# Linear Transformations

## Geometric Algorithms
## Lecture 7

CAS CS 132

# Recap Problem

*Find three vectors* $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ *in* $\mathbb{R}^3$ *such that*

» *every pair of vectors (i.e.,* $\{\mathbf{v}_1, \mathbf{v}_2\}$*,* $\{\mathbf{v}_1, \mathbf{v}_3\}$*,* $\{\mathbf{v}_2, \mathbf{v}_3\}$*)* ~~*are*~~ is *linearly independent*

» $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ *is linearly dependent*

# Answer

$r_1, r_2$ are lin. dep $\Longleftrightarrow$ colinear

$$r_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$r_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$r_3 = r_1 + r_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

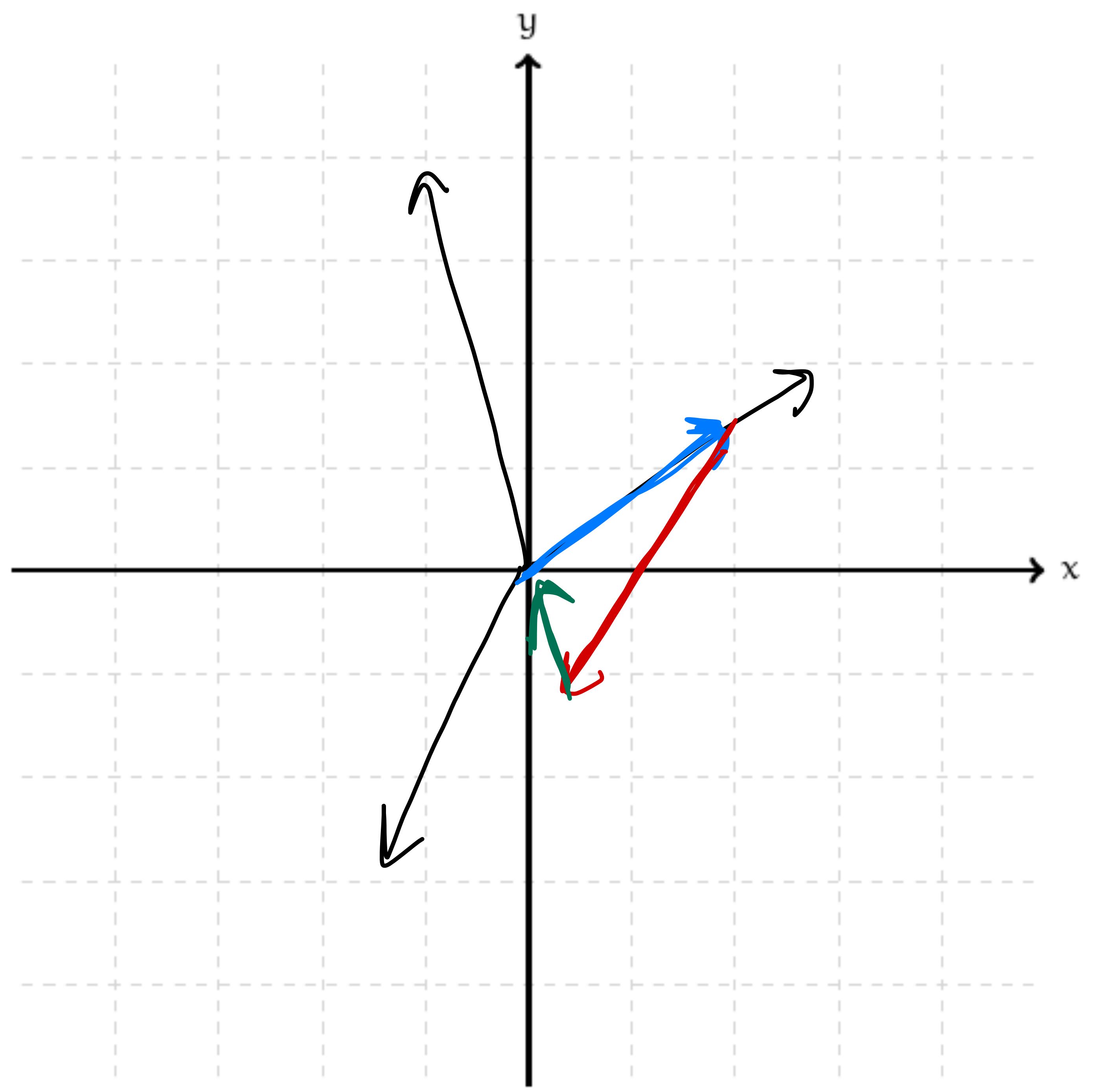
$$r_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$r_2 = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$$

$$r_3 = \begin{bmatrix} 3 \\ 6 \\ 8 \end{bmatrix}$$

# Demo: Geometry of Linear Dependence

$$a_1 \vec{v}_1 + a_2 \vec{v}_2 + a_3 \vec{v}_3 = \vec{0}$$



# Objectives

1. Introduce Matrix Transformations
2. Define Linear Transformations
3. Start looking at the Geometry of Linear Transformations
4. See an Non-Geometric Application

# Keywords

Transformations

Domain, Codomain

Image, Range

Matrix Transformations

Linear Transformations

Additivity, Homogeneity

Dilation, Contraction, Shearing, Rotation

# Introduction

# Recall: Spans (with Matrices)

**Definition.** The ***span*** of a set of vectors is the set of all possible linear combinations of them.

$$\mathsf{span}\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\} = \{[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n]\,\mathbf{v} : \mathbf{v} \in \mathbb{R}^n\}$$

# Recall: Spans (with Matrices)

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The span of the columns of a matrix $A$
is the set of of vectors resulting
from multiplying $A$ by any vector.

# Matrices as Transformations

Matrices allow us to *transform* vectors.

The transformed vector lies in the span of its columns.

$$\mathbf{x} \mapsto A\mathbf{x}$$

map a vector $\mathbf{x}$ to the vector $A\mathbf{v}$

# Example (Algebraic)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1(1) + 0 + 0 \\ 0 + 2(1) + 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1(2) + 0 + 0 \\ 0 + 2(3) + 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

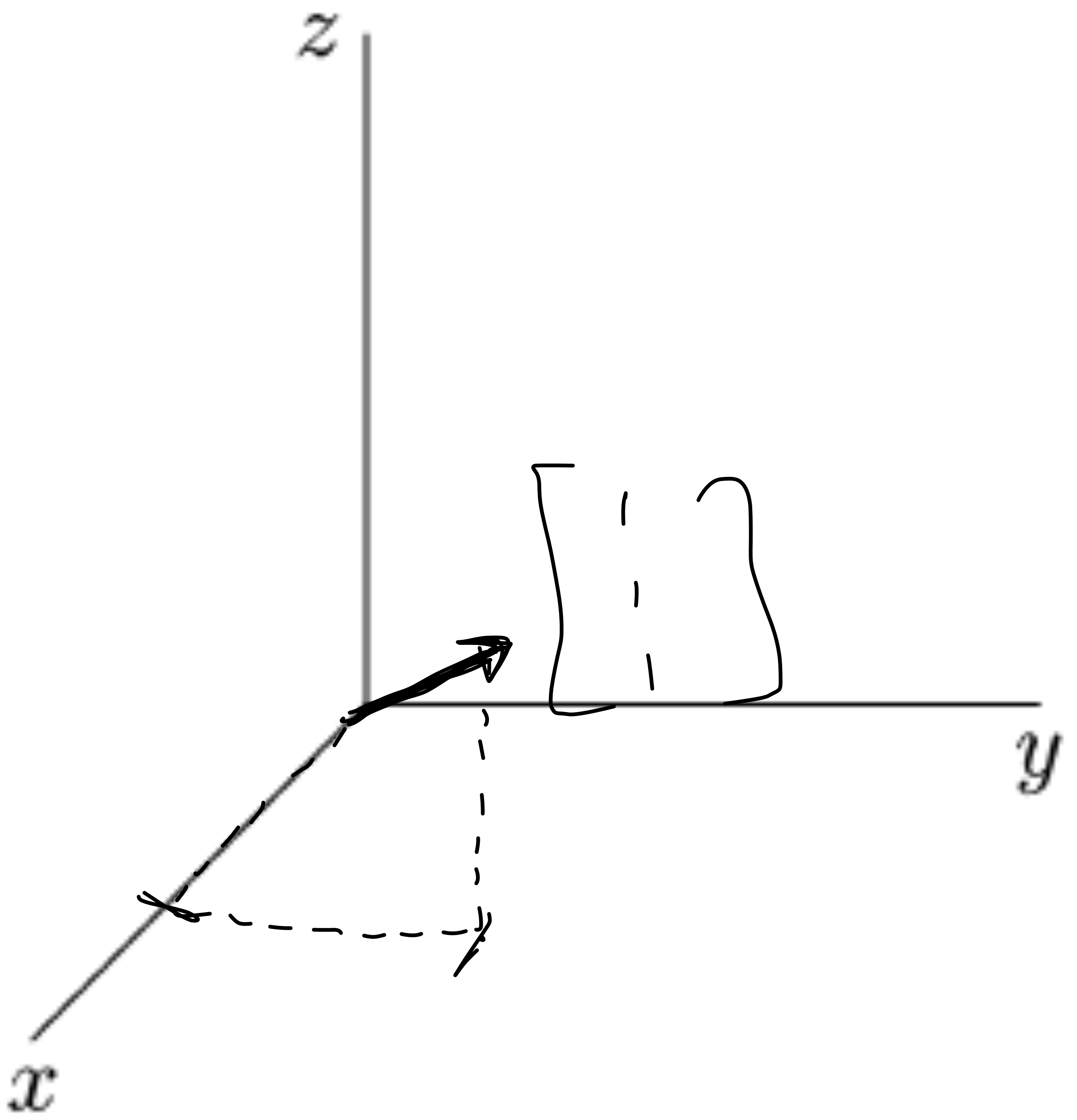
# Example (Algebraic)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1(x_1) + 0 + 0 \\ 0 + 2x_2 + 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_2 \end{bmatrix}$$

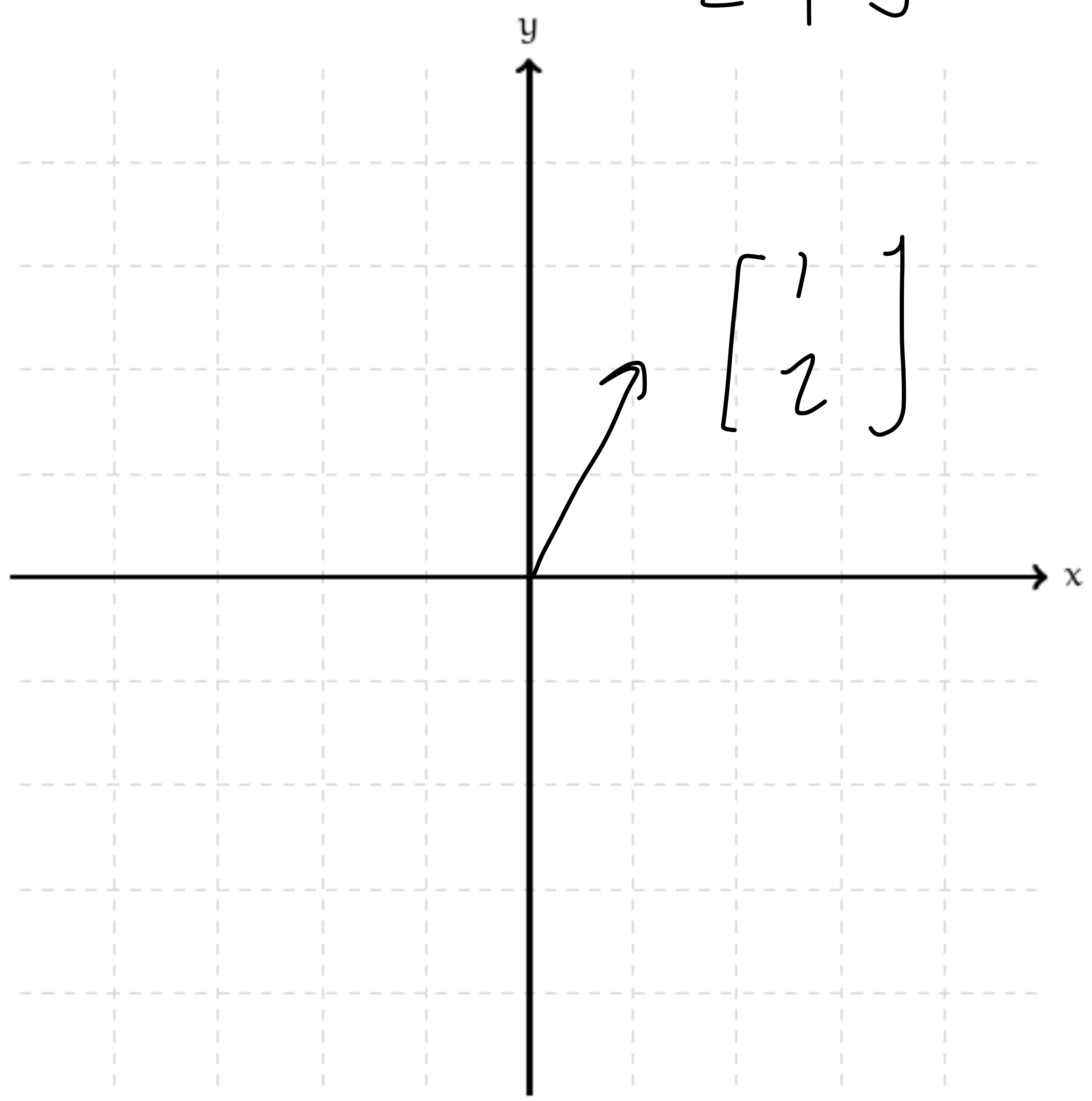
# Example (Geometric)

$$\begin{bmatrix}1 & 0 & 0\\0 & 2 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}x_1\\2x_2\end{bmatrix}$$

$$\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$$




$$\begin{bmatrix}1 & 0 & 0\\0 & 2 & 0\end{bmatrix}\mathbf{x}$$



!!Important!!

*The vector may be a different size after translation.*

# Recall: Matrix-Vector Multiplication and Dimension

`matrix-vector multiplication only works if the number of` *`columns`* `of the matrix matches the dimension of the vector`

$$\begin{array}{cc} & n \\ m & \begin{bmatrix} * & \cdots & * \\ * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \\ * & \cdots & * \end{bmatrix} \end{array} \begin{array}{cc} n & \begin{bmatrix} * \\ \vdots \\ * \end{bmatrix} \end{array} = \begin{array}{cc} m & \begin{bmatrix} * \\ * \\ \vdots \\ * \\ * \end{bmatrix} \end{array}$$

$(m \times n)$ $\qquad$ $\mathbb{R}^n$ $\qquad$ $\mathbb{R}^m$

# Motivating Questions

What kind of functions can we define in this way?

How do we interpret what the transformation does to a set of vectors?

How does this relate back to matrix equations?

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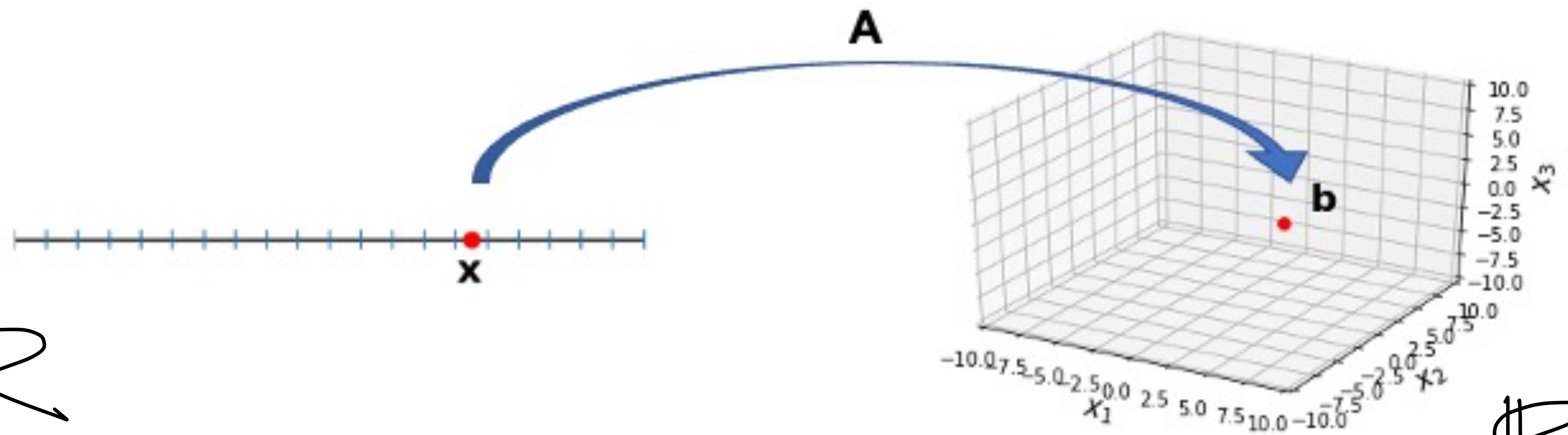
# A New Interpretation of the Matrix Equation

| | | |
|---|---|---|
| $A\mathbf{x} = \mathbf{b}$? | $\equiv$ | is there a vector which $A$ transforms into $\mathbf{b}$? |
| Solve $A\mathbf{x} = \mathbf{b}$ | $\equiv$ | find a vector which $A$ transforms into $\mathbf{b}$ |

# Question (Conceptual)

*Suppose a matrix transforms a vector according to the following picture. What is the size of the matrix?*

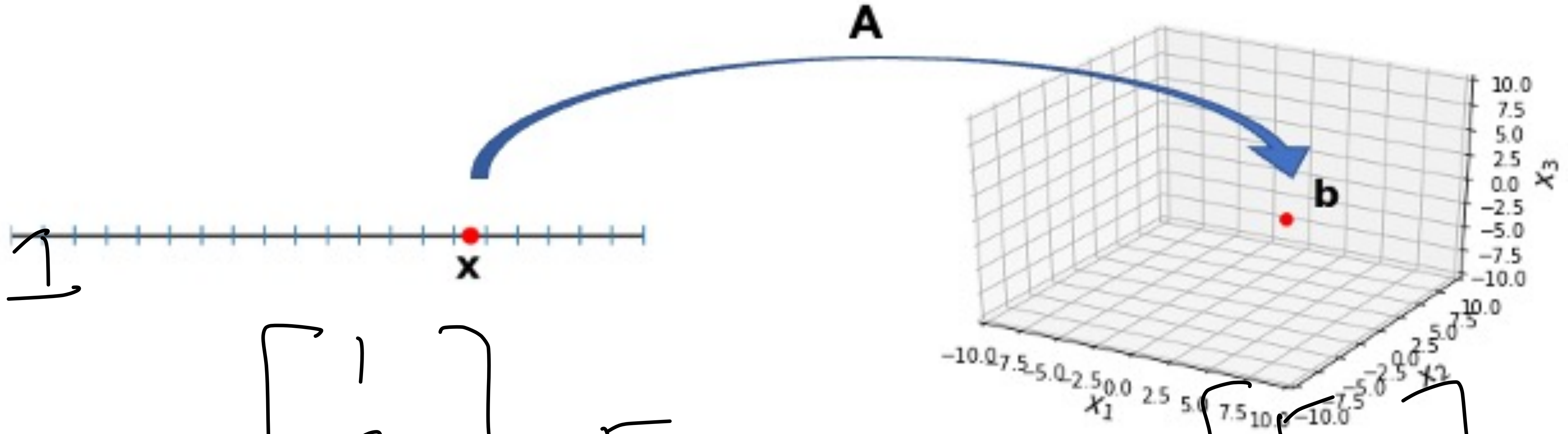



$\mathbb{R}$ $\mathbb{R}^3$

$$\underbrace{A}_{m \times n} x = b$$

$m$? $n$?

# Answer: $3 \times 1$




$\mathbb{R}^1$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} 5 = \begin{bmatrix} 5 \\ 10 \\ 15 \end{bmatrix}$$

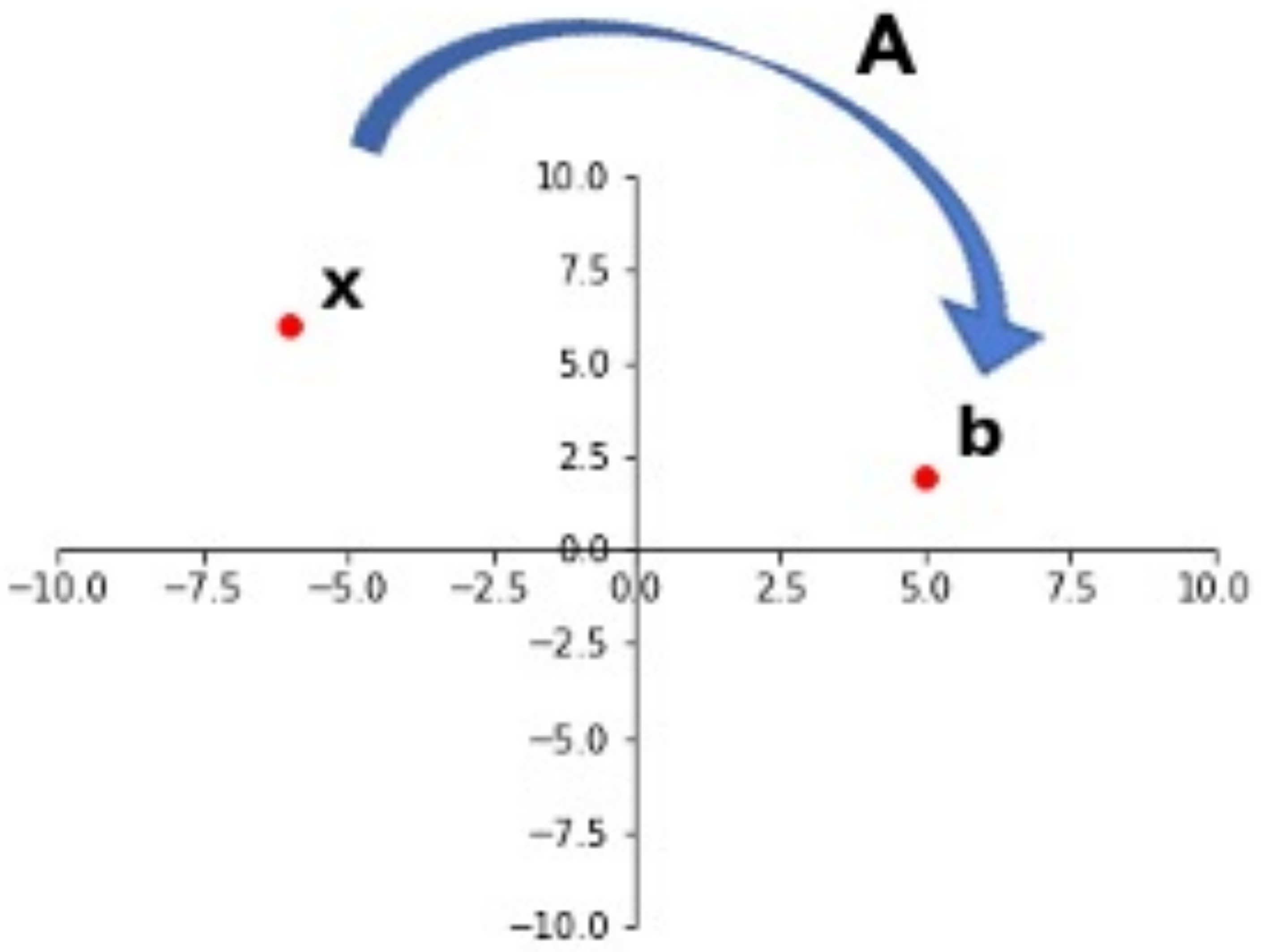
$$A \quad x \quad = \quad b$$

$$(3 \times 1) \; (1 \times 1) \qquad (3 \times 1) \qquad m = 3$$

$$(m \times n) \; (n \times 1) \qquad (m \times 1) \qquad n = 1$$

$\mathbb{R}^{\overset{3}{\cancel{n}}} \to \mathbb{R}^{\overset{3}{\cancel{n}}}$

Mapping between the same space can be viewed as a way of moving around points.



# Transformations

# Transformations in General

**Definition.** A ***transformation*** $T$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a function which maps every vector $\mathbf{v}$ in $\mathbb{R}^n$ to a vector $T(\mathbf{v})$ in $\mathbb{R}^m$.

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$$T : \mathbb{R}^n \to \mathbb{R}^m$$

domain codomain

$\mathbb{R}^1 \to \mathbb{R}^3$

dom. cod.

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$Ax = b$

$$\underbrace{T : \mathbb{R}^n}_{\text{domain}} \to \underbrace{\mathbb{R}^m}_{\text{codomain}}$$

**It's just a function, like in calculus.**

# Image and Range

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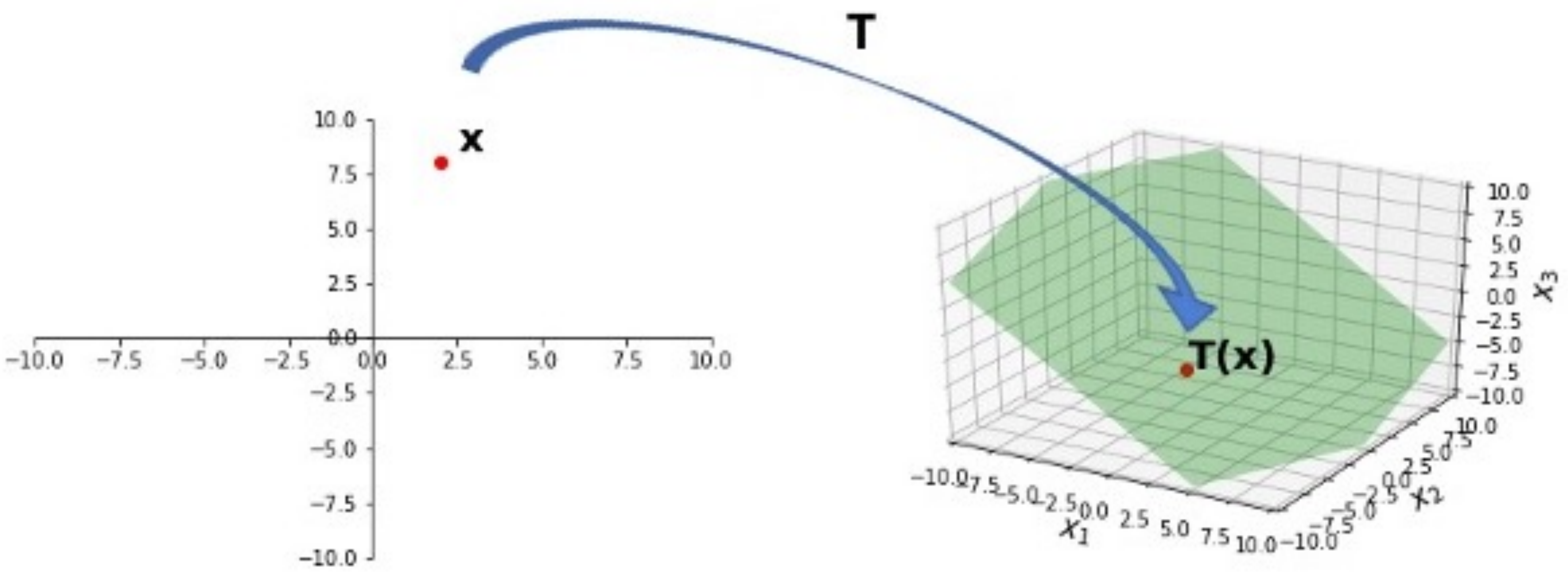
$$\text{ran}(T) = \{T(\mathbf{v}) : v \in \underbrace{\mathbb{R}^n}_{\text{domain}}\}$$

$$\text{image of } \mathbf{v} \text{ under } T \equiv \text{output of } T \text{ applied to } \mathbf{v}$$

$$\text{range of } T \equiv \text{all possible output of } T$$

# Codomain and Range

The codomain and range of a transformation may or may not be the same.



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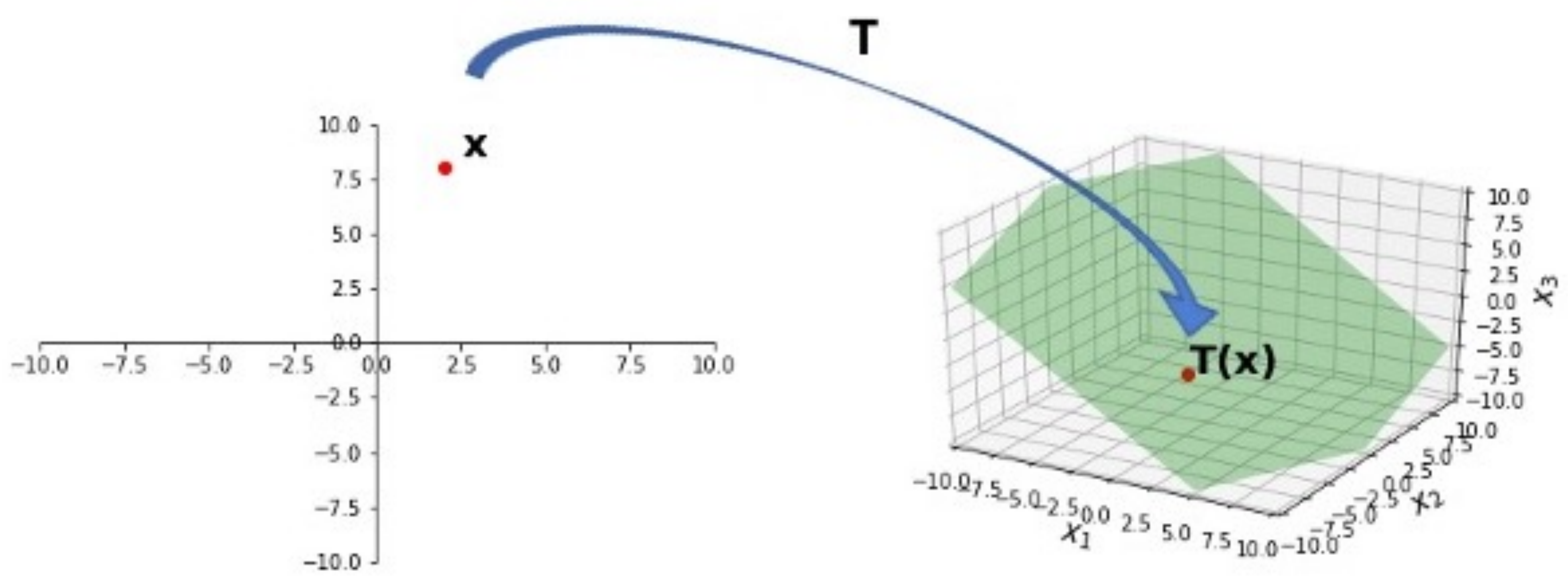


range: just the green plane

domain: $\mathbb{R}^2$

codomain: $\mathbb{R}^3$

$T : \mathbb{R}^{10} \to \mathbb{R}^{8}$

$T(\vec{v}) = \vec{0}$

codomain : $\mathbb{R}^{8}$

range$(T) = \{\vec{0}\}$

# Codomain and Range

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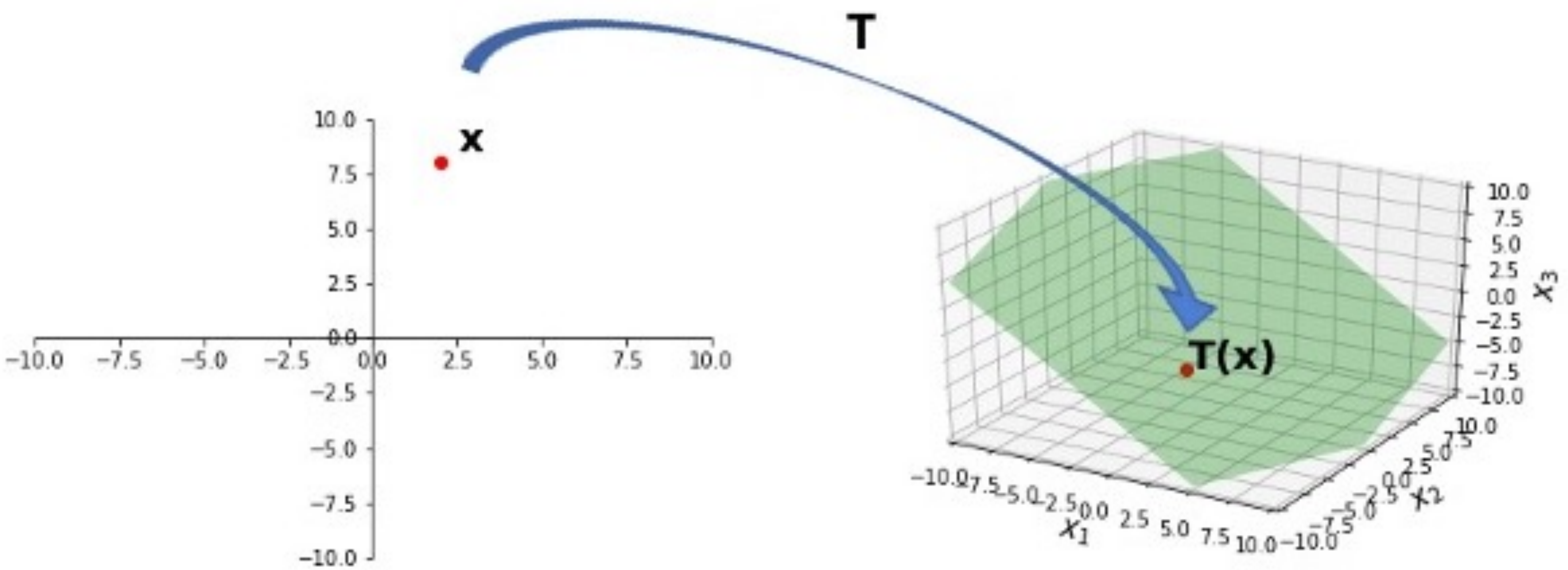


range: just the green plane

domain: $\mathbb{R}^2$

codomain: $\mathbb{R}^3$

The range is always contained in the codomain.

# Matrix Transformations

# Transformation of a Matrix

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The ***transformation of a*** $(m \times n)$ ***matrix*** $A$ is the function $T: \mathbb{R}^n \to \mathbb{R}^m$ such that

$$T(\mathbf{v}) = A\mathbf{v}$$

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given $\mathbf{v}$, return $A$ multiplied by $\mathbf{v}$

Transformation is Function not a matrix.

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e.g. $T(\mathbf{v}) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{v}$ $\quad T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

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The span of the columns of a matrix $A$ is the set of all possible *images* under $A$.

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$$\text{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\} = \text{ran}(\overset{T}{\cancel{[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n]}})$$

$$T(\vec{v}) = [\vec{a}_1 \quad \vec{a}_2 \quad \ldots \quad \vec{a}_n]\, \vec{v}$$

# Range and Span

The span of the columns of a matrix $A$ is the set of all possible *images* under $A$.

$$\operatorname{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\} = \operatorname{ran}(\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix})$$

The transformation of a vector $\mathbf{v}$ under the matrix $A$ always lies in the span of its columns.

# Example

$$T\left(\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} =$$

$$\begin{bmatrix} 2(1) + 1(-1) + 0 \\ 0(2) + 1(-1) + 0 \\ 1(2) + 3(-1) + 0 \end{bmatrix} =$$

$$\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

$$2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + (-1)\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} =$$

exercise

# Motivating Questions

What kind of functions can we define in this way?

How do we interpret what the transformation does to a set of vectors?

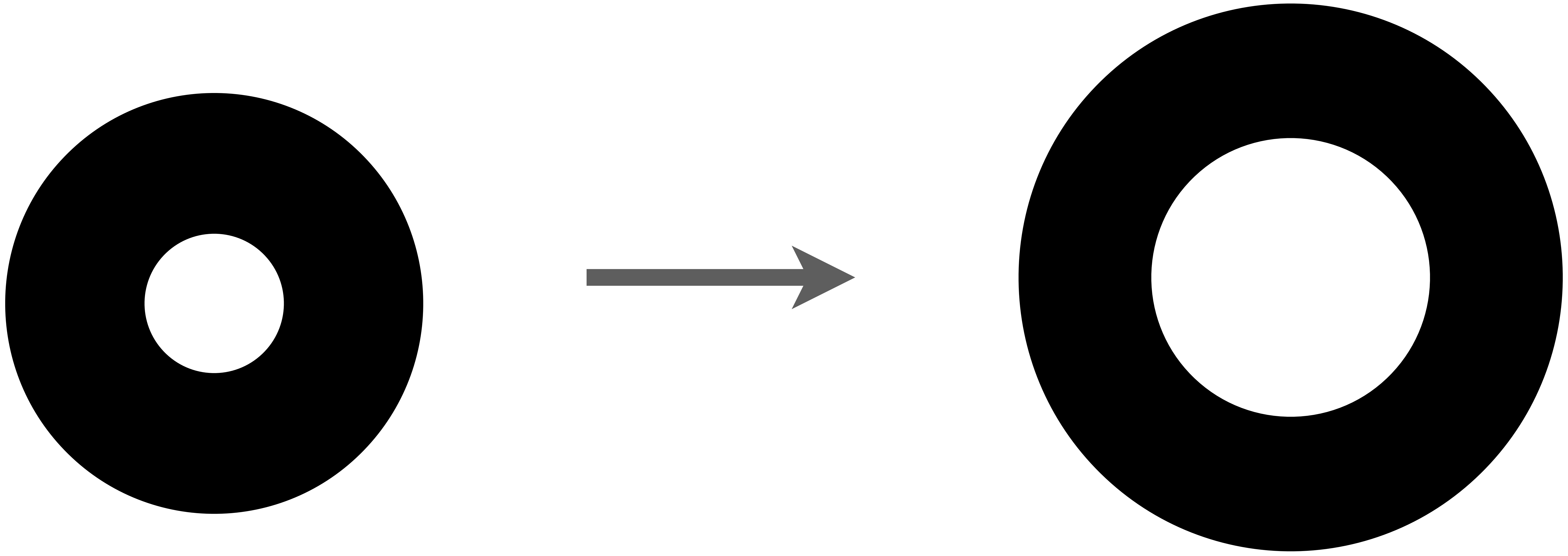
How does this relate back to matrix equations?

# Geometry of Matrix Transformations

# Motto

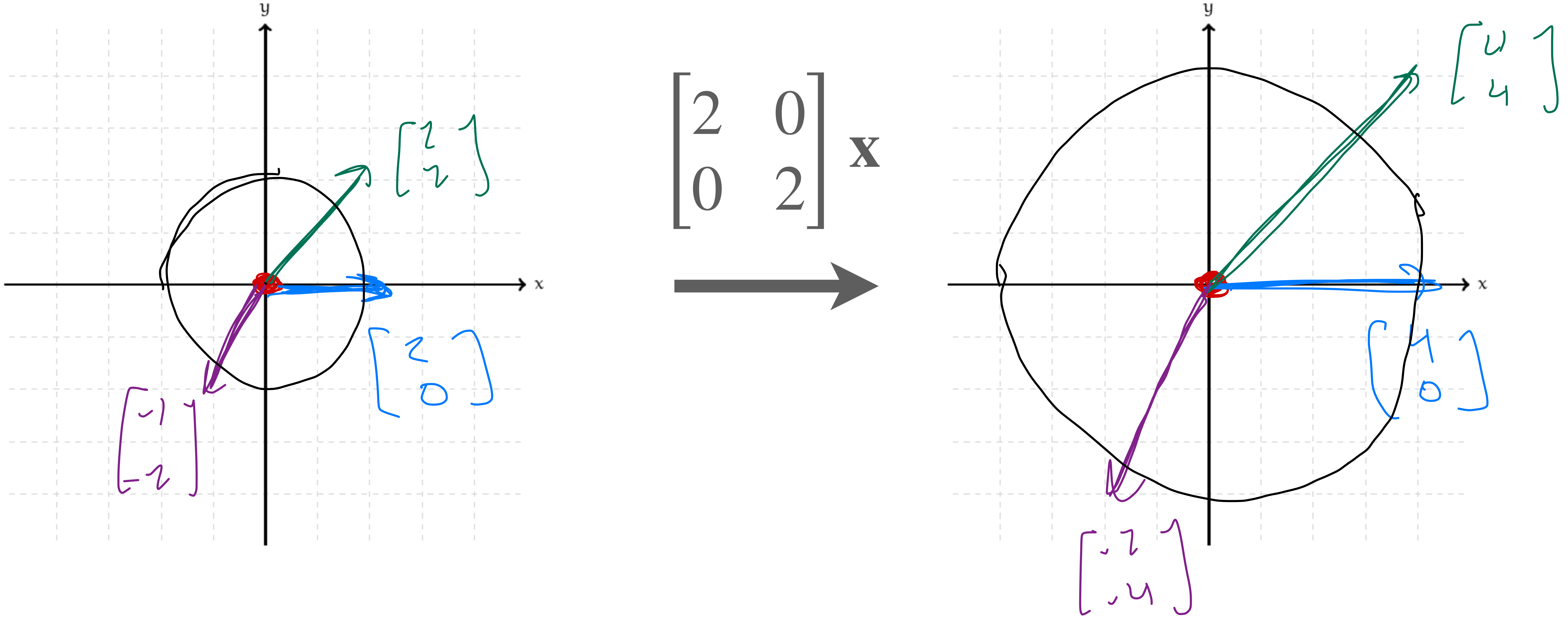
Matrix transformations change the "shape" of a set of set of vectors (points).

# Example: Dilation



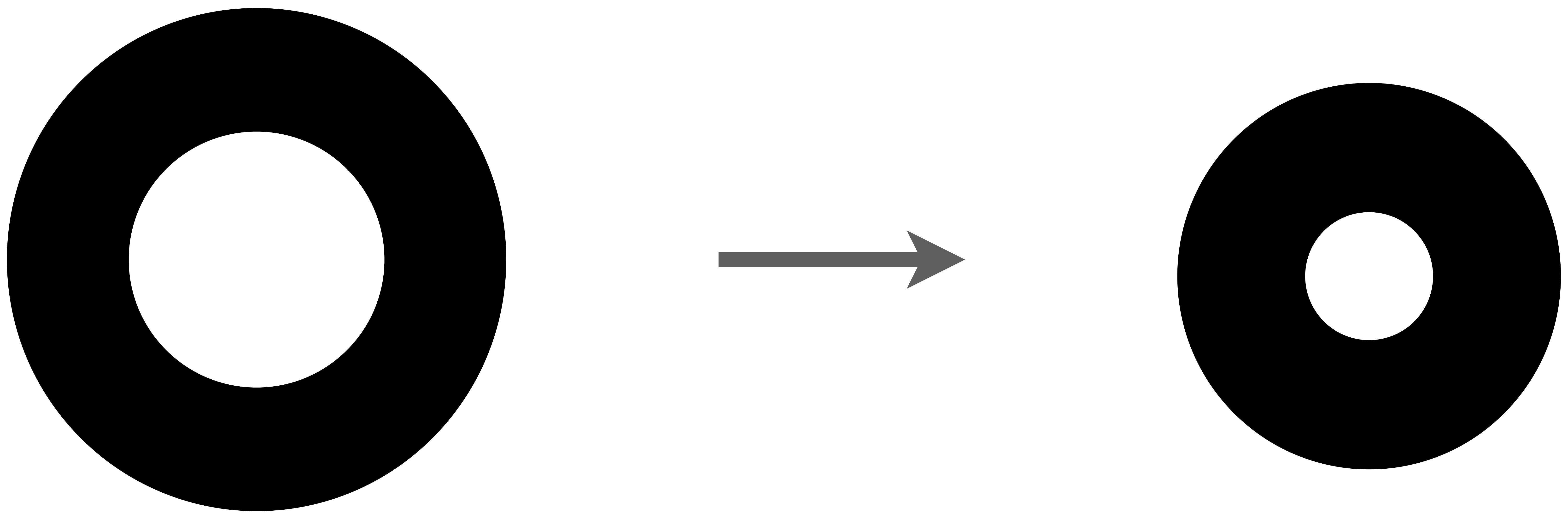
# Example: Dilation

$$\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} rx_1 \\ rx_2 \end{bmatrix}$$




if $r > 1$, then the transformation pushes points away from the origin.

# Example: Contraction



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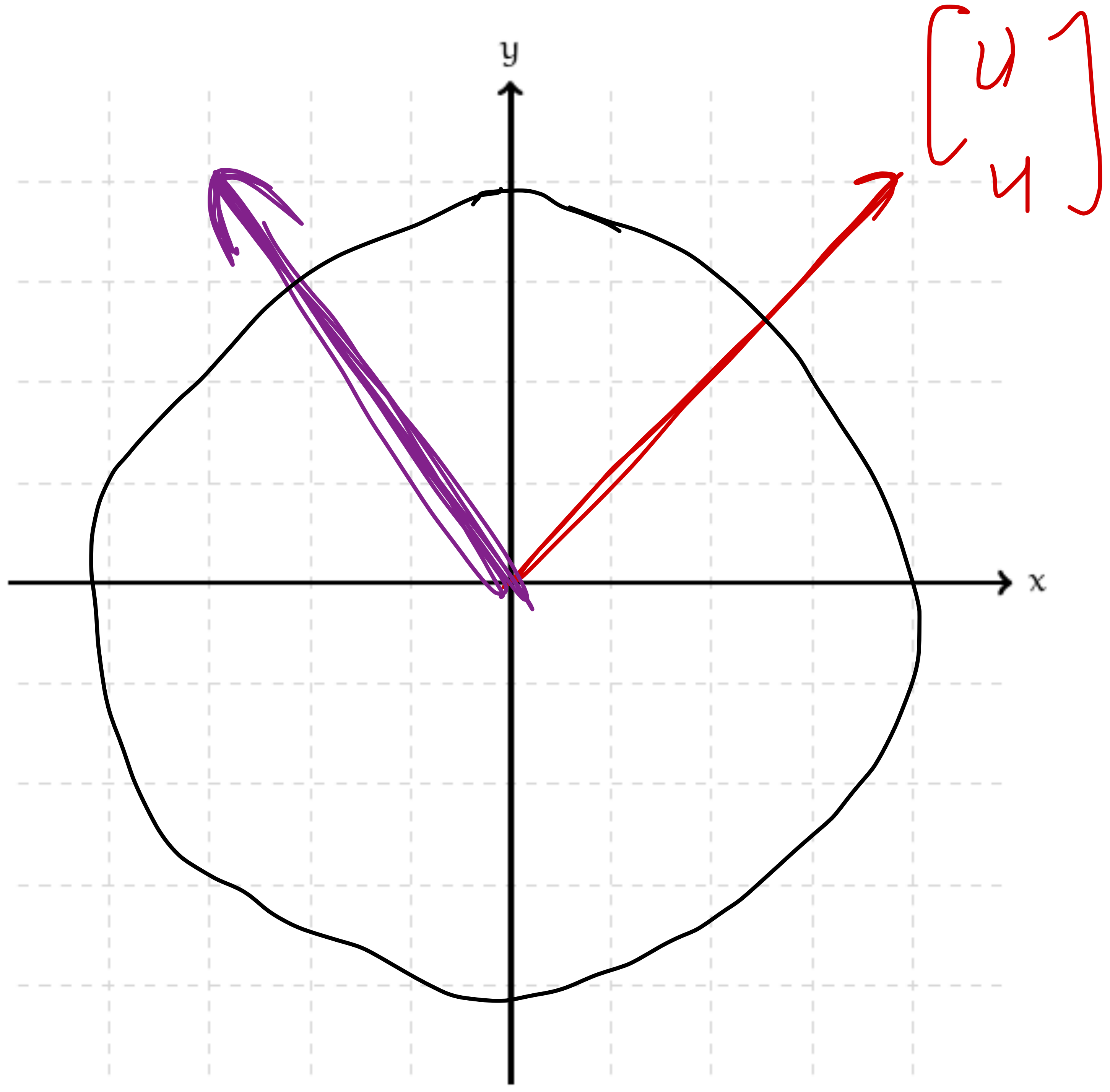
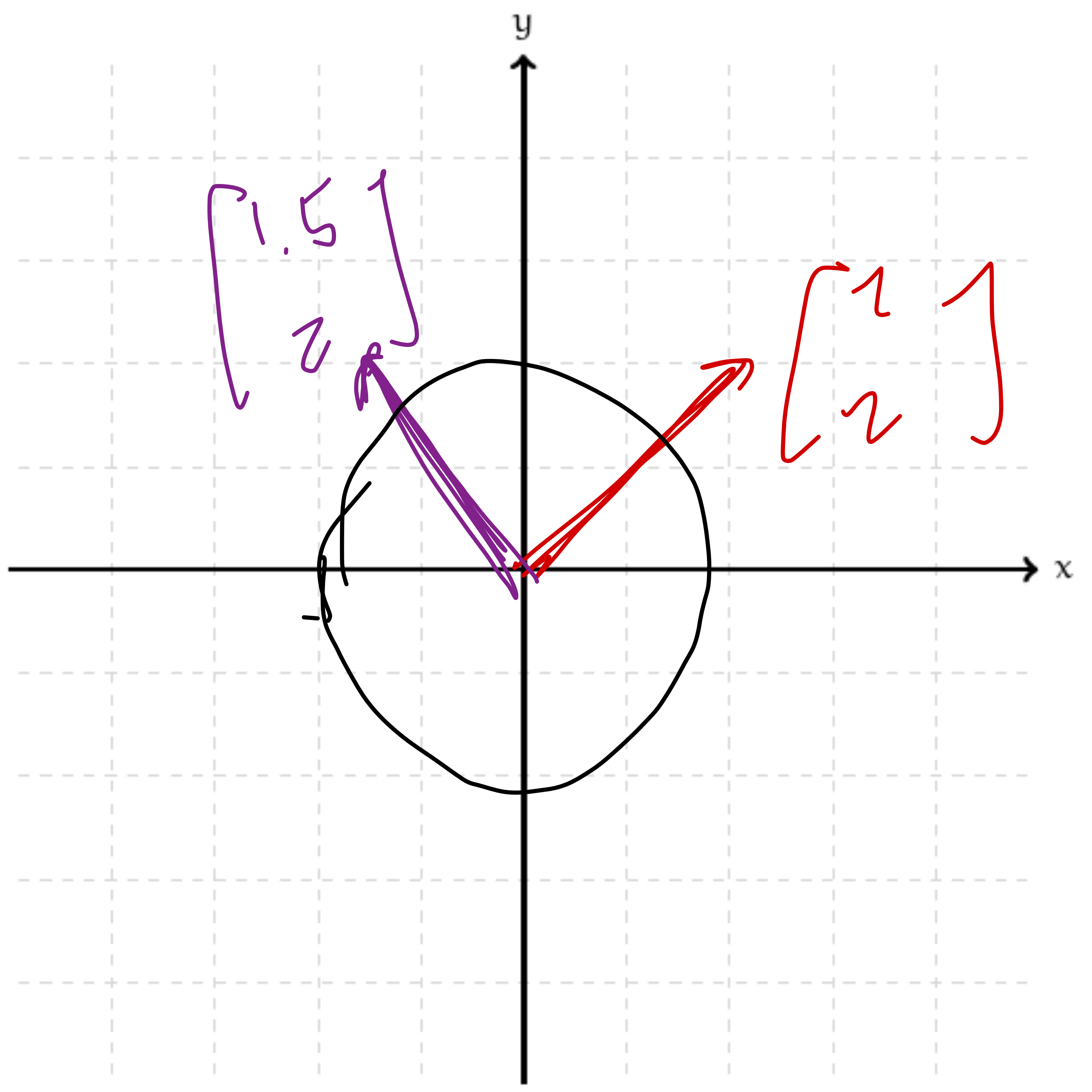
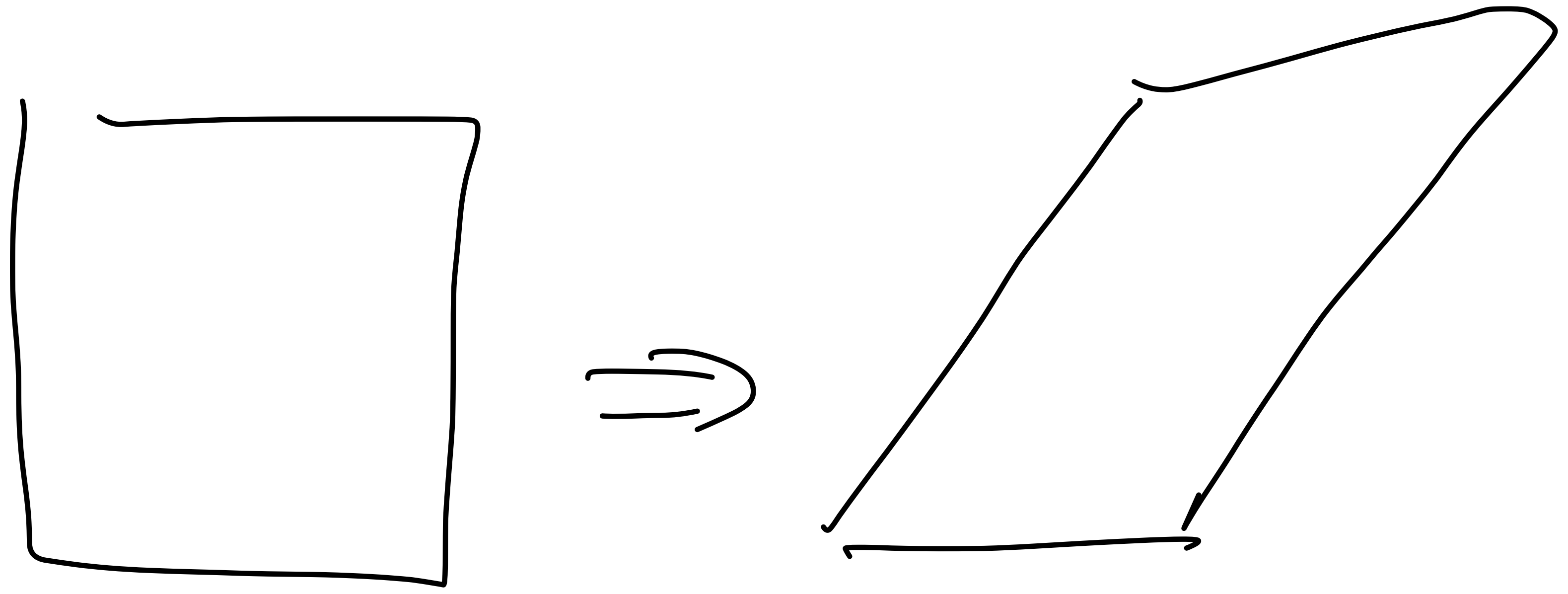



$$\begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \mathbf{x}$$



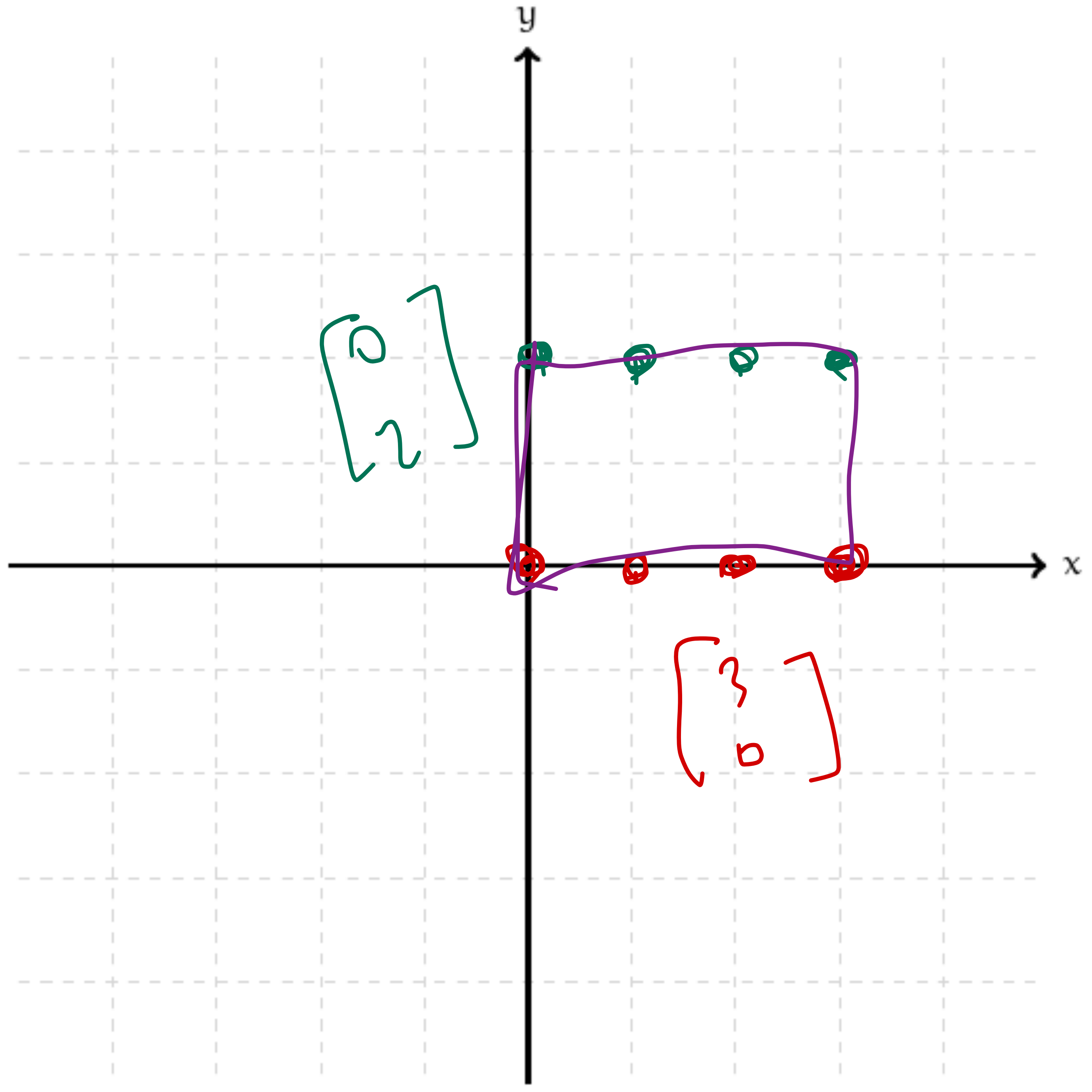

if $0 \leq r \leq 1$, then the transformation pulls points towards the origin.

# Example: Shearing

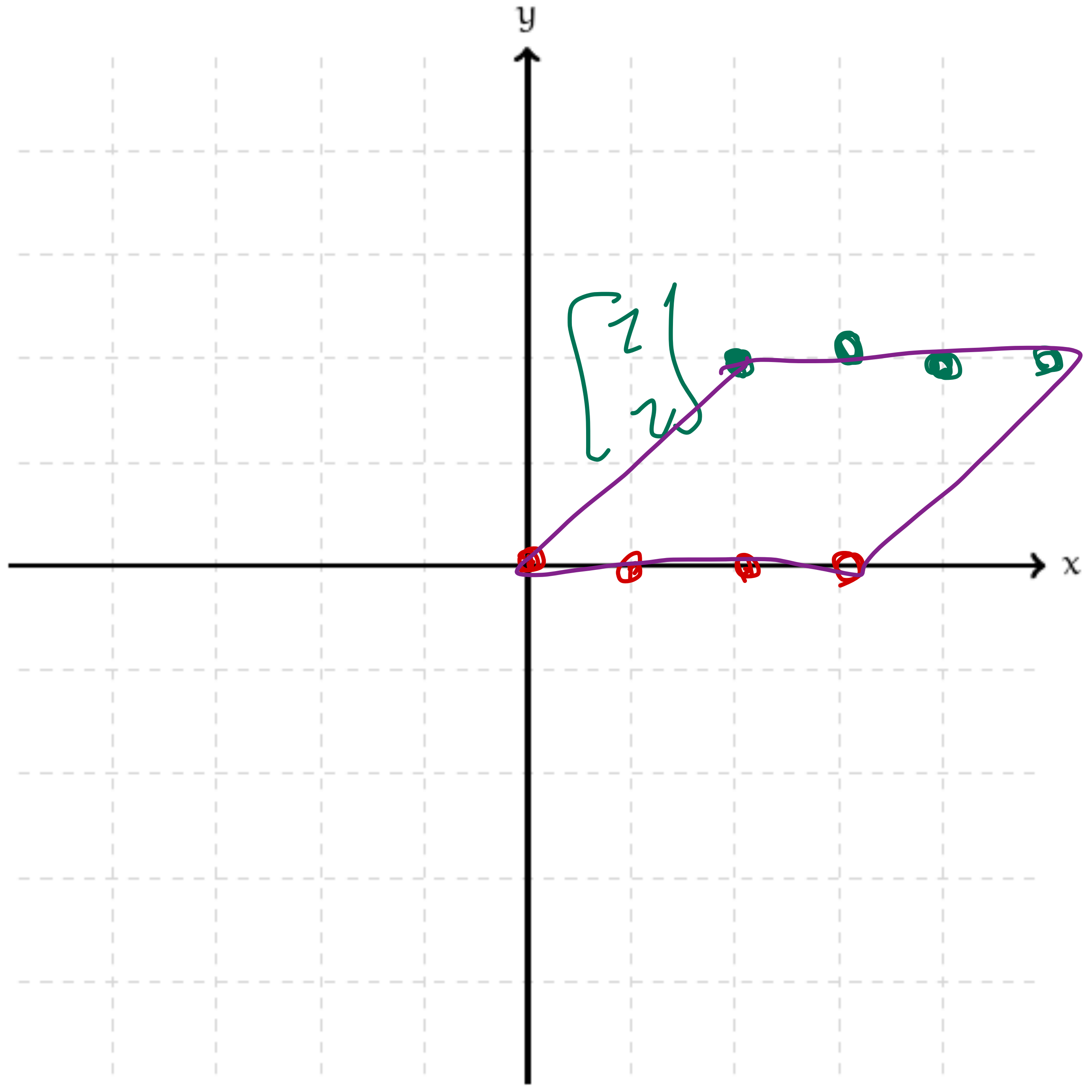


# Example: Shearing

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 \end{bmatrix}$$



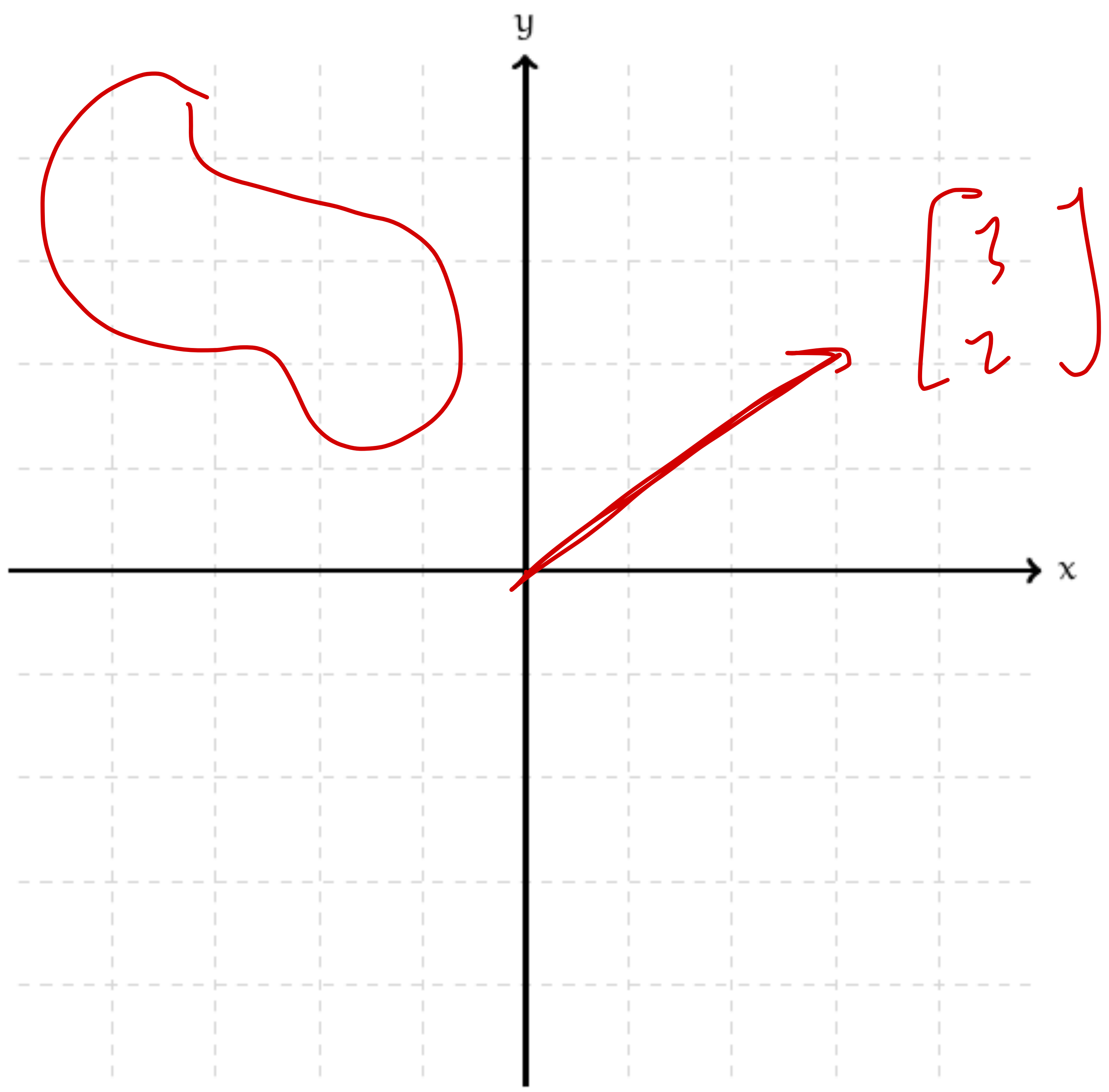

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{x}$$



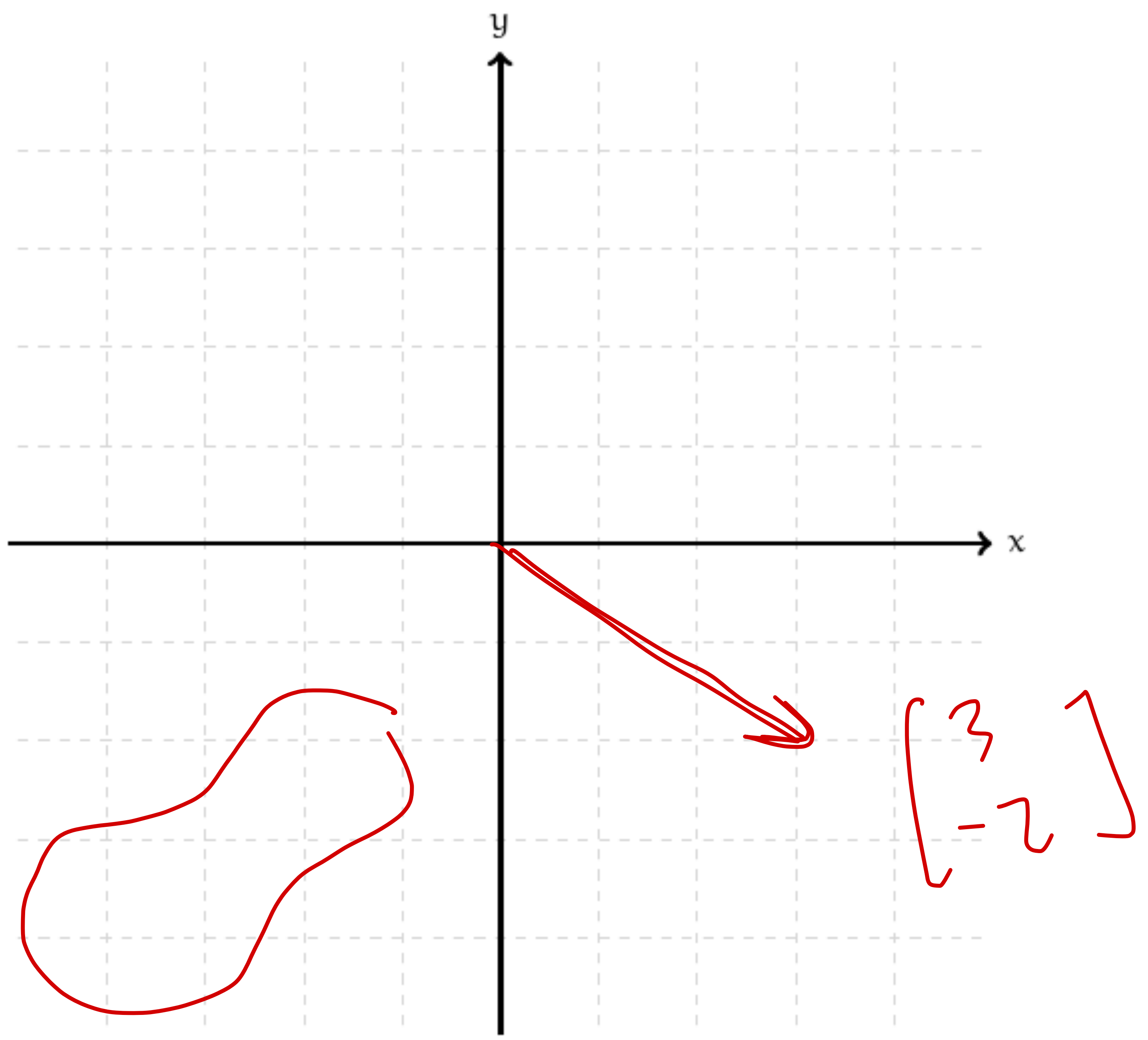

Imagine shearing like with rocks or metal.

# Question

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$$

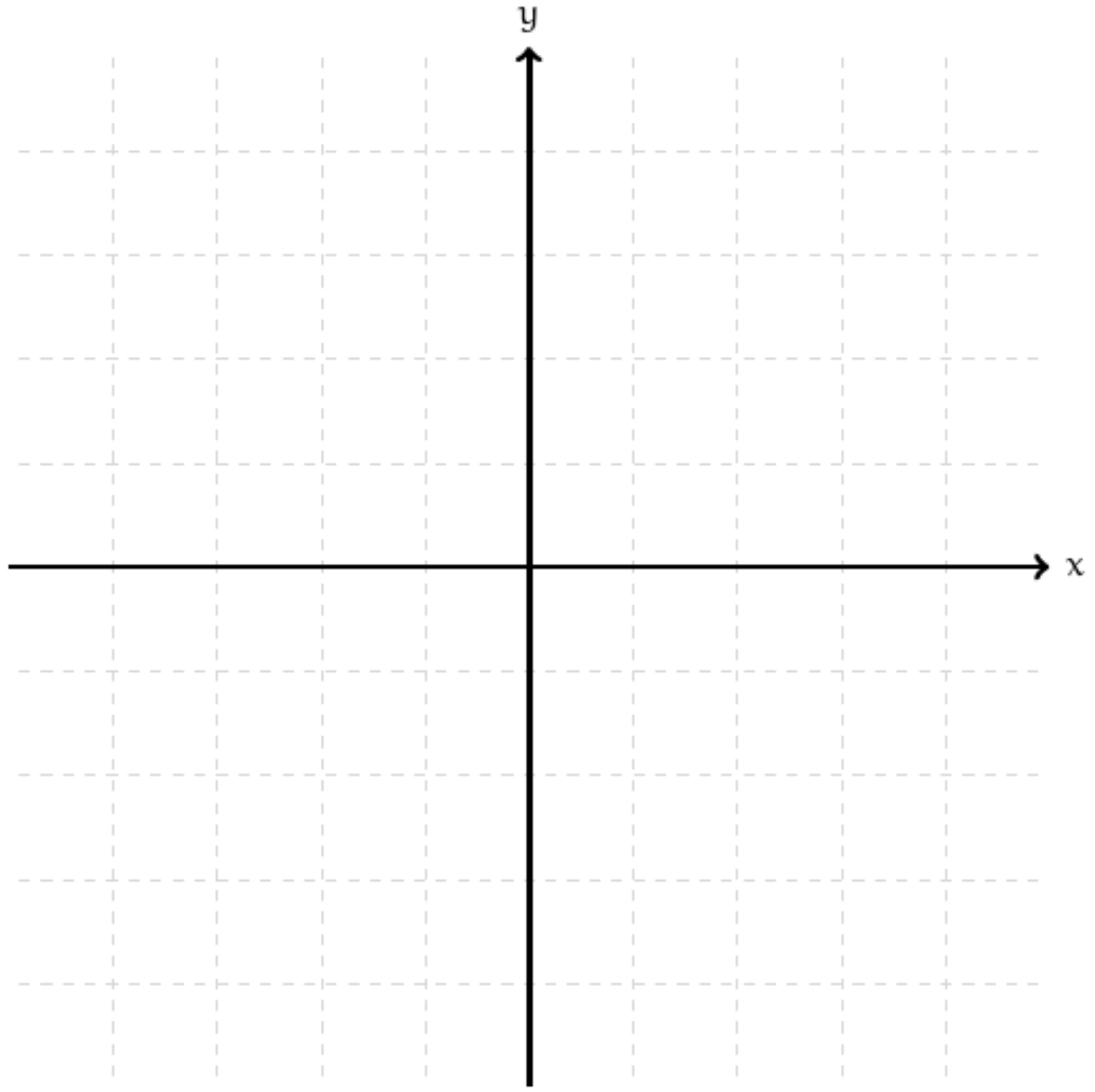



$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{x}$$

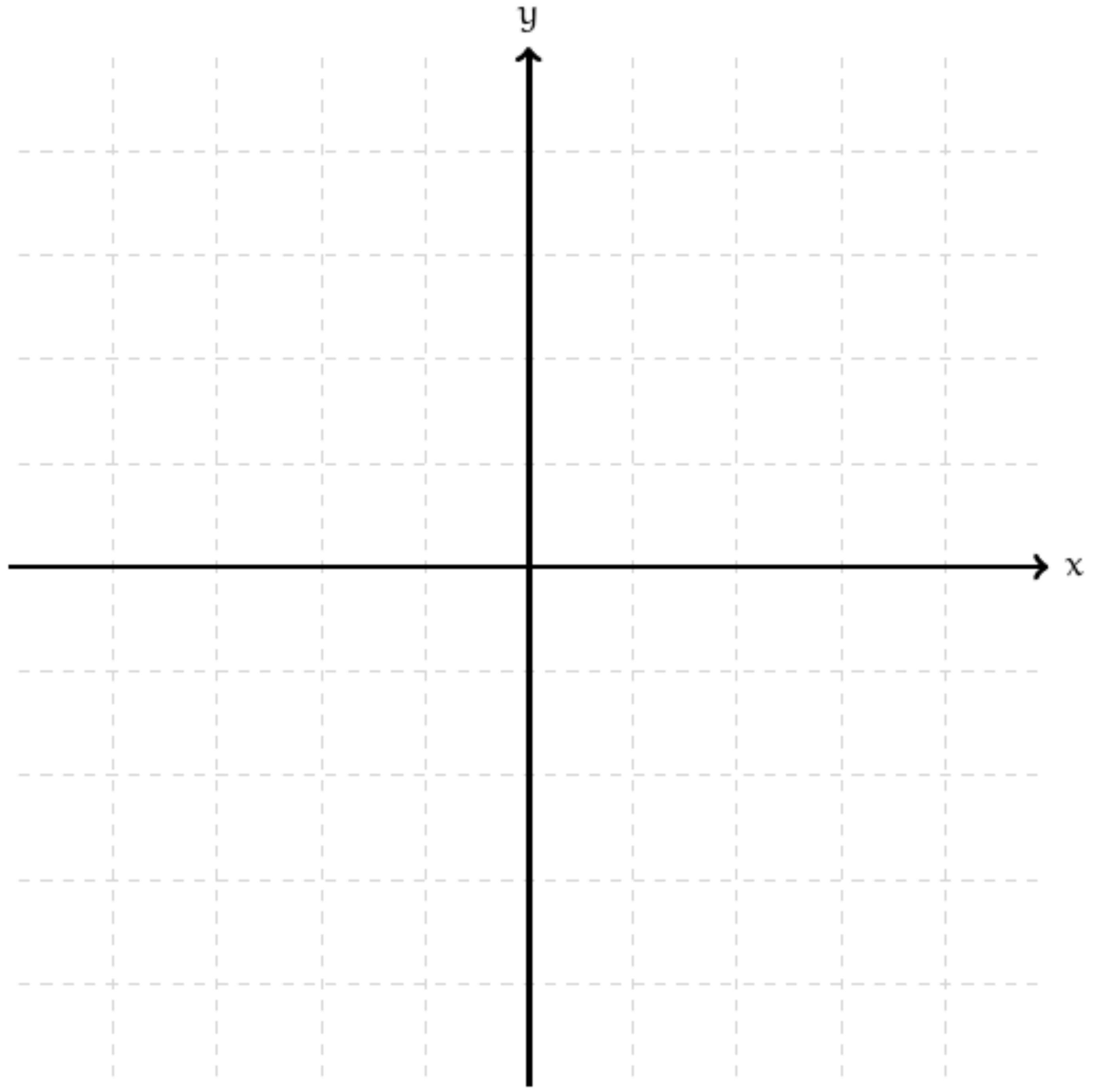



`Draw how this matrix transforms points. What kind of transformation does it represent?`

# Answer: Reflection




$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{x}$$



# Motivating Questions

What kind of functions can we define in this way?

How do we interpret what the transformation does to a set of vectors?

How does this relate back to matrix equations?

# Linear Transformations

# Recall: Algebraic Properties

Matrix–vector multiplication satisfies the following two properties:

1. $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$ (additivity)
2. $A(c\mathbf{v}) = c(A\mathbf{v})$ (homogeneity)

# Question

*Verify the following.*

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \left( \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \left( 2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right) = 2 \left( \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right)$$

# Answer

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \left( \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) =$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} =$$

# Answer

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \left( 2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right) =$$

# Linear Transformations

**Definition.** A transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is ***linear*** if it satisfies the following two properties.

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ (additivity)
2. $T(c\mathbf{v}) = cT(\mathbf{v})$ (homogeneity)

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Matrix transformations are linear transformations.

# Example: Identity

$$T(\mathbf{v}) = \mathbf{v}$$

$$T(\vec{u} + \vec{r}) = \vec{u} + \vec{r} = T(\vec{u}) + T(\vec{r}) \checkmark$$

$$T(c\vec{r}) = c\vec{r} = cT(\vec{r}) \checkmark$$

$T$ is linear

# Example: Zero

$$T(\mathbf{v}) = \mathbf{0}$$

$\checkmark$

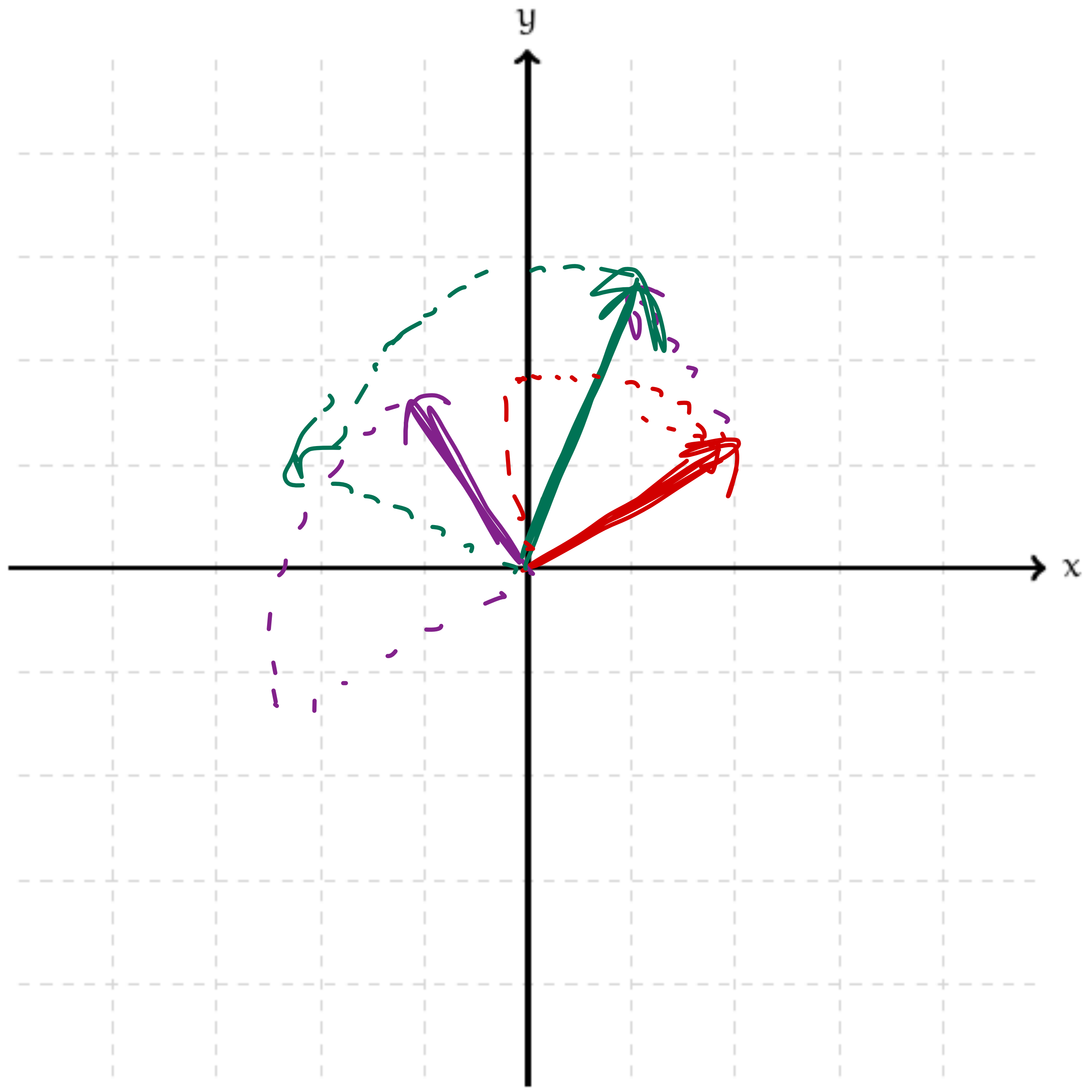
$$T(\vec{u} + \vec{r}) = \vec{0} = \vec{0} + \vec{0} = T(\vec{u}) + T(\vec{r})$$

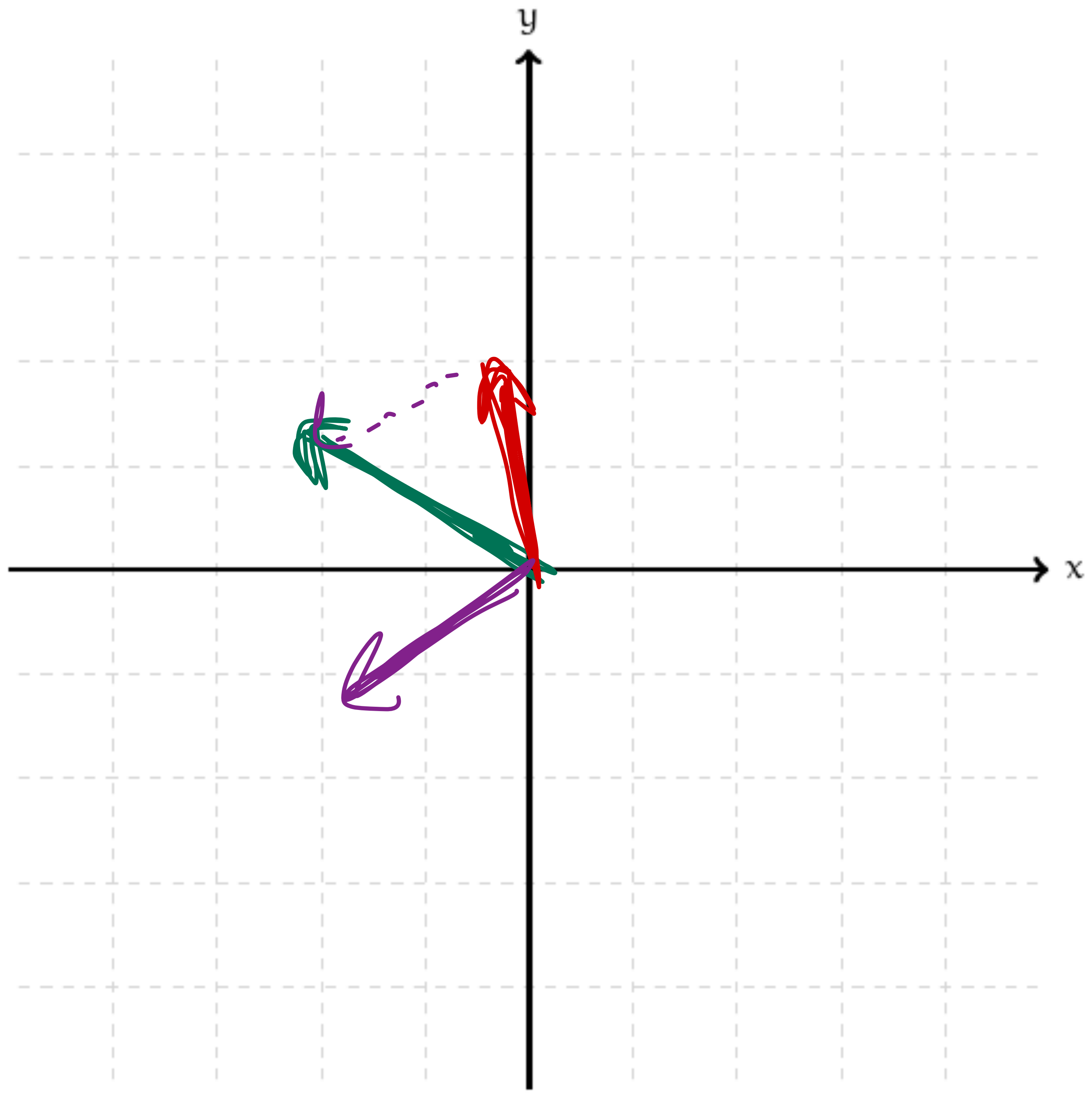
$$T(c\vec{r}) = \vec{0} = c\vec{0} = cT(\vec{r}) \quad \checkmark$$

$T$ is linear

# Example: Rotation

We'll see this on Thursday, but we can reason about it geometrically for now.



# Example: Indefinite Integrals

$$T(f) = \int f(x)dx$$

Disclaimer:
Advanced
Material

$$T(f+g) = \int (f+g)(x)dx = \int f(x) + g(x)dx = \int f(x)dx + \int g(x)dx = T(f) + T(g)$$

$$T(cf) = \int (cf)(x)dx = \int cf(x)dx = c\int f(x)dx = cT(f)$$

the same goes for derivatives
(how are functions vectors???)

# Example: Expectation

$$T(X) = \mathbb{E}[X]$$

Disclaimer:
Advanced
Material

This is exactly linearity of expectation.

(how are random variables vectors???)

# Non-Example: Squares

$$T(x) = x^2$$

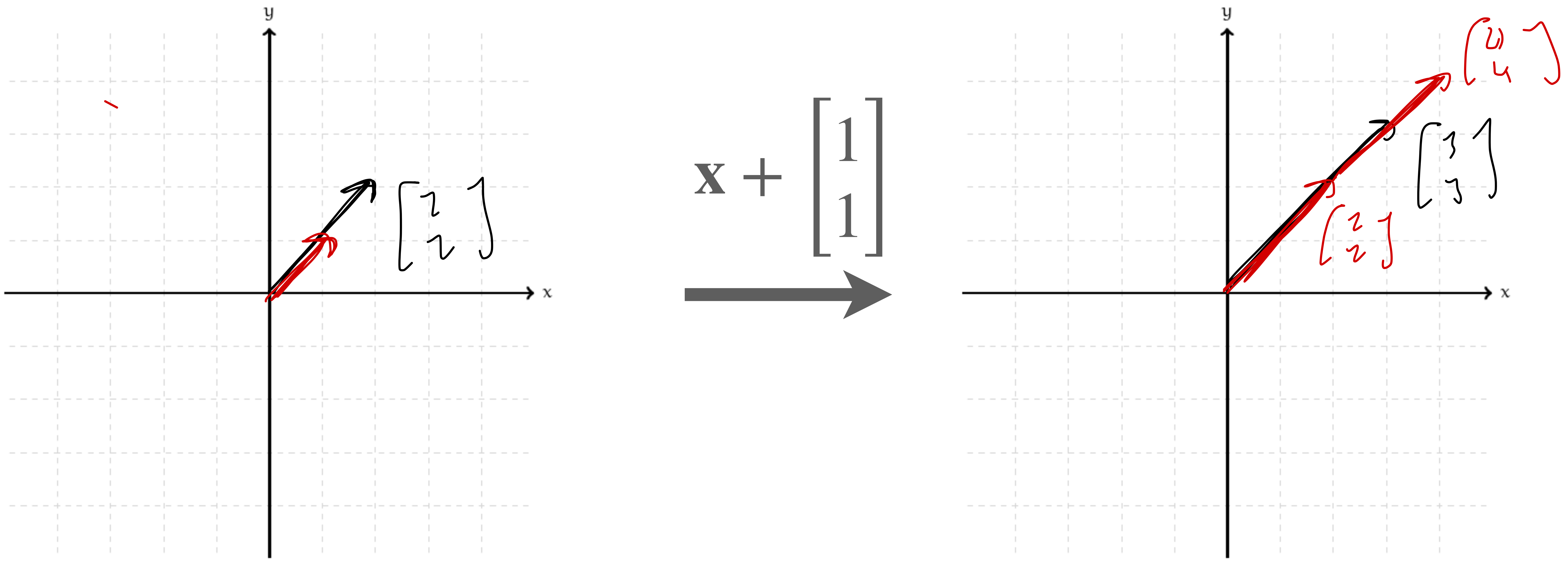
Note that $T : \mathbb{R}^1 \to \mathbb{R}^1$

$$T(1+1) = T(2) = 4$$

$$\neq$$

$$T(1) + T(1) = 1 + 1 = 2$$

# Non-Example: Translation



# Question

*Show that* $T(\mathbf{v}) = 5\mathbf{v}$ *is a linear transformation.*

*Show that* $T(x) = e^x$ *is not a linear transformation.*

# Answer

$$T(\mathbf{v}) = 5\mathbf{v}$$

# Answer

$$T(x) = e^x$$

# Properties of Linear Transformations

# The Zero Vector

$$T(\mathbf{0}) = \text{ ???}$$

# The Zero Vector

$$T(\mathbf{0}) = \mathbf{0}$$

# The Zero Vector

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Note: These may be different dimensions!

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# Verification

```
any matrix transformation:

rotation:

translation (non-example):
```

# A Single Condition

$$T(a\mathbf{v} + b\mathbf{u}) = aT(\mathbf{v}) + bT(\mathbf{u})$$

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$= aT(\mathbf{v}) + bT(\mathbf{u})$ (homogeneity for each term)

# A Single Condition

**Theorem.** A transformation $T: \mathbb{R}^m \to \mathbb{R}^n$ is linear if and only if for any vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^m$ and any real numbers $a$ and $b$,

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It's often easiest to show this single condition.

# Linear Combinations

$$T(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_n\mathbf{v}_n) = a_1T(\mathbf{v}_1) + a_2T(\mathbf{v}_2) + \ldots + a_nT(\mathbf{v}_n)$$

We can generalize this condition to any linear combination.

# Linear Combinations

$$T\left(\sum_{i=1}^{n} a_i \mathbf{v}_i\right) = \sum_{i=1}^{n} a_i T(\mathbf{v}_i)$$

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# Linear Combinations

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We can generalize this condition to any linear combination.

This is the most useful form.

# Application: Unit Cost Matrices

# A Question for a Business Student

Suppose you have a company that produces two products B and C.

For each product you know how much you spend per dollar on **material** (M), **labor** (L) and **overhead** (O).

$$\begin{array}{cc} B & C \end{array}$$
$$\begin{bmatrix} .45 & .40 \\ .25 & .30 \\ .15 & .15 \end{bmatrix} \begin{array}{c} M \\ L \\ O \end{array}$$

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*How much are you spending, in total, on each cost, given that you made $s_1$ dollars worth of B and $s_2$ dollars worth of C?*

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*How much are you spending, in total, on each cost, given that you made $s_1$ dollars worth of B and $s_2$ dollars worth of C?*

Solution. Use matrix transformations.

# As a Matrix Transformation

$$T(\mathbf{x}) = \begin{bmatrix} 0.45 & 0.40 \\ 0.25 & 0.30 \\ 0.15 & 0.25 \end{bmatrix} \mathbf{x}$$

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$$T\left(\begin{bmatrix} s_1 \\ s_2 \end{bmatrix}\right) = s_1 \begin{bmatrix} 0.45 \\ 0.25 \\ 0.15 \end{bmatrix} + s_2 \begin{bmatrix} 0.40 \\ 0.30 \\ 0.15 \end{bmatrix} = \begin{bmatrix} \texttt{total material cost} \\ \texttt{total labor cost} \\ \texttt{total overhead cost} \end{bmatrix}$$

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This is much more valuable if we had *a lot* of products and a complex collection of costs.

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This is a very powerful *algorithmic* idea.

# Summary

Matrices can be viewed as linear transformations.

Matrix transformations change the "shape" of points sets.

Linear transformations behave well with respect to linear combinations.