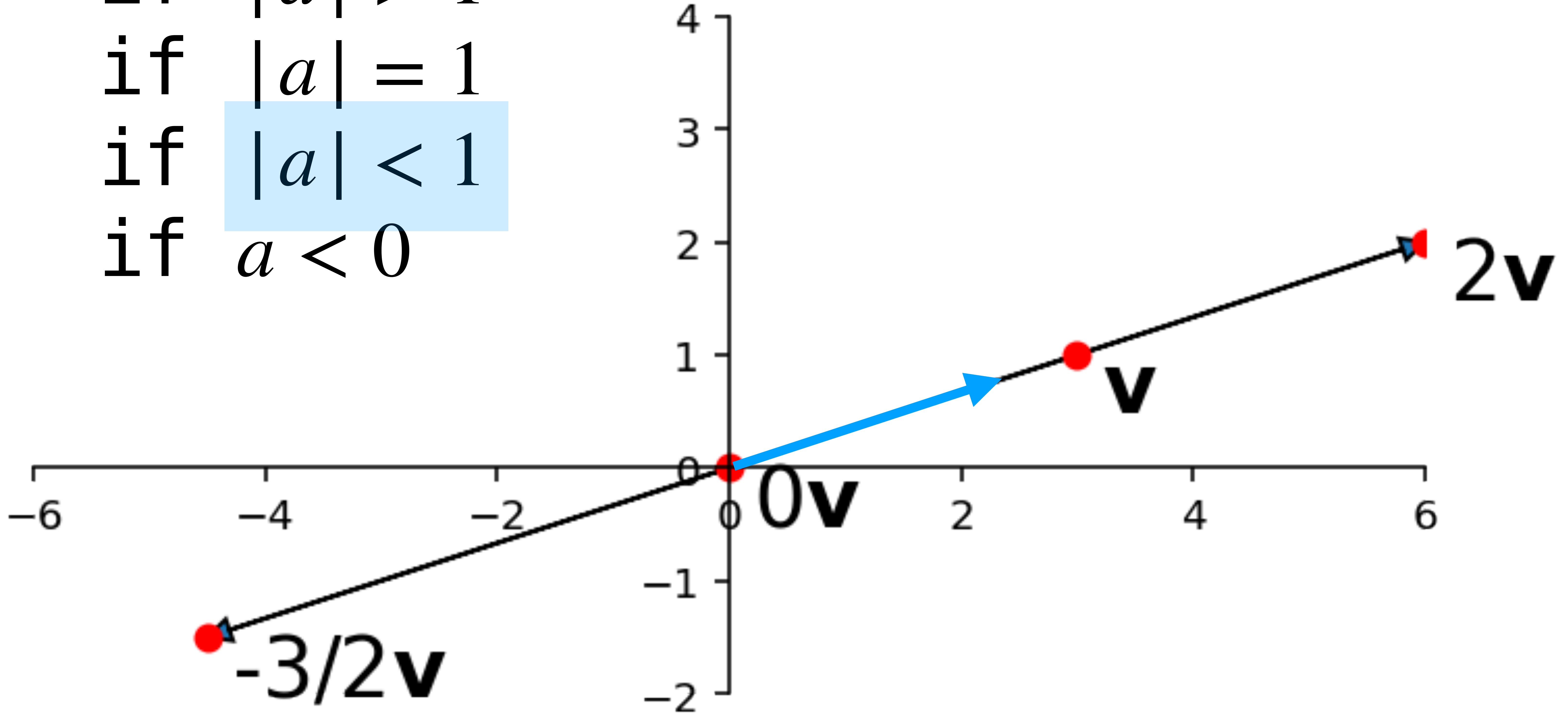
longer	if $\lvert a \rvert > 1$
the same length	if $\lvert a \rvert = 1$
shorter	if $\lvert a \rvert < 1$
reversed	if $a < 0$



Recall Vector Scaling (Geometrically)

```
longer               if
the same length      if
shorter              if
reversed             if
```

longer	if $\|a\| > 1$
the same length	if $\|a\| = 1$
shorter	if $\|a\| < 1$
reversed	if $a < 0$



Recall Vector Scaling (Geometrically)

longer	if $\lvert a \rvert > 1$
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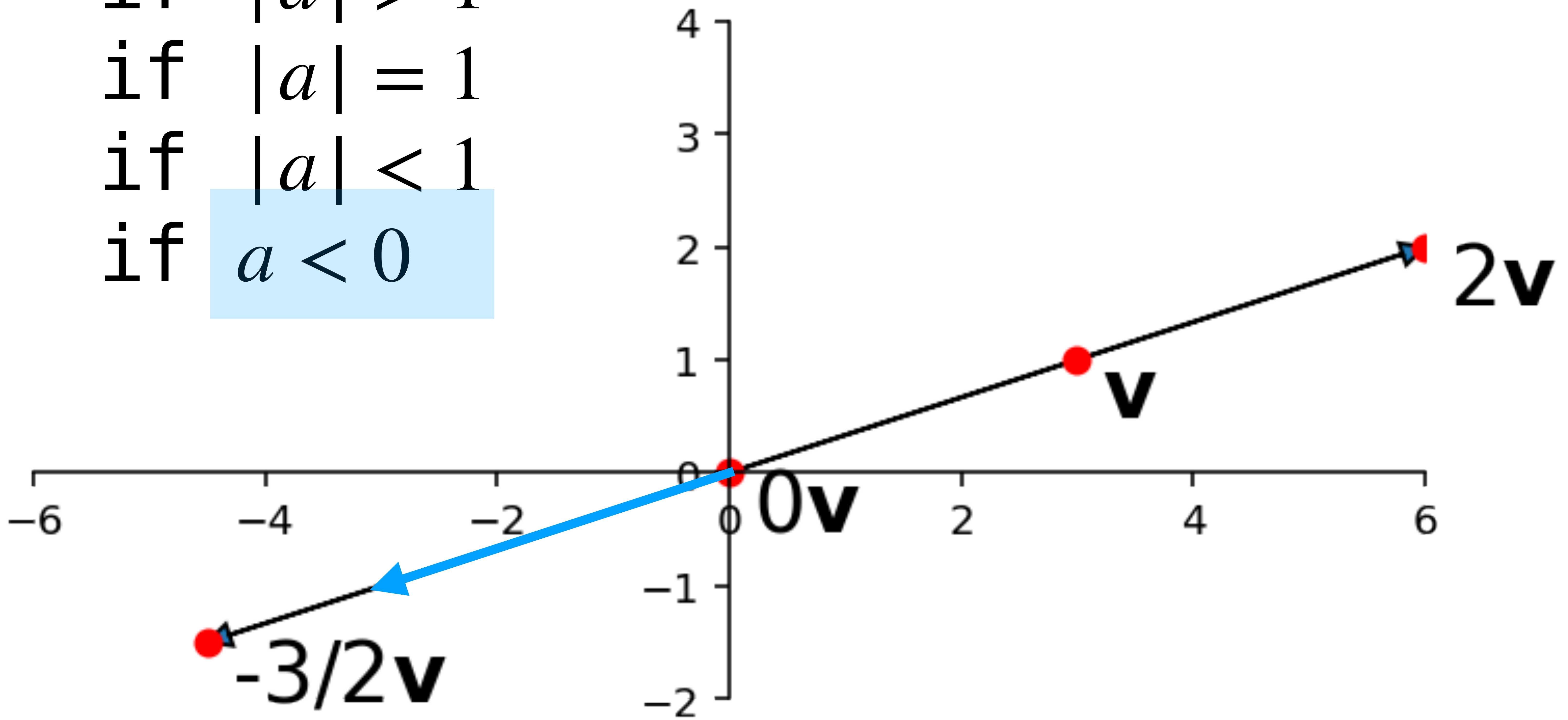


Recall: The Fundamental Connection

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$$

augmented matrix

Why not view these as a vector too?

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m$$

system of linear equations

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{1m} \end{bmatrix} + x_2 \begin{bmatrix} a_{21} \\ a_{21} \\ \vdots \\ a_{2m} \end{bmatrix} + \ldots + x_n \begin{bmatrix} a_{n1} \\ a_{n2} \\ \vdots \\ a_{nm} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

vector equation

Solutions as Vectors

Observation. a solution is, in essence, an ordered list of numbers

so it can be represented as a vector

Can we view a linear system as a single equation with matrices and vectors?

How do matrices and vectors "interface"?

Matrix-Vector Multiplication

Matrix-Vector "Interface"

multiplication

what does $A\mathbf{v}$ mean when A is a matrix and $\mathbf{v}$ is a vector?

Matrix-Vector Multiplication (Pictorially)

$$As$$

Matrix-Vector Multiplication (Pictorially)

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

Matrix-Vector Multiplication (Pictorially)

$$s_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{1m} \end{bmatrix} + s_2 \begin{bmatrix} a_{21} \\ a_{21} \\ \vdots \\ a_{2m} \end{bmatrix} + \dots + s_n \begin{bmatrix} a_{n1} \\ a_{n2} \\ \vdots \\ a_{nm} \end{bmatrix}$$

`a linear combination of the columns where`
$\mathbf{s}$ `defines the weights`

Why keeping track of matrix size is important

this only works if the number of *columns* of the matrix matches the number of *rows* of the vector

$$\underbrace{\begin{bmatrix} * & \cdots & * \\ * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \\ * & \cdots & * \end{bmatrix}}_{m \text{ rows},\ n \text{ columns}} \begin{bmatrix} * \\ \vdots \\ * \end{bmatrix} = \begin{bmatrix} * \\ * \\ \vdots \\ * \\ * \end{bmatrix}$$

$(m \times n)$ $(n \times 1)$ $(m \times 1)$

Non-Example

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} + 3???$$

THESE DON'T MATCH

(2×2) (3×1)

Example

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

THESE MATCH

$$(2 \times 2) \quad (2 \times 1)$$

Matrix-Vector Multiplication

Definition. Given a $(m \times n)$ matrix A with columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$, and a vector $\mathbf{v}$ in $\mathbb{R}^n$, we define

$$A\mathbf{v} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + \ldots v_n\mathbf{a}_n$$

$A\mathbf{v}$ is a linear combination of the columns of A with weights given by $\mathbf{v}$

Algebraic Properties

Unlike the properties of vectors operations, the algebraic properties of matrix–vector multiplication are **very important.**

1. $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v}$
2. $A(c\mathbf{v}) = c(A\mathbf{v})$

There are only two, please memorize them...

Derivation of (1) for A in $\mathbb{R}^{n\times 3}$

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3]\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right)$$

Derivation of (1) for A in $\mathbb{R}^{n\times 3}$

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3]\left(\begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix}\right)$$

by vector addition

Derivation of (1) for A in $\mathbb{R}^{n\times 3}$

$$(u_1 + v_1)\mathbf{a}_1 + (u_2 + v_2)\mathbf{a}_2 + (u_3 + v_3)\mathbf{a}_3$$

`by matrix vector multiplication`

Derivation of (1) for A in $\mathbb{R}^{n\times 3}$

$$u_1\mathbf{a}_1 + v_1\mathbf{a}_1 + u_2\mathbf{a}_2 + v_2\mathbf{a}_2 + u_3\mathbf{a}_3 + v_3\mathbf{a}_3$$

`by vector scaling (distribution)`

Derivation of (1) for A in $\mathbb{R}^{n\times 3}$

$$(u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + u_3\mathbf{a}_3) + (v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + v_3\mathbf{a}_3)$$

`by rearranging`

Derivation of (1) for A in $\mathbb{R}^{n\times 3}$

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3] \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

`by matrix vector multiplication`

Derivation of (1) for A in $\mathbb{R}^{n\times 3}$

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3]\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right)$$

equals

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3]\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3]\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

fin

A Common Error

$$A\mathbf{v} \neq \mathbf{v}A$$

it is **important** that we write our matrix-vectors multiplications with the matrix on the <u>left</u>

this may feel artificial now, since the RHS is meaningless to us now, but it won't be for long

Looking forward a bit

Remember. column vectors are matrices with 1 column

Eventually we'll be able to view all of these as matrix operations

Question

Compute the following matrix-vector multiplication

$$\begin{bmatrix} 2 & -3 & 4 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix}$$

Answer

$$\begin{bmatrix} 2 & -3 & 4 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix}$$

Answer

$$5\begin{bmatrix}2\\-1\end{bmatrix}+5\begin{bmatrix}-3\\1\end{bmatrix}+4\begin{bmatrix}4\\0\end{bmatrix}$$

Answer

$$\begin{bmatrix} 10 \\ -5 \end{bmatrix} + \begin{bmatrix} -15 \\ 5 \end{bmatrix} + \begin{bmatrix} 16 \\ 0 \end{bmatrix}$$

Answer

$$\begin{bmatrix} 11 \\ 0 \end{bmatrix}$$

A Tip

$$\begin{bmatrix} 2 & -3 & 4 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} ? \\ ? \end{bmatrix}$$

A Tip

$$\begin{bmatrix} 2 & -3 & 4 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} ? \\ ? \end{bmatrix}$$

$$5(2) + 5(-3) + 4(4) = 11$$

A Tip

$$\begin{bmatrix} 2 & -3 & 4 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 11 \\ ? \end{bmatrix}$$

$$5(-1) + 5(1) + 4(0) = 0$$

A Tip

$$\begin{bmatrix} 2 & -3 & 4 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 11 \\ 0 \end{bmatrix}$$

Calculating $A\mathbf{v}$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ \vdots \\ ? \end{bmatrix}$$

$$v_1 = a_{11}s_1 + a_{12}s_2 + \ldots + a_{1n}s_n = \sum_{i=1}^{n} a_{1i}s_i$$

Calculating $A\mathbf{v}$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} v_1 \\ ? \\ \vdots \\ ? \end{bmatrix}$$

$$v_2 = a_{21}s_1 + a_{22}s_2 + \ldots + a_{2n}s_n = \sum_{i=1}^{n} a_{2i}s_i$$

Calculating $A\mathbf{v}$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ ? \end{bmatrix}$$

$$v_m = a_{m1}s_1 + a_{m2}s_2 + \ldots + a_{mn}s_n = \sum_{i=1}^{n} a_{mi}s_i$$

Calculating $A\mathbf{v}$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$$

Row-Column Rule

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} a_{1i}s_i \\ \sum_{i=1}^{n} a_{2i}s_i \\ \vdots \\ \sum_{i=1}^{n} a_{mi}s_i \end{bmatrix}$$

Inner product: $\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \sum_{i=1}^{n} a_i s_i$

Calculating $A\mathbf{v}$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} a_{1i} s_i \\ \sum_{i=1}^{n} a_{2i} s_i \\ \vdots \\ \sum_{i=1}^{n} a_{mi} s_i \end{bmatrix}$$

The ith entry of the $A\mathbf{s}$ is the inner product of the ith row of A and $\mathbf{s}$

The Matrix Equation

Recall: Vector Equations

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{b}$$

Question. Can $\mathbf{b}$ be written as a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$?

The Idea. think of the weights as *unknowns*

we can use the same idea for matrix-vector multiplication

The Matrix Equation

$$A\mathbf{x} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n]\,\mathbf{x} = \mathbf{b}$$

Can $\mathbf{b}$ be written as a linear combination of **the columns of A**?

The Idea. write the "vector part" of our matrix-vector multiplication as an *unknown*

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

HOW TO: The Matrix Equation

Question. Does $A\mathbf{x} = \mathbf{b}$ have a solution?

Question. Is $A\mathbf{x} = \mathbf{b}$ consistent?

Question. write down a solution to the equation $A\mathbf{x} = \mathbf{b}$

HOW TO: The Matrix Equation

Question. write down a solution to the equation $A\mathbf{x} = \mathbf{b}$

Solution. we can write this as:

(matrix equation) $[\mathbf{a}_1 \quad \mathbf{a}_2 \ldots \quad \mathbf{a}_n]\,\mathbf{x} = \mathbf{b}$

(vector equation) $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots x_n\mathbf{a}_n = \mathbf{b}$

(augmented matrix) $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}$

!!they all have the same solution set!!

HOW TO: The Matrix Equation

Question. write down a solution to the equation $A\mathbf{x} = \mathbf{b}$

Solution.

use Gaussian elimination (or other means) to convert $[\mathbf{a}_1 \ \ \mathbf{a}_2 \ \ \dots \ \ \mathbf{a}_n \ \ \mathbf{b}]$ to reduced echelon form

then read off a solution from the reduced echelon form

Span (Revisited)

Recall: Span

Definition. the ***span*** of a set of vectors is the set of all possible linear combinations of them

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = \{\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots \alpha_n\mathbf{v}_n : \alpha_1, \alpha_2, \ldots, \alpha_n \text{ are in } \mathbb{R}\}$$

$\mathbf{u} \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ exactly when $\mathbf{u}$ can be expressed as a linear combination of those vectors

Spans (with Matrices)

Definition. the ***span*** of the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ is:

$$\text{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\} = \{[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n]\,\mathbf{v} : \mathbf{v} \in \mathbb{R}^n\}$$

the span of the columns of a matrix A
is the set of of vectors resulting
from multiplying A by any vector

(we will soon start thinking of A as a way of *transforming* vectors)

A Note on the Geometry of Spans

Last time.

The span of one vectors is a line

The span of two vectors is a plane

A Note on the Geometry of Spans

Updated.

The span of one vectors ~~is a~~ can be line

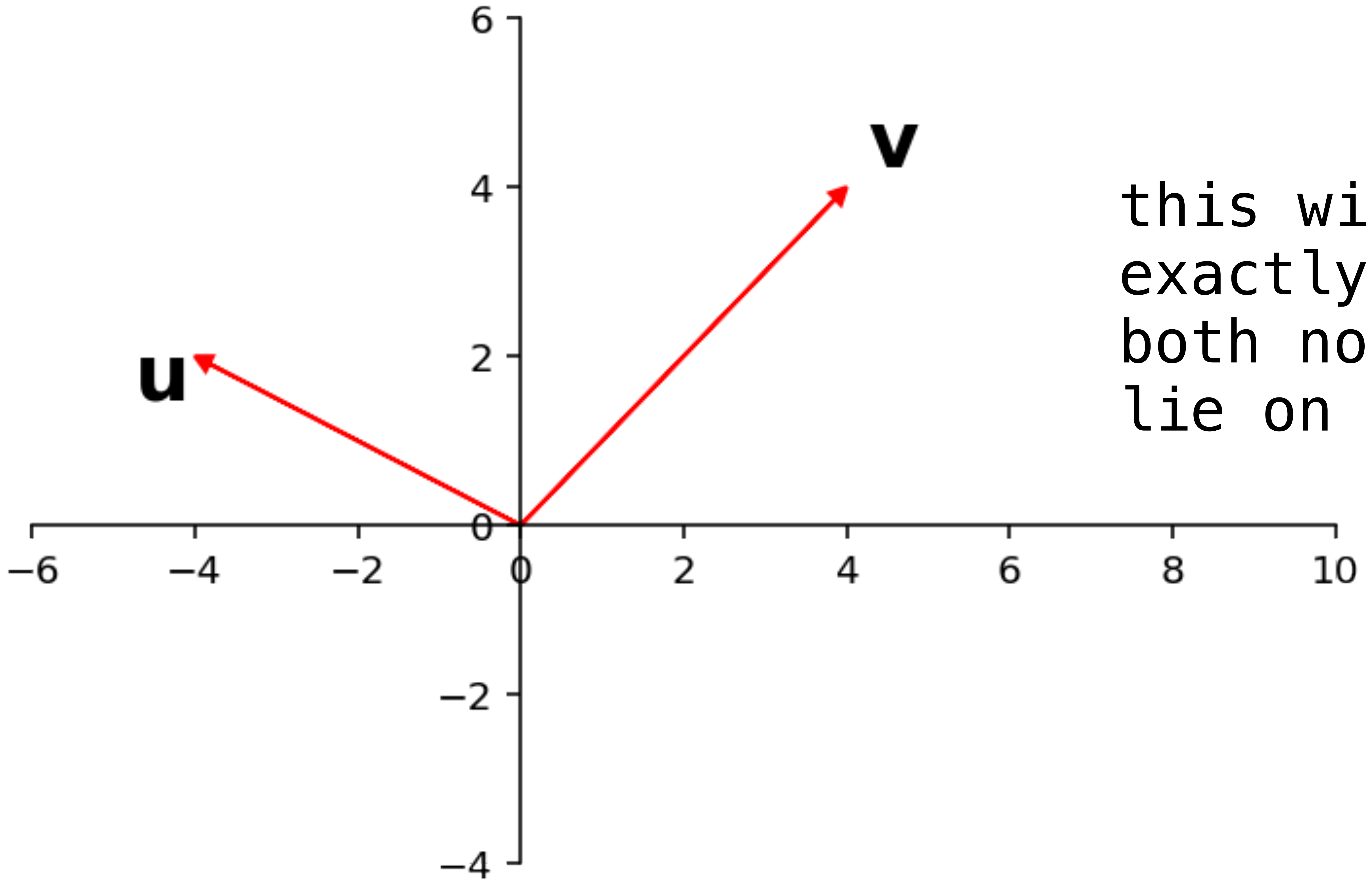
The span of two vectors ~~is a~~ can be plane

but not always

demo
(from ILA)

Spanning $\mathbb{R}^2$

if two (or more) vectors in $\mathbb{R}^2$ span a plane, they must span all of $\mathbb{R}^2$ (why?)



this will happen exactly if they are both nonzero and don't lie on the same line

What about $\mathbb{R}^n$?

A Thought Experiment

suppose I give you the augmented matrix of a linear system but I cover up the last column

$$\begin{bmatrix} 1 & 2 & 3 & \text{[redacted]} \\ 2 & 1 & 0 & \end{bmatrix}$$

A Thought Experiment

`then we reduce it to echelon form`

$$\begin{bmatrix} 1 & 2 & 3 & \text{[redacted]} \\ 2 & 1 & 0 & \end{bmatrix}$$

$$R_2 \leftarrow R_2 - 2R_1$$

A Thought Experiment

then we reduce it to echelon form

$$\begin{bmatrix} 1 & 2 & 3 & \text{[redacted]} \\ 0 & -3 & -6 & \end{bmatrix}$$

Does it have a solution?

A Thought Experiment

then we reduce it to echelon form

$$\begin{bmatrix} 1 & 2 & 3 & \text{[redacted]} \\ 0 & -3 & -6 & \end{bmatrix}$$

Yes. It doesn't have an inconsistent row

A Thought Experiment

`what about this system?`

$$\begin{bmatrix} 1 & 1 & 2 & \text{[redacted]} \\ 2 & 2 & 4 & \end{bmatrix}$$

$$R_2 \leftarrow R_2 - 2R_1$$

A Thought Experiment

`what about this system?`

$$\begin{bmatrix} 1 & 1 & 2 & \text{[redacted]} \\ 0 & 0 & 0 & \end{bmatrix}$$

`it depends...`

Pivots and Spanning $\mathbb{R}^m$

$$\begin{bmatrix} 1 & 2 & 3 & \text{[redacted]} \\ 2 & 1 & 0 & \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & \text{[redacted]} \\ 0 & -3 & -6 & \end{bmatrix}$$

`if it doesn't matter what the last column is, then` **`every choice must be possible`**

`every vector in` $\mathbb{R}^2$ **`can be written as a linear combination of`** $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, **`and`** $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$

Spanning $\mathbb{R}^m$

Theorem. For any $m \times n$ matrix, the following are logically equivalent

1. For every $\mathbf{b}$ in $\mathbb{R}^m$, $A\mathbf{x} = \mathbf{b}$ has a solution
2. The columns of A span $\mathbb{R}^m$
3. A has a pivot position in every row

HOW TO: Spanning $\mathbb{R}^m$

Question. Does the set of vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ from $\mathbb{R}^m$ span all if $\mathbb{R}^m$?

Solution. Reduce $[\mathbf{a}_1 \ \ \mathbf{a}_2 \ \ \ldots \ \ \mathbf{a}_n]$ to echelon form and check if every row has a pivot

Question

Do $\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$ *and* $\begin{bmatrix} 0 \\ 1 \\ 2023 \end{bmatrix}$ *span all of* $\mathbb{R}^3$*?*

Answer: No

the matrix

$$\begin{bmatrix} 2 & 0 \\ 2 & 1 \\ 3 & 2023 \end{bmatrix}$$

cannot have more than 2 pivot positions

Not spanning $\mathbb{R}^m$

$$\begin{bmatrix} 1 & 1 & 2 & \blacksquare \\ 2 & 2 & 4 & \blacksquare \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & \blacksquare \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}$$

in this case the choice matters

we can't make the last column $[0 \;\; 0 \;\; 0 \;\; \blacksquare]$ for nonzero $\blacksquare$

but we can make the last column <u>parameters</u> to find equations that must hold

Not spanning $\mathbb{R}^m$

$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 2 & 2 & 4 & b_2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \end{bmatrix}$$

as long as $(-2)b_1 + b_2 = 0$, the system is consistent

this gives use a <u>linear equation</u> which describes the span of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$

HOW TO: Not spanning $\mathbb{R}^m$

Question. Suppose the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ in $\mathbb{R}^m$ do not span all of $\mathbb{R}^m$. Give a linear equation which describes their span.

Solution. Reduce the matrix

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}$$

to echelon form, where $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ is a parameter. Then write down the equation given by the row of the form $[0 \;\; 0 \;\; \ldots \;\; 0 \;\; \blacksquare]$.

Question (Understanding Check)

True *or* ***False,*** *the echelon form of any matrix has at most one column of the form* $[0 \;\; 0 \;\; \ldots \;\; 0 \;\; \blacksquare]$ *where* $\blacksquare$ *is nonzero.*

Answer: True

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

leading entry not to the right

this is not in echelon form

Question

Give a linear equation for the span of the vectors $\begin{bmatrix}1\\2\\0\end{bmatrix}$ *and* $\begin{bmatrix}-1\\-1\\-1\end{bmatrix}$.

Answer

$$\begin{bmatrix} 1 & -1 & b_1 \\ 2 & -1 & b_2 \\ 0 & -1 & b_3 \end{bmatrix}$$

Answer

$$\begin{bmatrix} 1 & -1 & b_1 \\ 0 & 2 & b_2 - 2b_1 \\ 0 & -1 & b_3 \end{bmatrix}$$

$R_2 \leftarrow R_2 - 2R_1$

Answer

$$\begin{bmatrix} 1 & -1 & b_1 \\ 0 & 2 & b_2 - 2b_1 \\ 0 & 0 & b_3 + (1/2)(b_2 - 2b_1) \end{bmatrix}$$

$$R_3 \leftarrow R_3 - (1/2)R_2$$

Answer

$$0 = b_3 + (1/2)(b_2 - 2b_1)$$

Answer

$$b_1 - (1/2)b_2 - b_3 = 0$$

Answer

$$x_1 - (1/2)x_2 - x_3 = 0$$

Taking Stock

Four Representations

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$$

augmented matrix

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

matrix equation

they all have the same solution sets

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

system of linear equations

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{1m} \end{bmatrix} + x_2 \begin{bmatrix} a_{21} \\ a_{21} \\ \vdots \\ a_{2m} \end{bmatrix} + \dots + x_n \begin{bmatrix} a_{n1} \\ a_{n2} \\ \vdots \\ a_{nm} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

vector equation

Summary

Matrix and vectors can be multiplied together to get new vectors

The matrix equation is another representation of systems of linear equations

Looking forward: Matrices *transform* vectors