Linear Transformations

Geometric Algorithms
Lecture 7

CAS CS 132

Recap Problem

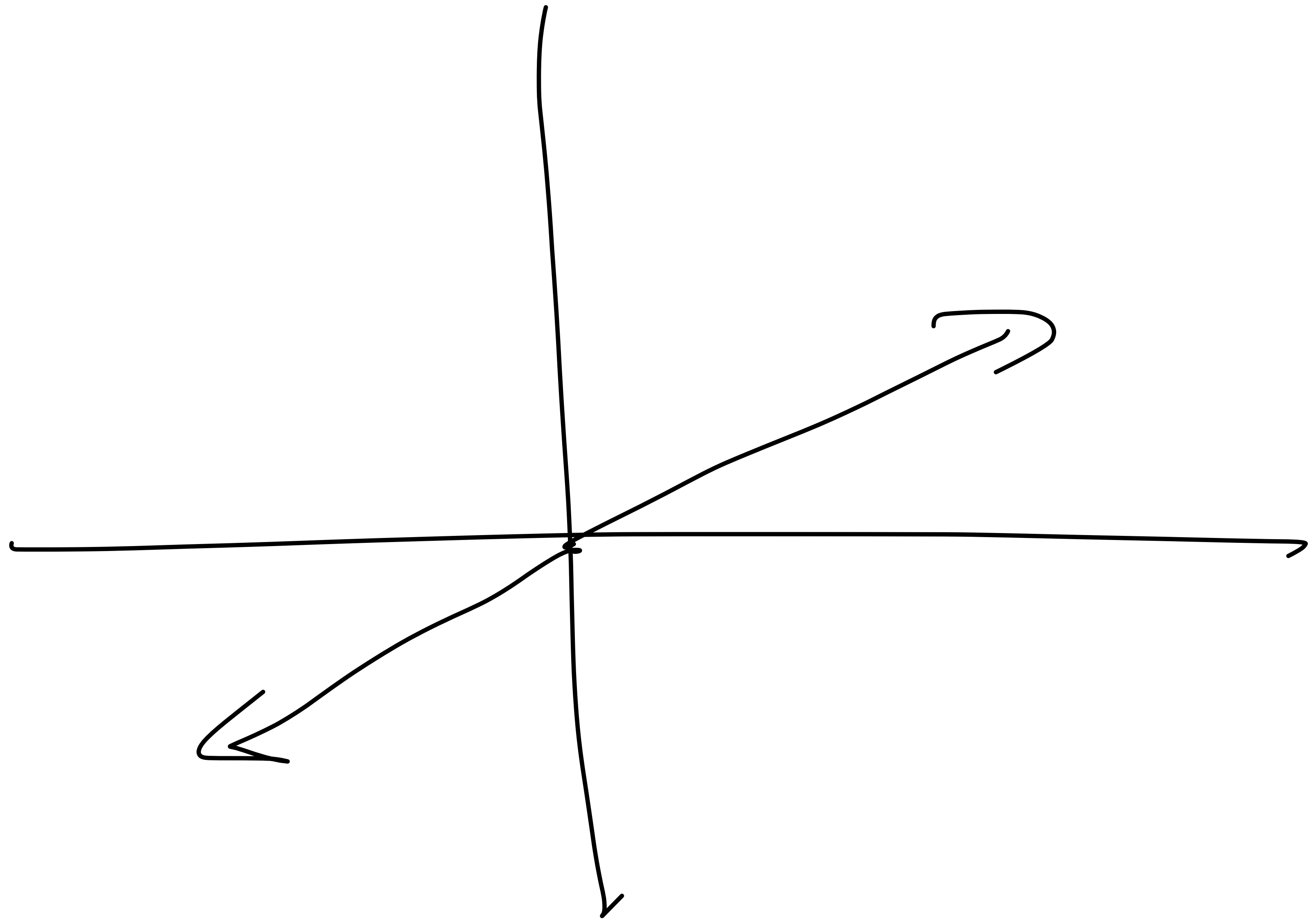
Find three vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ *in* $\mathbb{R}^3$ *such that*

» *every pair of vectors (i.e.,* $\{\mathbf{v}_1, \mathbf{v}_2\}$*,* $\{\mathbf{v}_1, \mathbf{v}_3\}$*,* $\{\mathbf{v}_2, \mathbf{v}_3\}$*) are linearly independent*

» $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ *is linearly dependent*

Answer

r_1, r_2 are lin dep. $\iff$ they are colinear.



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

$$v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$r_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$r_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$r_3 = r_1 + r_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

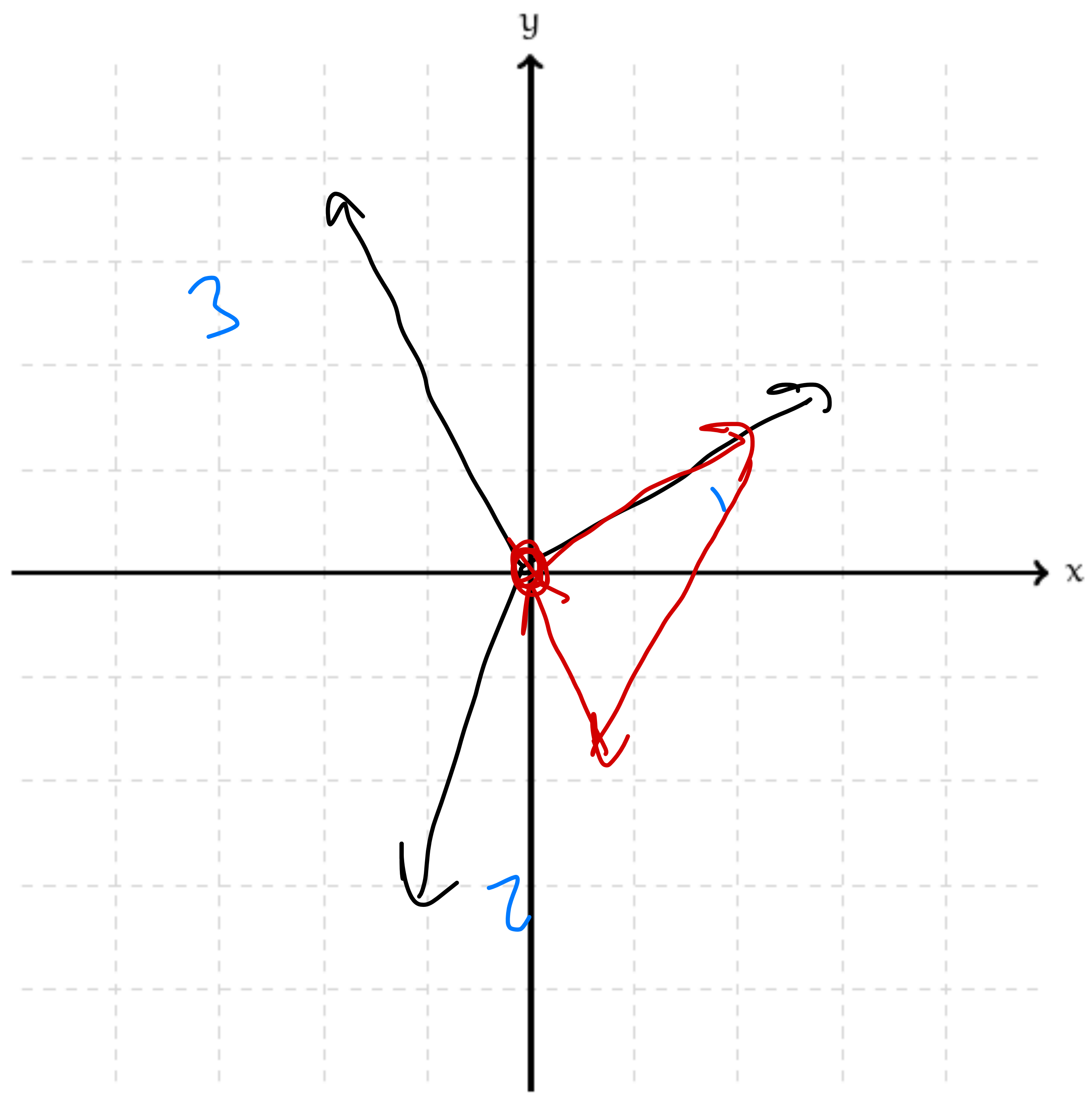
$$r_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$r_2 = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$$

$$r_3 = \begin{bmatrix} 3 \\ 6 \\ 8 \end{bmatrix}$$

Demo: Geometry of Linear Dependence

$a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 = \vec{0}$



Objectives

1. Introduce Matrix Transformations
2. Define Linear Transformations
3. Start looking at the Geometry of Linear Transformations
4. See an Non-Geometric Application

Keywords

Transformations

Domain, Codomain

Image, Range

Matrix Transformations

Linear Transformations

Additivity, Homogeneity

Dilation, Contraction, Shearing, Rotation

Introduction

Recall: Spans (with Matrices)

Definition. The ***span*** of a set of vectors is the set of all possible linear combinations of them.

$$\text{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\} = \{[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n]\,\mathbf{v} : \mathbf{v} \in \mathbb{R}^n\}$$

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The span of the columns of a matrix A is the set of of vectors resulting from multiplying A by any vector.

Matrices as Transformations

Matrices allow us to *transform* vectors.

The transformed vector lies in the span of its columns.

$$\mathbf{x} \mapsto A\mathbf{x}$$

map a vector $\mathbf{x}$ to the vector $A\mathbf{v}$

Example (Algebraic)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1(1) + 0 + 0 \\ 0 + 2(1) + 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2(1) + 0 + 0 \\ 0 + 2(3) + 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

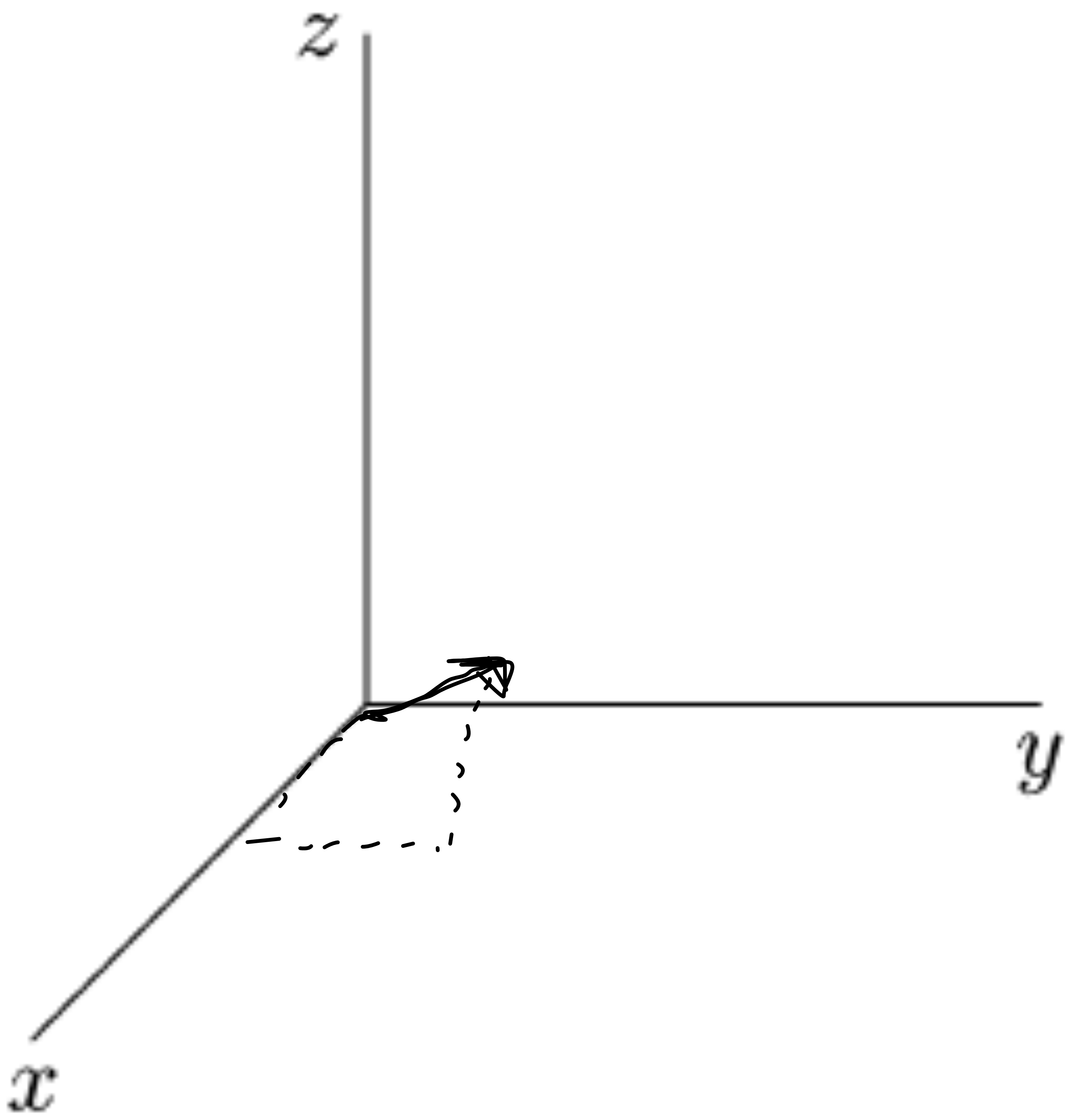
Example (Algebraic)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1(x_1) + 0 + 0 \\ 0 + 2(x_2) + 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_2 \end{bmatrix}$$

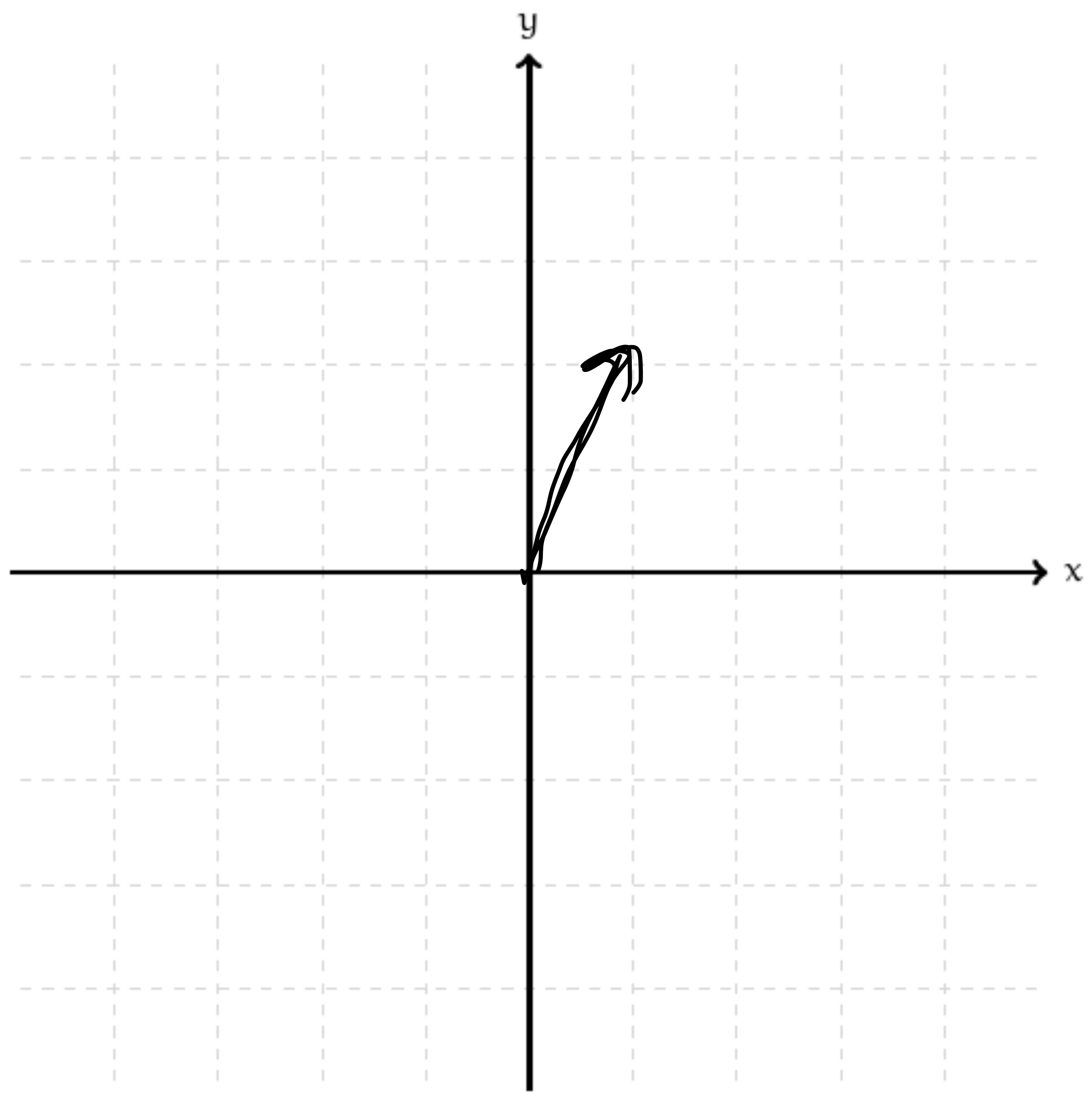
Example (Geometric)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_2 \end{bmatrix}$$

$$\cdots \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \mathbf{x}$$



!!Important!!

The vector may be a different size after translation.

Recall: Matrix-Vector Multiplication and Dimension

`matrix-vector multiplication only works if the number of` *`columns`* `of the matrix matches the dimension of the vector`

$$
\overbrace{\begin{bmatrix} * & \cdots & * \\ * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \\ * & \cdots & * \end{bmatrix}}^{n} m \quad n \begin{bmatrix} * \\ \vdots \\ * \end{bmatrix} = m \begin{bmatrix} * \\ * \\ \vdots \\ * \\ * \end{bmatrix}
$$

$$(m \times n) \qquad \mathbb{R}^n \qquad \mathbb{R}^m$$

Motivating Questions

What kind of functions can we define in this way?

How do we interpret what the transformation does to a set of vectors?

How does this relate back to matrix equations?

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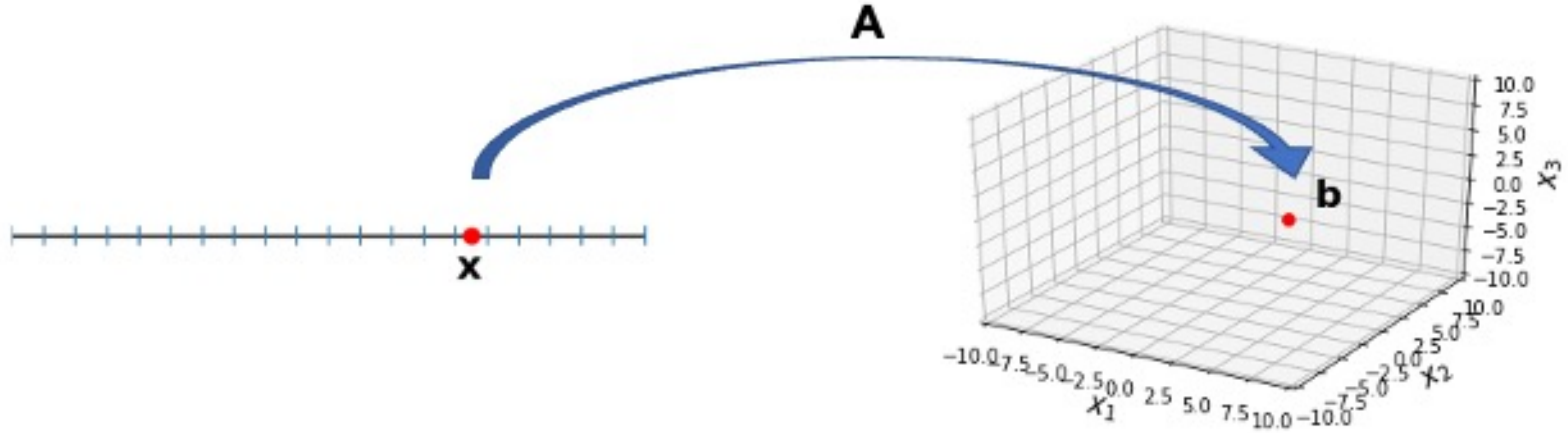
A New Interpretation of the Matrix Equation

$A\mathbf{x} = \mathbf{b}$?	$\equiv$	is there a vector which A transforms into $\mathbf{b}$?
Solve $A\mathbf{x} = \mathbf{b}$	$\equiv$	find a vector which A transforms into $\mathbf{b}$

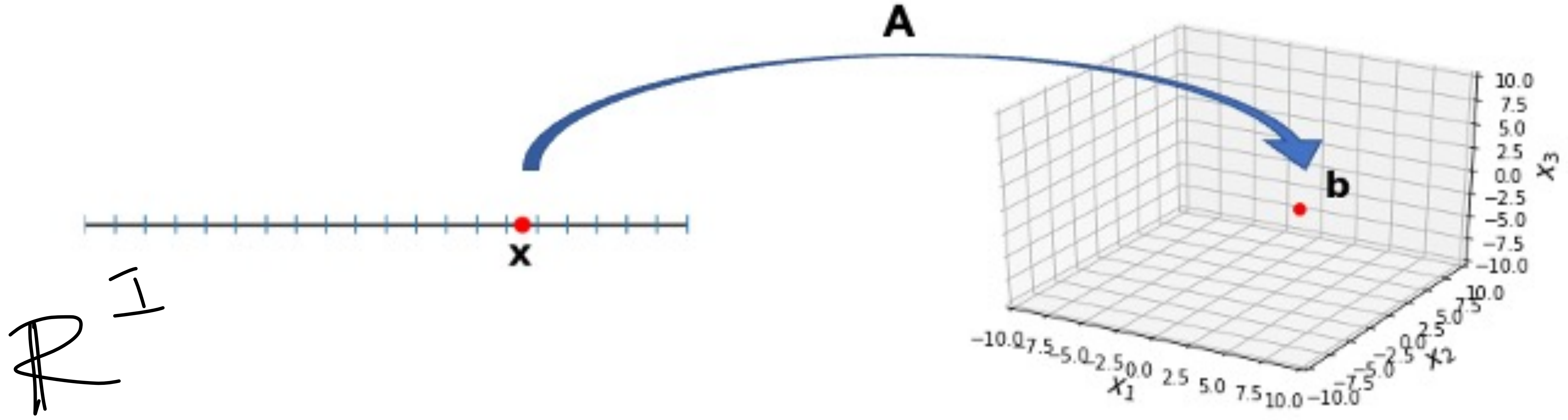
Question (Conceptual)

Suppose a matrix transforms a vector according to the following picture. What is the size of the matrix?



$(m \times n)$ what is m? n?

Answer: 3×1



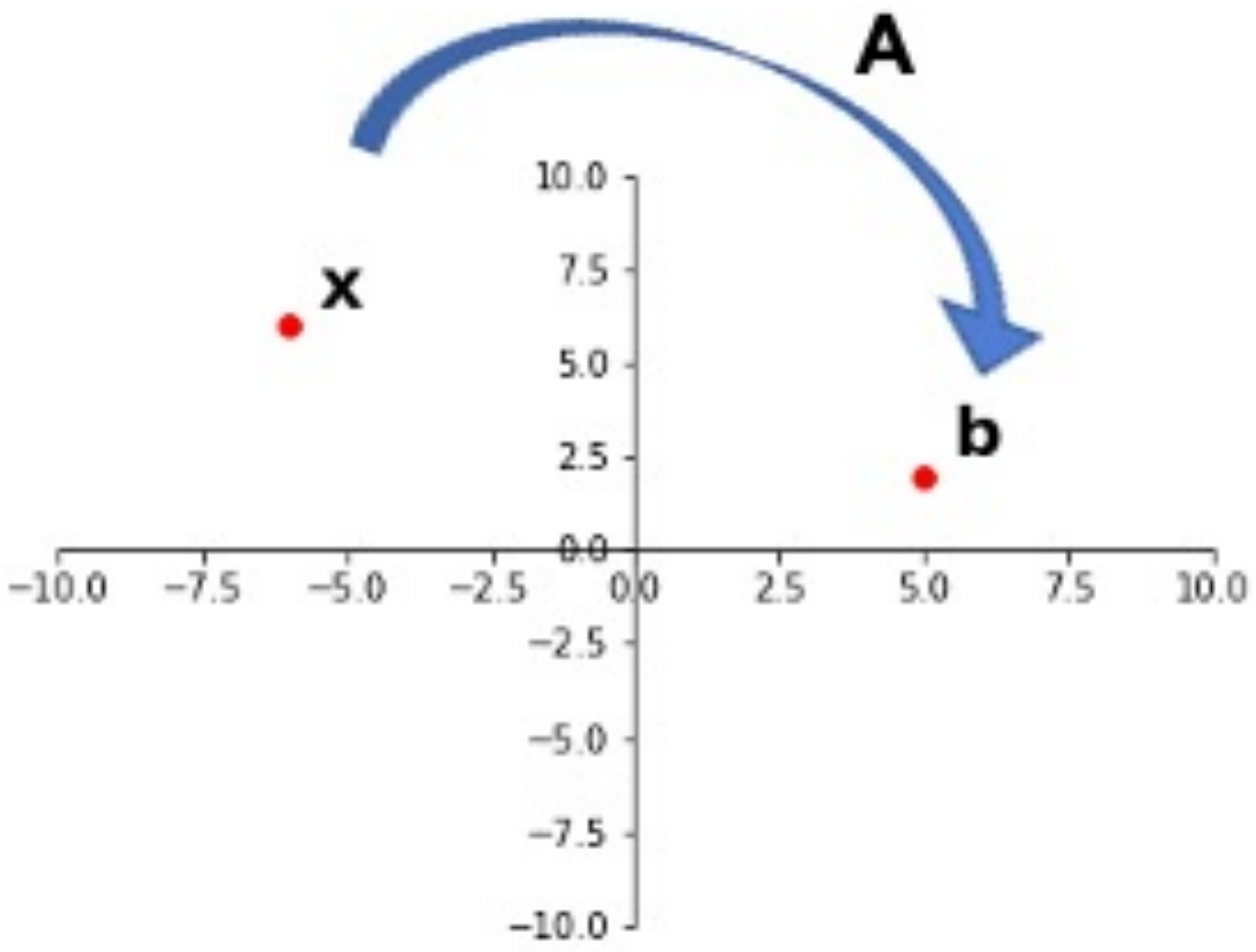
$$A \quad x \quad = \quad b$$

$$(m \times n) \quad (n \times 1) \quad (m \times 1)$$

$$(3 \times 1) \quad n = 1 \quad m = 3$$

$$\mathbb{R}^{n} \to \mathbb{R}^{n}$$

Mapping between the same space can be viewed as a way of moving around points.



Transformations

Transformations in General

Definition. A ***transformation*** T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a function which maps every vector $\mathbf{v}$ in $\mathbb{R}^n$ to a vector $T(\mathbf{v})$ in $\mathbb{R}^m$.

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$$T : \mathbb{R}^n \to \mathbb{R}^m$$

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domain codomain

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$$T : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

domain → codomain

It's just a function, like in calculus.

Image and Range

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Definition. The ***range*** of a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is the set of all possible images under T.

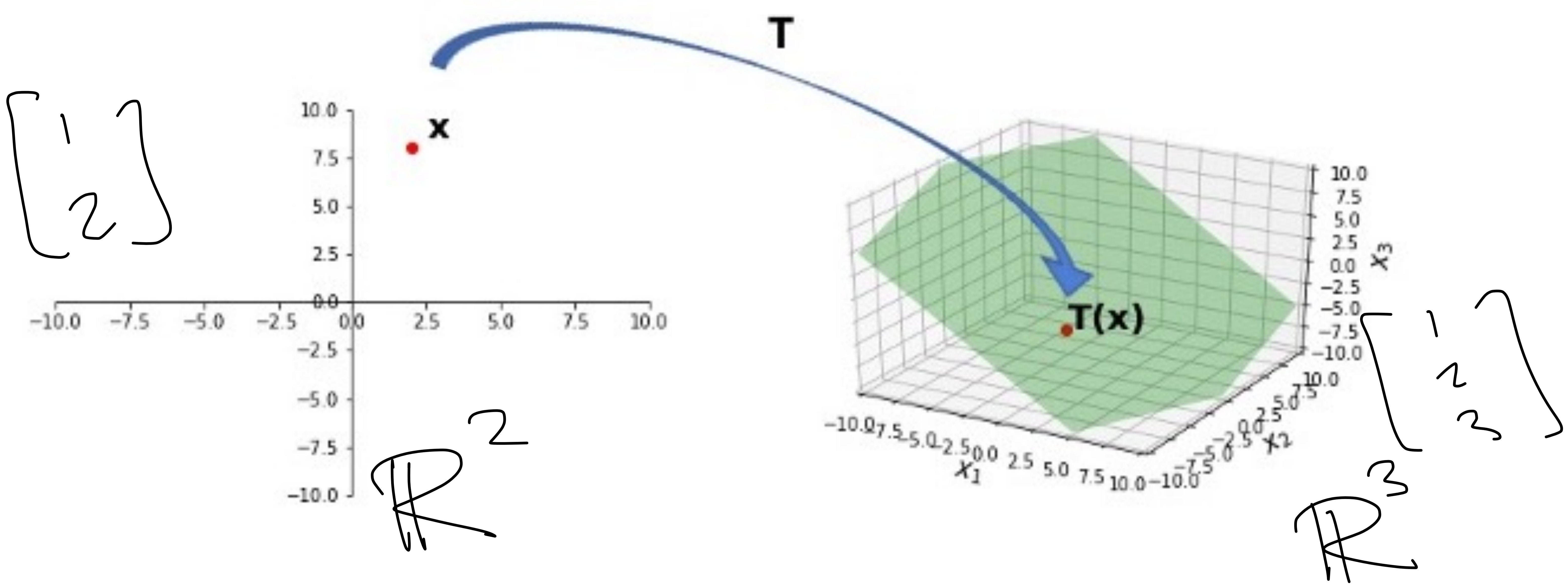
$$\text{ran}(T) = \{T(\mathbf{v}) : v \in \mathbb{R}^n\}$$

image of $\mathbf{v}$ under T $\equiv$ output of T applied to $\mathbf{v}$

range of T $\equiv$ all possible output of T

Codomain and Range

The codomain and range of a transformation may or may not be the same.



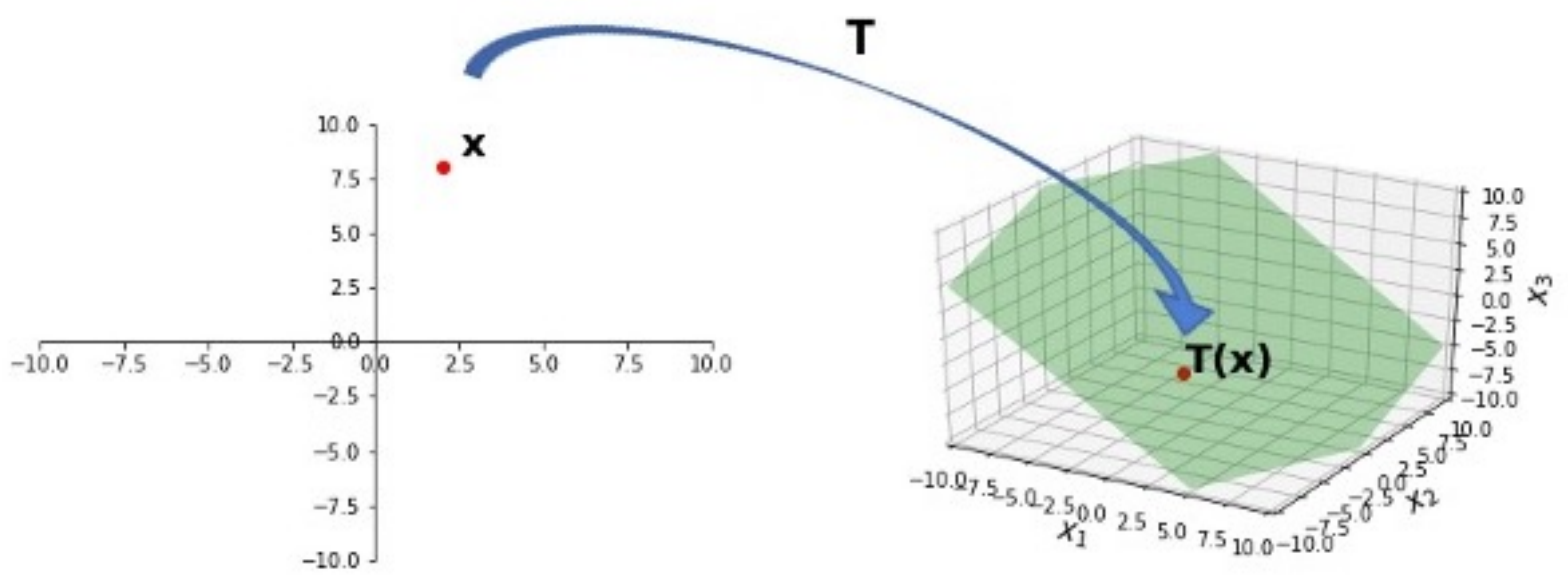
Codomain and Range

$T: \mathbb{R}^4 \to \mathbb{R}^2$

$T(\vec{r}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\text{ran}(T) = \{\vec{0}\}$

The codomain and range of a transformation may or may not be the same.



range: just the green plane

domain: $\mathbb{R}^2$

codomain: $\mathbb{R}^3$

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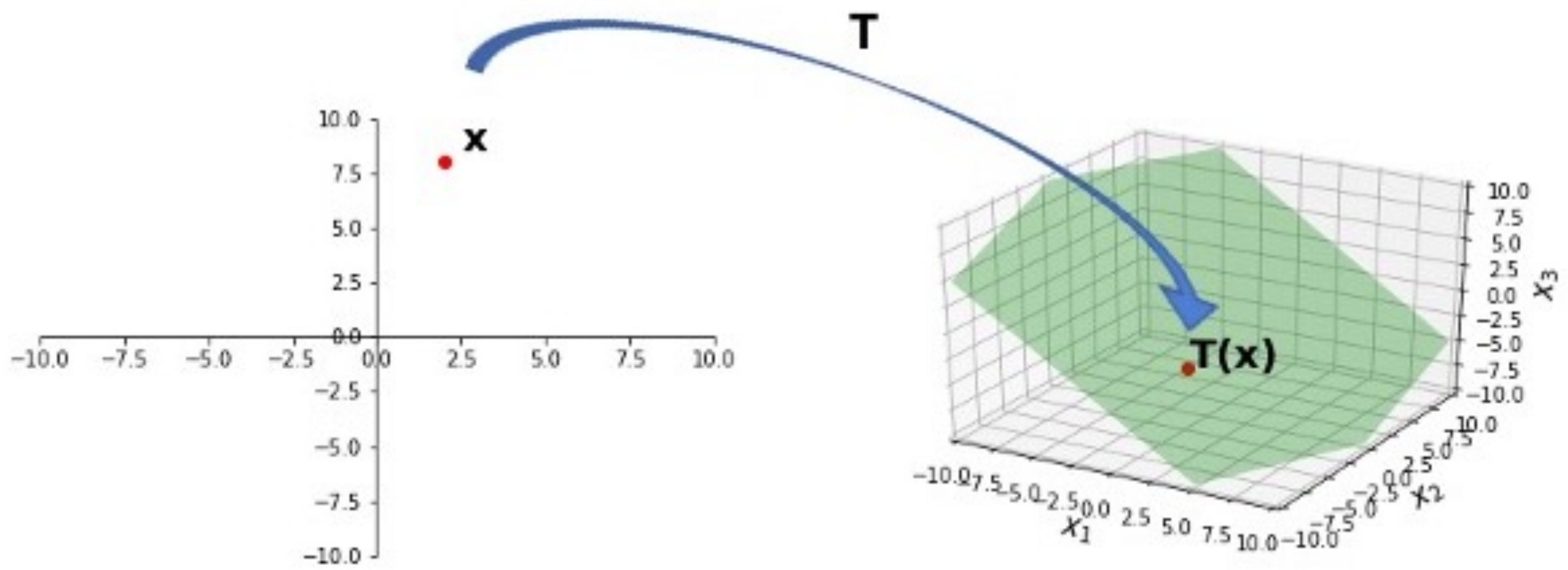

range: just the green plane

domain: $\mathbb{R}^2$

codomain: $\mathbb{R}^3$

The range is always contained in the codomain.

Matrix Transformations

Transformation of a Matrix

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The ***transformation of a*** $(m \times n)$ ***matrix*** A is the function $T : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$T(\mathbf{v}) = A\mathbf{v}$$

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given $\mathbf{v}$, return A multiplied by $\mathbf{v}$

transformation is <u>function</u> not a matrix.

Transformation of a Matrix

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$$T(\mathbf{v}) = A\mathbf{v}$$

given $\mathbf{v}$, return A multiplied by $\mathbf{v}$

e.g. $T(\mathbf{v}) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{v}$

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Range and Span

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The span of the columns of a matrix A is the set of all possible *images* under A.

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$$\text{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\} = \text{ran}([\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n])$$

where $T(\vec{v}) = [\vec{a}_1 \; \vec{a}_2 \ldots \vec{a}_n]\vec{r}$

Range and Span

The span of the columns of a matrix A is the set of all possible *images* under A.

$$\operatorname{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\} = \operatorname{ran}([\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n])$$

The transformation of a vector $\mathbf{v}$ under the matrix A always lies in the span of its columns.

Example

$$\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 2\\ 1 & 3 & 0\end{bmatrix}\begin{bmatrix}2\\ -1\\ 0\end{bmatrix} =$$

$$\begin{bmatrix}2 + (-1) + 0\\ 0(2) + 1(-1) + 2(0)\\ 2 + -3 + 0\end{bmatrix} = \begin{bmatrix}1\\ -1\\ -1\end{bmatrix}$$

$$2\begin{bmatrix}1\\ 0\\ 1\end{bmatrix} + (-1)\begin{bmatrix}1\\ 1\\ 3\end{bmatrix} + 0\begin{bmatrix}1\\ 2\\ 0\end{bmatrix} =$$

exercise

$$= \begin{bmatrix}1\\ -1\\ -1\end{bmatrix}$$

Motivating Questions

What kind of functions can we define in this way?

How do we interpret what the transformation does to a set of vectors?

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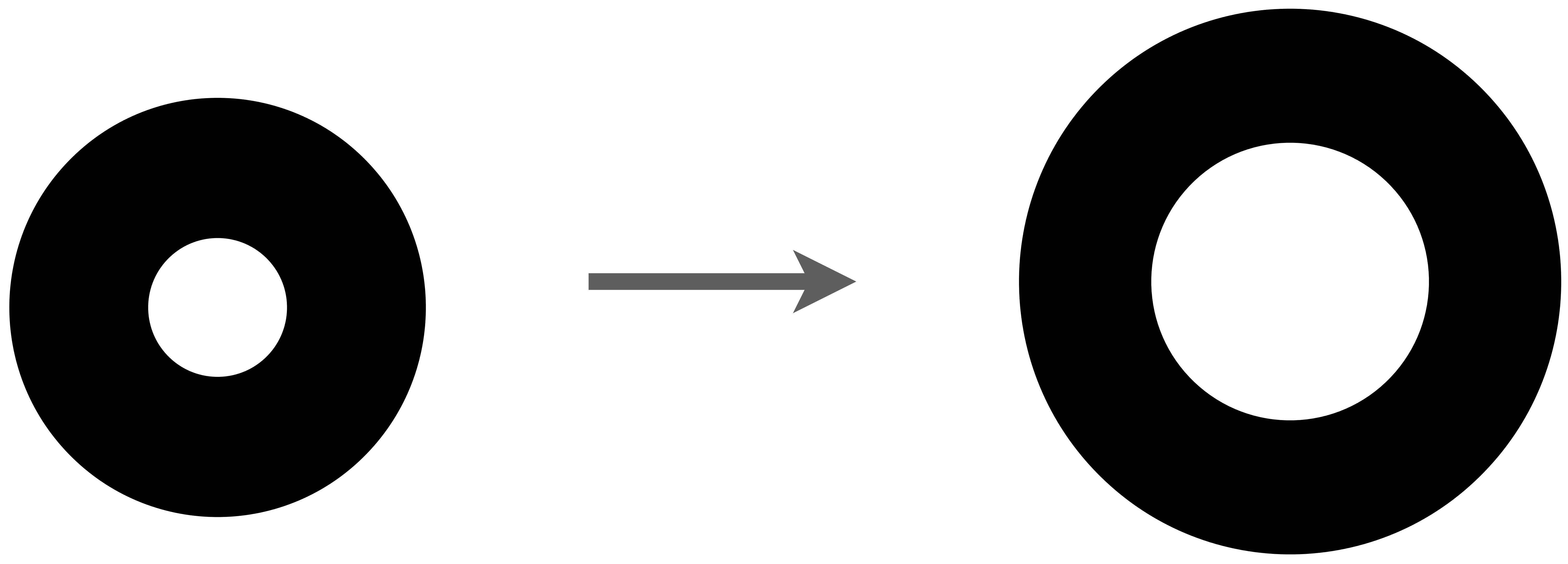
Geometry of Matrix Transformations

Motto

Matrix transformations change the "shape" of a set of set of vectors (points).

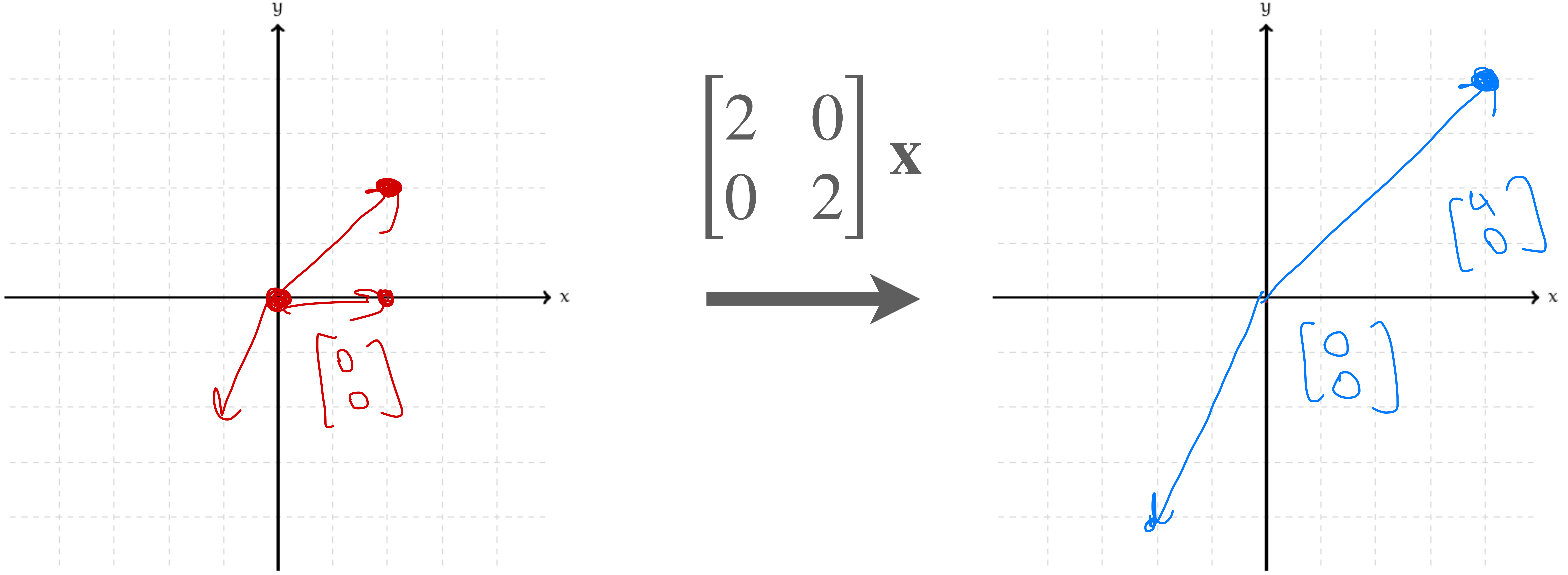
Example: Dilation



Example: Dilation

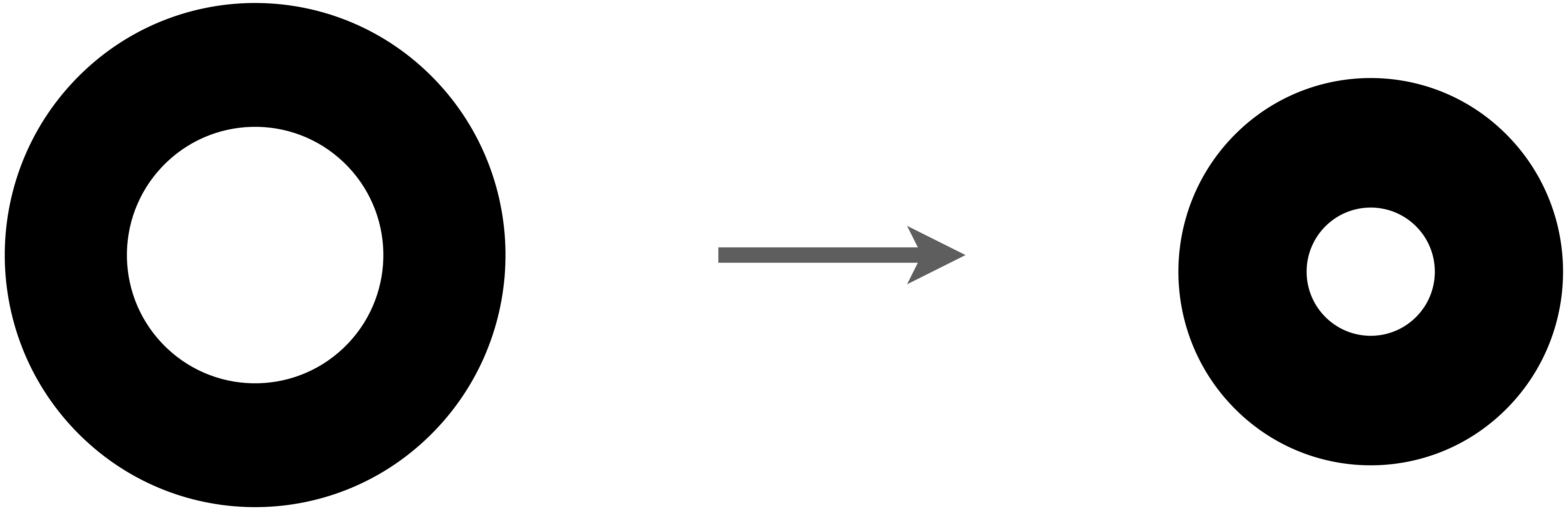
$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} rx_1 \\ rx_2 \end{bmatrix}$$



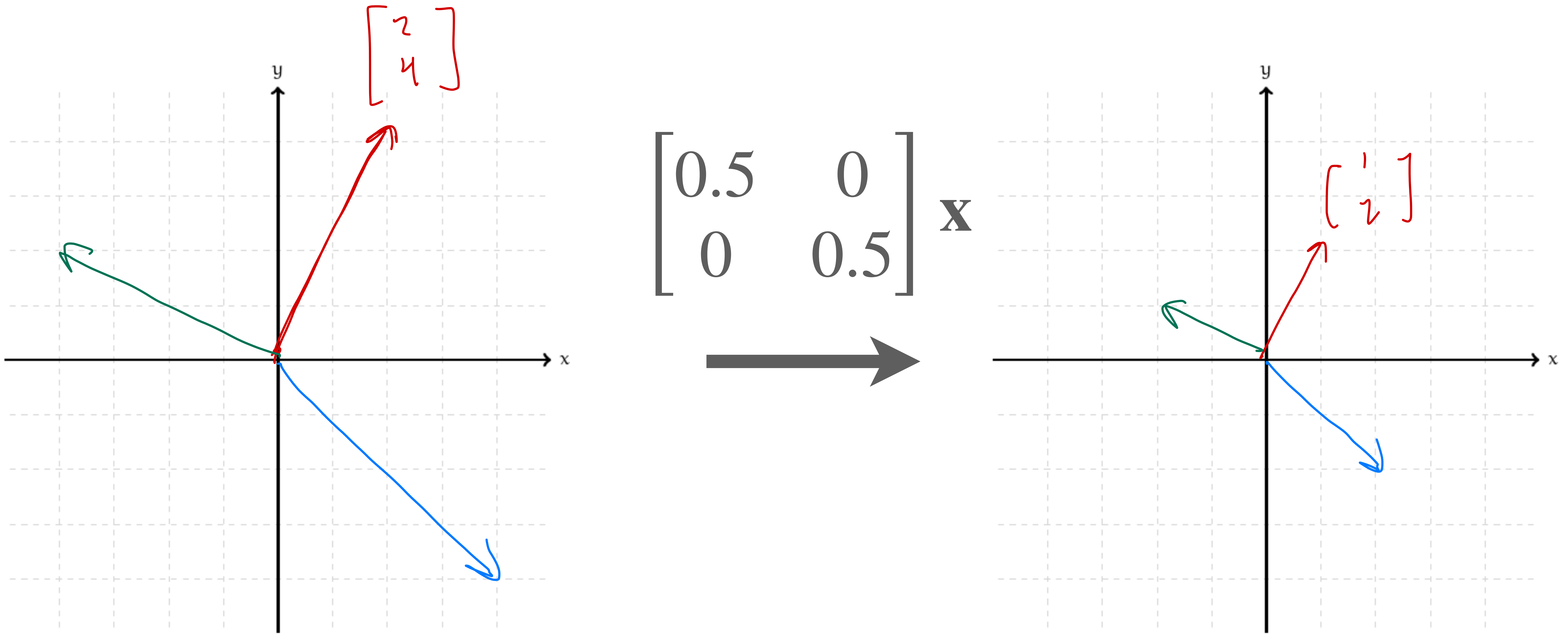
if $r > 1$, then the transformation pushes points away from the origin.

Example: Contraction



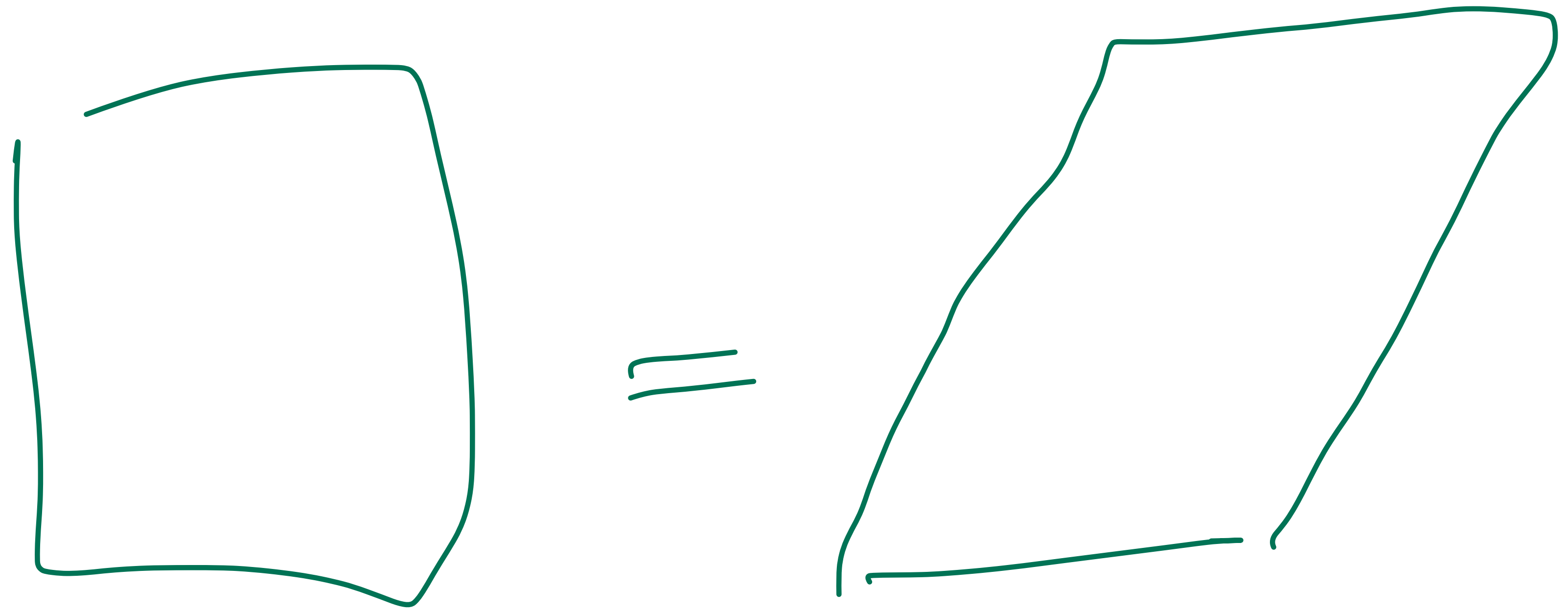
Example: Contraction

$$\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} rx_1 \\ rx_2 \end{bmatrix}$$



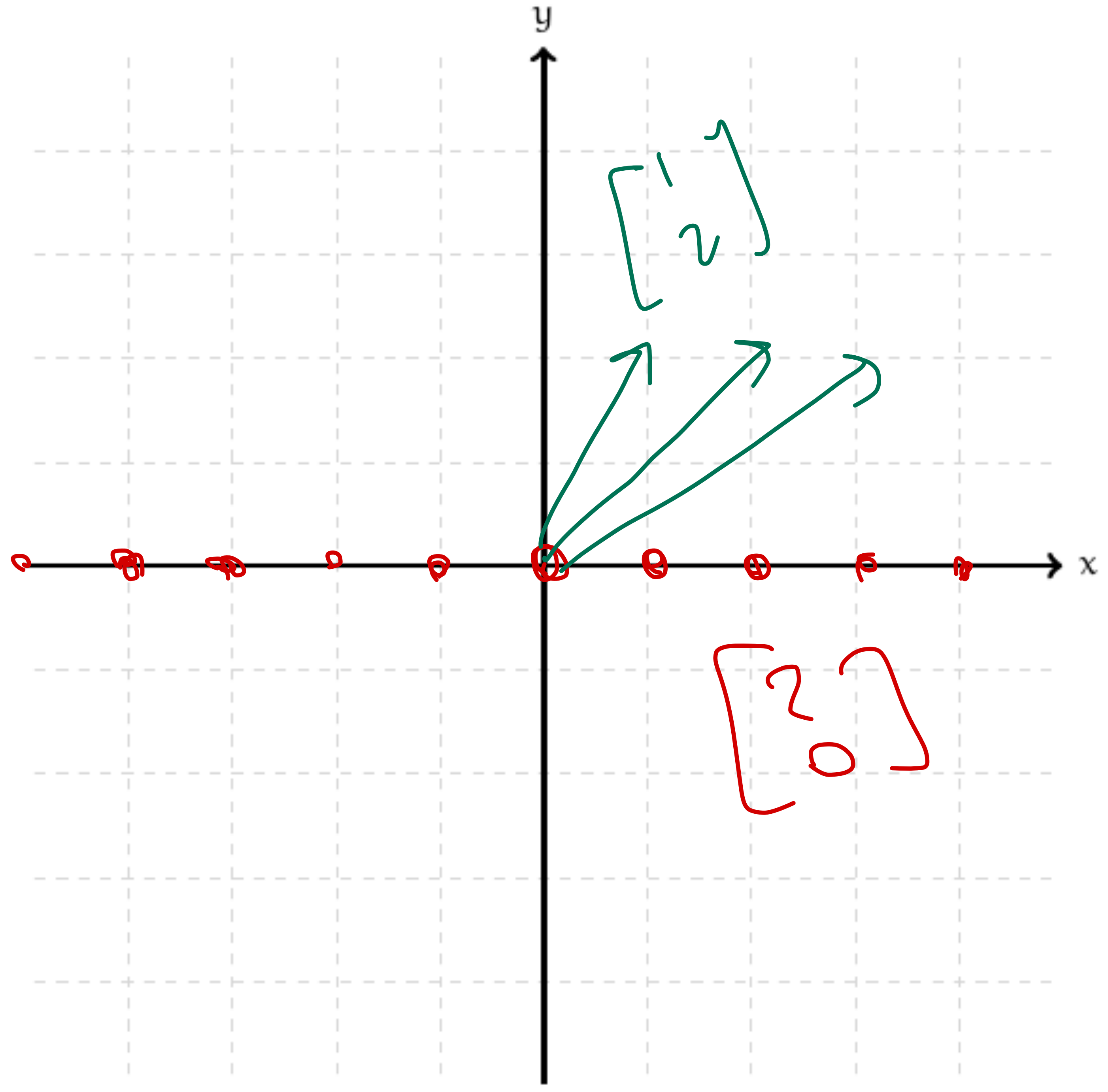
if $0 \leq r \leq 1$, then the transformation pulls points towards the origin.

Example: Shearing

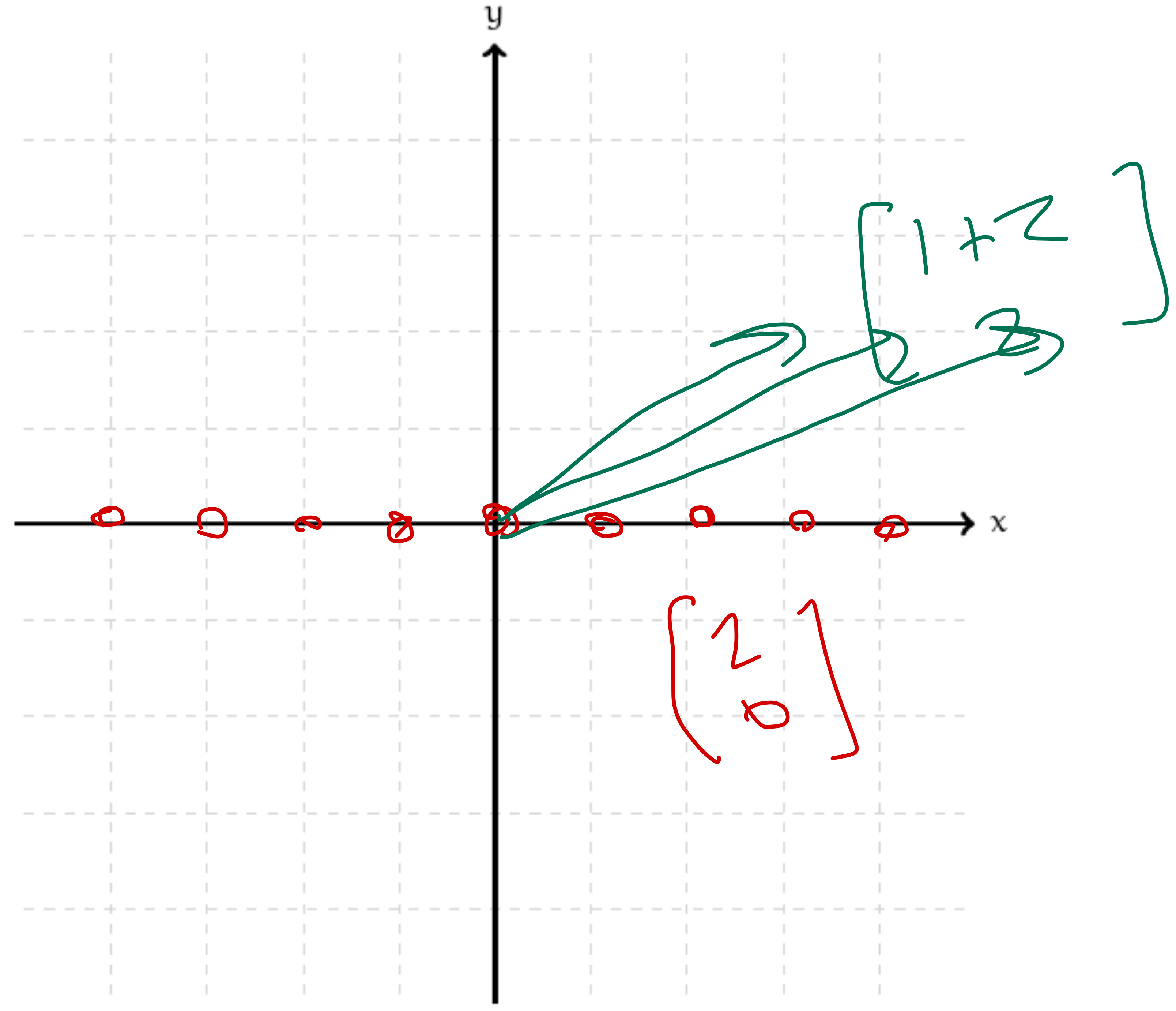


Example: Shearing

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 \end{bmatrix}$$



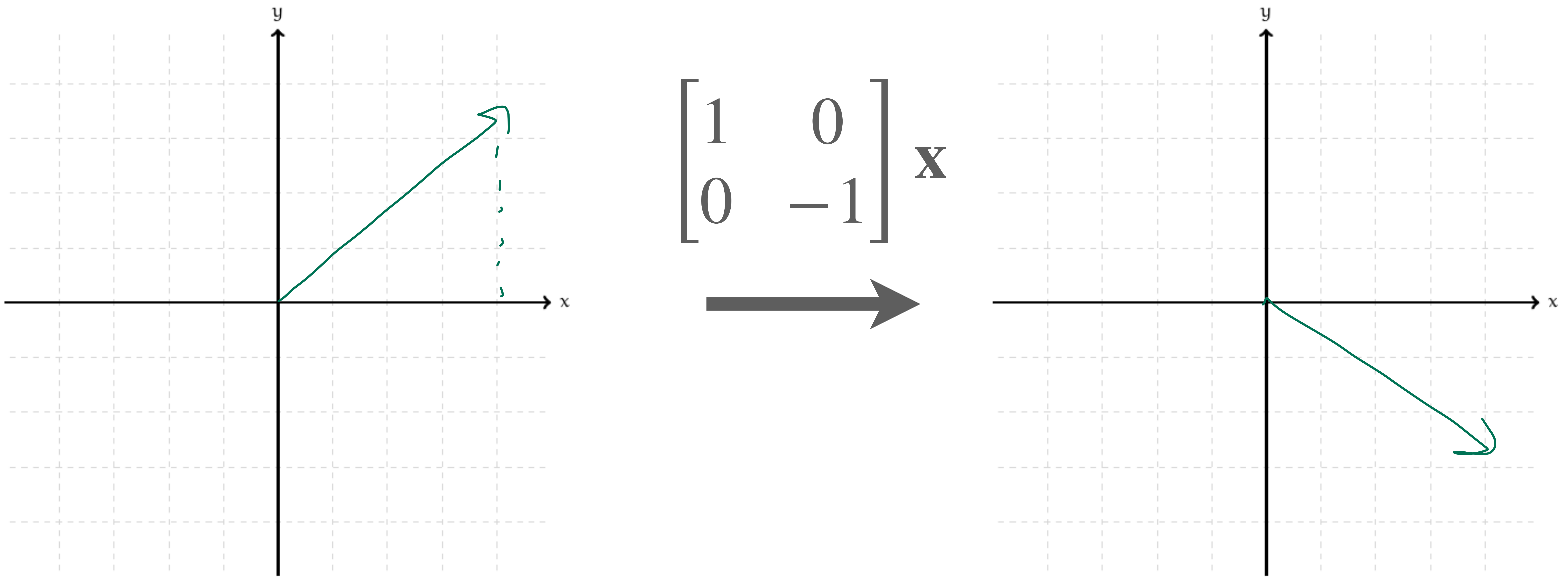
$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{x}$$



Imagine shearing like with rocks or metal.

$$\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}x_1\\ -x_2\end{bmatrix}$$

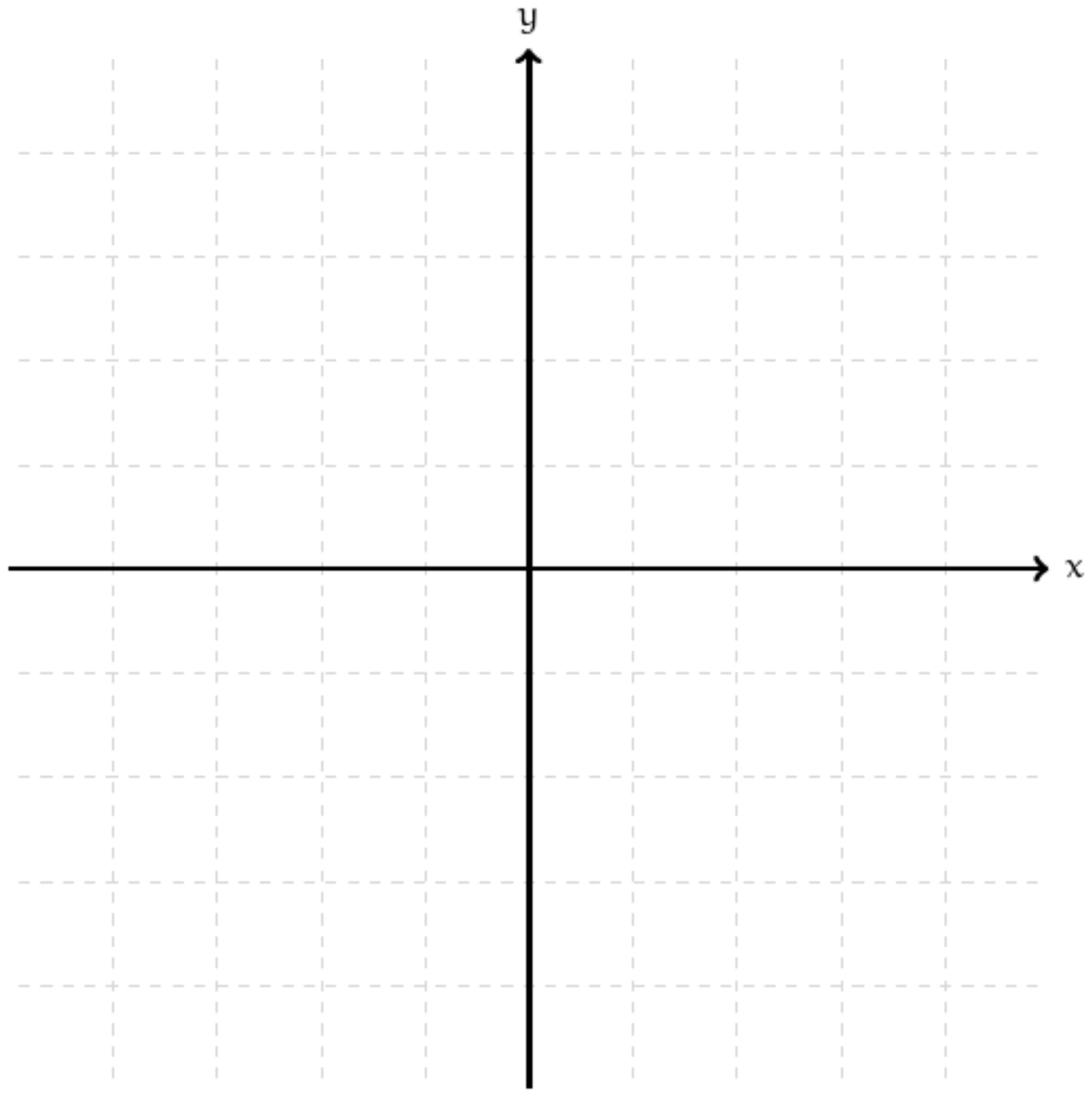
Question



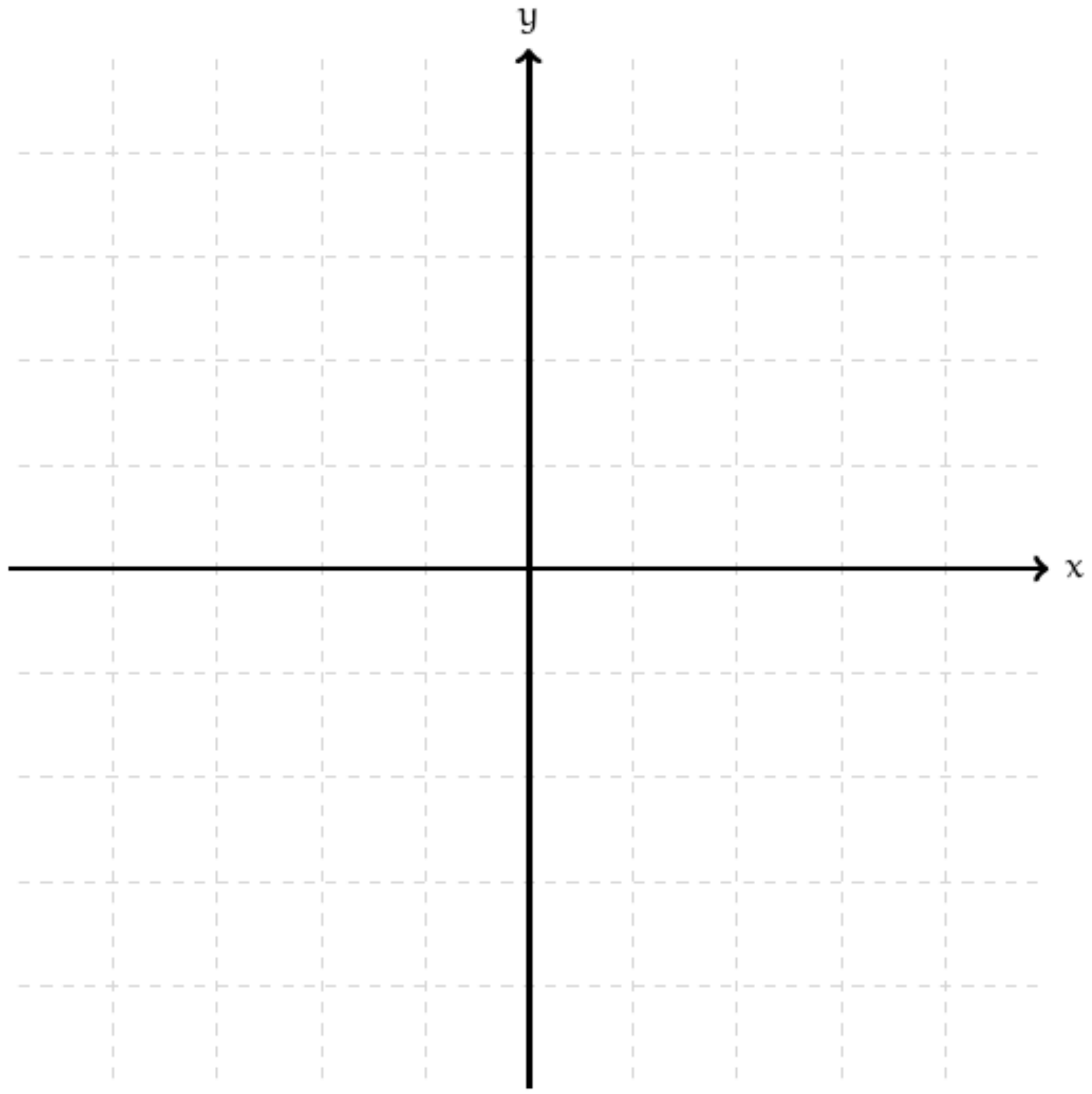
$$\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\mathbf{x}$$

Draw how this matrix transforms points. What kind of transformation does it represent?

Answer: Reflection



$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{x}$$



Motivating Questions

What kind of functions can we define in this way?

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Linear Transformations

Recall: Algebraic Properties

Matrix–vector multiplication satisfies the following two properties:

1. $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$ (additivity)
2. $A(c\mathbf{v}) = c(A\mathbf{v})$ (homogeneity)

Question

Verify the following.

$$\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\left(\begin{bmatrix}2\\3\end{bmatrix}+\begin{bmatrix}1\\1\end{bmatrix}\right)=\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix}+\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}$$

$$\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\left(2\begin{bmatrix}2\\3\end{bmatrix}\right)=2\left(\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix}\right)$$

Answer

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) =$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} =$$

Answer

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \left(2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right) =$$

Linear Transformations

Definition. A transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is ***linear*** if it satisfies the following two properties.

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ (additivity)
2. $T(c\mathbf{v}) = cT(\mathbf{v})$ (homogeneity)

Linear Transformations

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1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ (additivity)
2. $T(c\mathbf{v}) = cT(\mathbf{v})$ (homogeneity)

Matrix transformations are linear transformations.

Example: Identity

$$T(\mathbf{v}) = \mathbf{v}$$

$$T(\vec{v} + \vec{r}) = \vec{u} + \vec{r} = T(\vec{u}) + T(\vec{r})$$

$$T(c\vec{r}) = c\vec{r} = cT(\vec{r})$$

Example: Zero

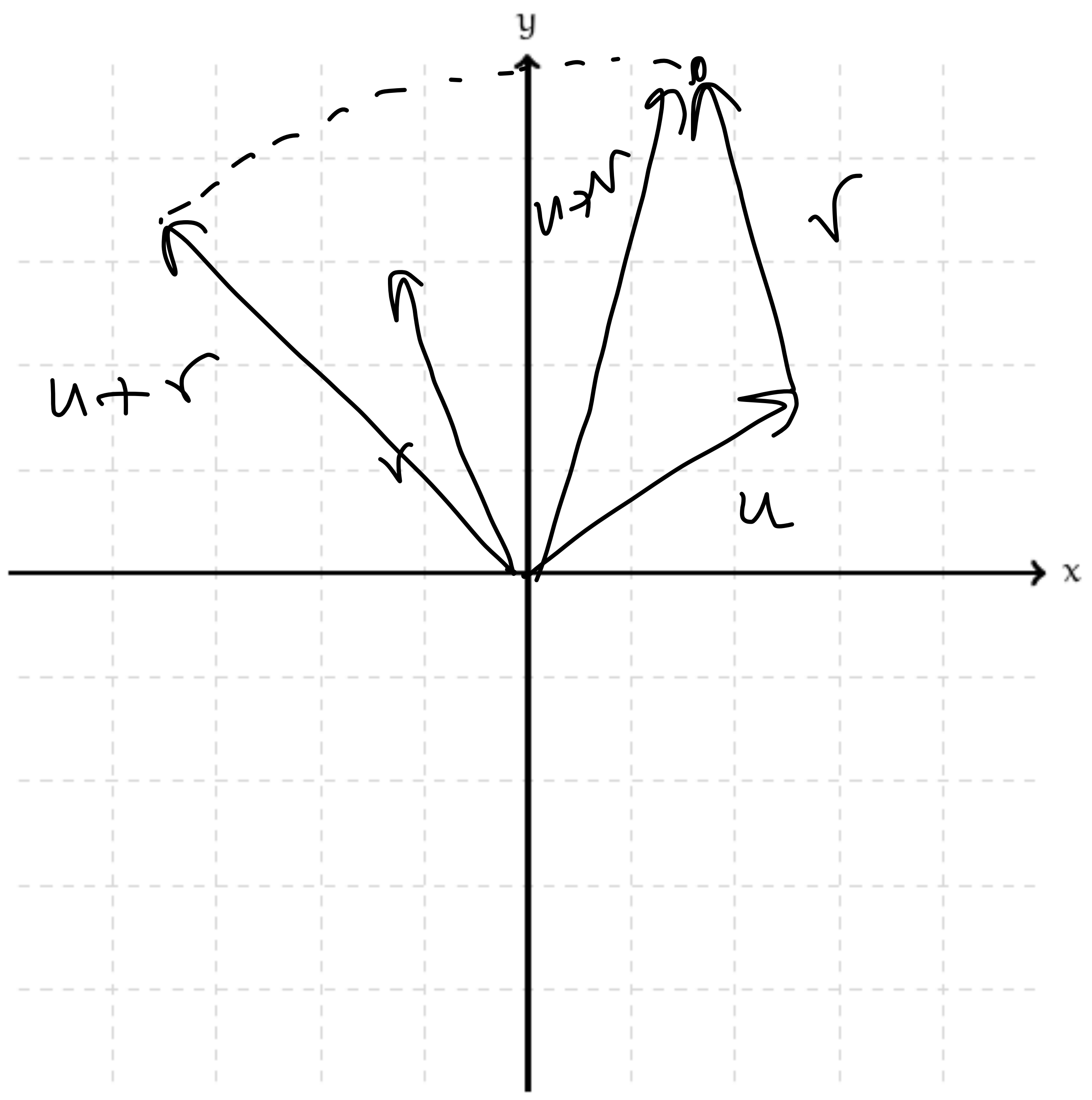
$$T(\mathbf{v}) = \mathbf{0}$$

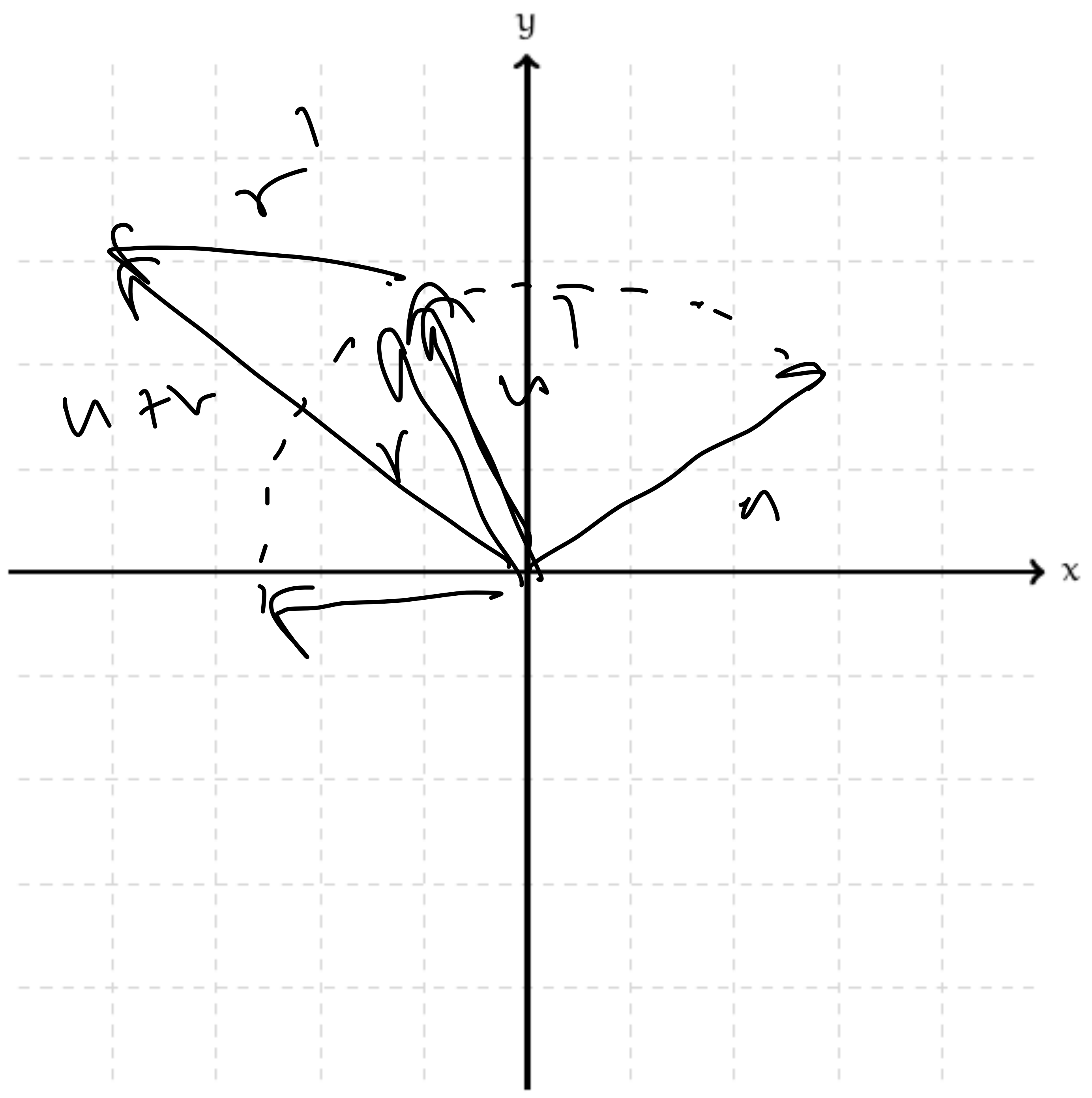
$$T(\vec{u} + \vec{r}) = \vec{0} = \vec{0} + \vec{0} = T(\vec{u}) + T(\vec{v})$$

$$T(c\vec{r}) = \vec{0} = c\vec{0} = cT(\vec{r})$$

Example: Rotation

We'll see this on Thursday, but we can reason about it geometrically for now.





Example: Indefinite Integrals

$$T(f) = \int f(x)dx$$

Disclaimer: Advanced Material

$$T(f+g) = \int (f+g)(x)dx = \int f(x) + g(x)dx = \int f(x)dx + \int g(x)dx = T(f) + T(g)$$

$$T(cf) = \int (cf)(x)dx = \int cf(x)dx = c\int f(x)dx = cT(f)$$

the same goes for derivatives
(how are functions vectors???)

Example: Expectation

$$T(X) = \mathbb{E}[X]$$

Disclaimer:
Advanced
Material

This is exactly linearity of expectation.

(how are random variables vectors???)

Non-Example: Squares

$$T(x) = x^2$$

Note that $T: \mathbb{R}^1 \to \mathbb{R}^1$

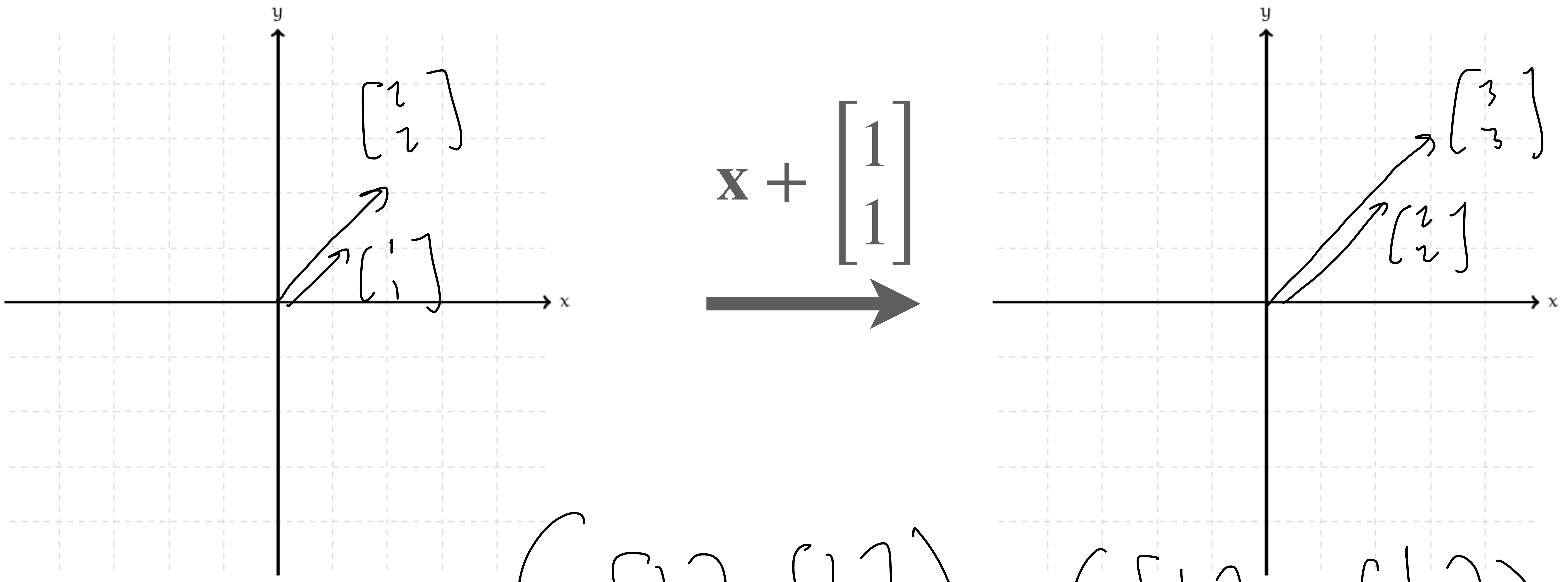
$$T(1+1) = T(2) = 4$$

$$\neq$$

$$T(1) + T(1) = 2$$

Non-Example: Translation

$$\left(\begin{bmatrix}1\\1\end{bmatrix}+\begin{bmatrix}1\\1\end{bmatrix}\right)+\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}3\\3\end{bmatrix}$$



$$\left(\begin{bmatrix}1\\1\end{bmatrix}+\begin{bmatrix}1\\1\end{bmatrix}\right)+\left(\begin{bmatrix}1\\1\end{bmatrix}+\begin{bmatrix}1\\1\end{bmatrix}\right)=\begin{bmatrix}4\\4\end{bmatrix}$$

Question

Show that $T(\mathbf{v}) = 5\mathbf{v}$ *is a linear transformation.*

Show that $T(x) = e^x$ *is not a linear transformation.*

Answer

$$T(\mathbf{v}) = 5\mathbf{v}$$

Answer

$$T(x) = e^x$$

Properties of Linear Transformations

The Zero Vector

$$T(\mathbf{0}) = \text{ ???}$$

The Zero Vector

$$T(\mathbf{0}) = \mathbf{0}$$

The Zero Vector

$$T(\mathbf{0}) = \mathbf{0}$$

The zero vector is *fixed* by linear transformations.

It can't move anywhere.

The Zero Vector

$$T(\mathbf{0}) = \mathbf{0}$$

Note: These may be different dimensions!

The zero vector is *fixed* by linear transformations.

It can't move anywhere.

Verification

```
any matrix transformation:

rotation:

translation (non-example):
```

A Single Condition

$$T(a\mathbf{v} + b\mathbf{u}) = aT(\mathbf{v}) + bT(\mathbf{u})$$

A Single Condition

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We can combine our linearity conditions:

A Single Condition

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We can combine our linearity conditions:

$T(a\mathbf{v} + b\mathbf{u})$

A Single Condition

$$T(a\mathbf{v} + b\mathbf{u}) = aT(\mathbf{v}) + bT(\mathbf{u})$$

We can combine our linearity conditions:

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A Single Condition

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$= aT(\mathbf{v}) + bT(\mathbf{u})$ `(homogeneity for each term)`

A Single Condition

Theorem. A transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is linear if and only if for any vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^m$ and any real numbers a and b,

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It's often easiest to show this single condition.

Linear Combinations

$$T(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_n\mathbf{v}_n) = a_1T(\mathbf{v}_1) + a_2T(\mathbf{v}_2) + \ldots + a_nT(\mathbf{v}_n)$$

We can generalize this condition to any linear combination.

Linear Combinations

$$T\left(\sum_{i=1}^{n} a_i \mathbf{v}_i\right) = \sum_{i=1}^{n} a_i T(\mathbf{v}_i)$$

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We can generalize this condition to any linear combination.

This is the most useful form.

Application: Unit Cost Matrices

A Question for a Business Student

Suppose you have a company that produces two products B and C.

For each product you know how much you spend per dollar on **material** (M), **labor** (L) and **overhead** (O).

$$\begin{array}{cc} B & C \end{array}$$

$$\begin{bmatrix} .45 & .40 \\ .25 & .30 \\ .15 & .15 \end{bmatrix} \begin{array}{c} M \\ L \\ O \end{array}$$

A Question for a Business Student

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B	C	
.45	.40	M
.25	.30	L
.15	.15	O

A Question for a Business Student

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How much are you spending, in total, on each cost, given that you made s_1 dollars worth of B and s_2 dollars worth of C?

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How much are you spending, in total, on each cost, given that you made s_1 dollars worth of B and s_2 dollars worth of C?

Solution. Use matrix transformations.

As a Matrix Transformation

$$T(\mathbf{x}) = \begin{bmatrix} 0.45 & 0.40 \\ 0.25 & 0.30 \\ 0.15 & 0.25 \end{bmatrix} \mathbf{x}$$

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This is much more valuable if we had *a lot* of products and a complex collection of costs.

Moral: Data Manipulation

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We can manipulate data (linearly) via linear transformations (which we will see, means via matrix multiplication).

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We can write down a *single* matrix which we can multiply every time.

This is a very powerful *algorithmic* idea.

Summary

Matrices can be viewed as linear transformations.

Matrix transformations change the "shape" of points sets.

Linear transformations behave well with respect to linear combinations.