The Matrix of a Linear Transformation

Geometric Algorithms

Lecture 8

CAS CS 132

Objectives

1. Recap some of the previous lectures material
2. See the general properties of linear transformations
3. Show that matrix transformations and linear transformations are really the same thing
4. See more the geometry of linear transformations
5. Relate the properties of matrix equations to properties of linear transformations

Keywords

matrix of a linear transformation

standard basis vectors (standard coordinate vectors)

2D linear transformations

the unit square

one-to-one

onto

Recap

Recall: Matrices as Transformations

Matrices allow us to *transform* vectors.

The transformed vector lies in the span of its columns.

$$\mathbf{x} \mapsto A\mathbf{x}$$

map a vector $\mathbf{x}$ to the vector $A\mathbf{v}$

Recall: Transformation of a Matrix

The ***transformation of a*** $(m \times n)$ ***matrix*** A is the function $T: \mathbb{R}^n \to \mathbb{R}^m$ such that

$$T(\mathbf{v}) = A\mathbf{v}$$

given $\mathbf{v}$, return A multiplied by $\mathbf{v}$

e.g. $T(\mathbf{v}) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{v}$

Recall: Motivating Questions

What kind of functions can we define in this way?

How do we interpret what the transformation does to a set of vectors?

How does this relate back to matrix equations?

Recall: Linear Transformations

Definition. A transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is ***linear*** if it satisfies the following two properties.

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ (additivity)
2. $T(c\mathbf{v}) = cT(\mathbf{v})$ (homogeneity)

Recall: Linear Transformations

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Matrix transformations are linear transformations.

Recall: Examples

Examples of Linear Transformations:

» identity, constant zero
» dilation, contraction, shearing, reflection
» rotation (more on that today)
» *(advanced) integrals, derivatives, expectation*

Non-Examples of Linear Transformations:

» squares, translation

Example

$$T\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = \begin{bmatrix} v_1 + v_2 \\ v_3 \\ v_2 - v_3 \\ v_1 \end{bmatrix}$$

domain: $\mathbb{R}^3$ codomain: $\mathbb{R}^4$

want to show:

verify homogeneity: $T\left(c\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = cT\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right)$

$$cT\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = c\begin{bmatrix} v_1 + v_2 \\ v_3 \\ v_2 - v_3 \\ v_1 \end{bmatrix}$$

$$T\left(c\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = T\left(\begin{bmatrix} cv_1 \\ cv_2 \\ cv_3 \end{bmatrix}\right) = \begin{bmatrix} cv_1 + cv_2 \\ cv_3 \\ cv_2 - cv_3 \\ cv_1 \end{bmatrix} = \begin{bmatrix} c(v_1 + v_2) \\ cv_3 \\ c(v_2 - v_3) \\ cv_1 \end{bmatrix}$$

Question

Show that $T(\mathbf{v}) = 5\mathbf{v}$ *is a linear transformation.*

Show that $T(x) = 2^x$ *is not a linear transformation.*

$\mathbb{R}^1 \to \mathbb{R}^1$

Answer

$$T(\mathbf{v}) = 5\mathbf{v}$$

$$T(\vec{v}+\vec{w}) = 5(\vec{v}+\vec{w}) =$$

$$T(\vec{v}) + T(\vec{w}) = 5\vec{v} + 5\vec{w}$$

$$5(2\vec{v}) =$$

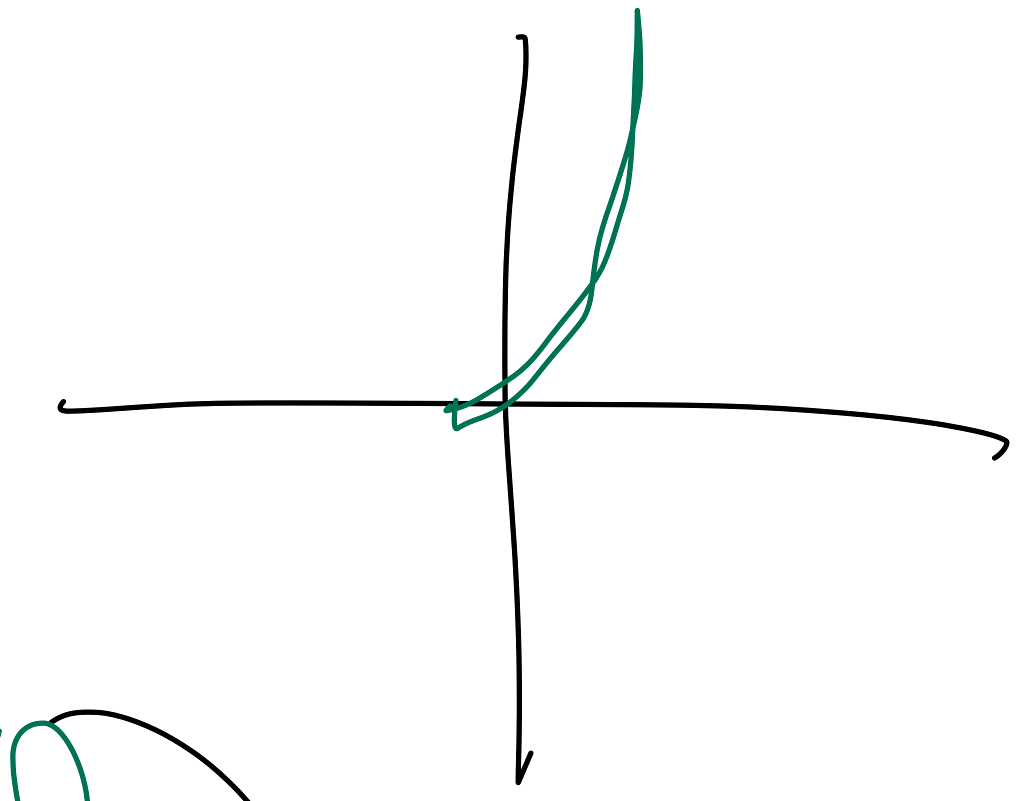
$$10\vec{v} = 2(5\vec{v})$$

$$5(c\vec{v}) = T(c\vec{v})$$

$$(5c)\vec{v} = (c5)\vec{v} = c(5\vec{v})$$

Answer

$$T(x) = 2^x$$



$$T(0+0) \neq T(0) + T(0)$$

$$T(0+0) = T(0) = 1$$

$$T(0) + T(0) = 1 + 1 = 2$$

$$1 \neq 2$$

Properties of Linear Transformations

The Zero Vector

$$T(\mathbf{0}) = \text{???}$$

The Zero Vector

$$T(\mathbf{0}) = \mathbf{0}$$

The Zero Vector

$$\vec{0} \parallel$$

$$T(\mathbf{0}) = \mathbf{0}$$

$$T(\vec{v}) - T(\vec{v}) = T(\vec{v} + (-\vec{v})) = T(\vec{0})$$

$$T(\vec{0}) = T(0\vec{v}) = 0\,T(\vec{v}) = \vec{0}$$

The zero vector is *fixed* by linear transformations.

It can't move anywhere.

non-linear: $T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

The Zero Vector

$$T(\mathbf{0}) = \mathbf{0}$$

Note: These may be different dimensions!

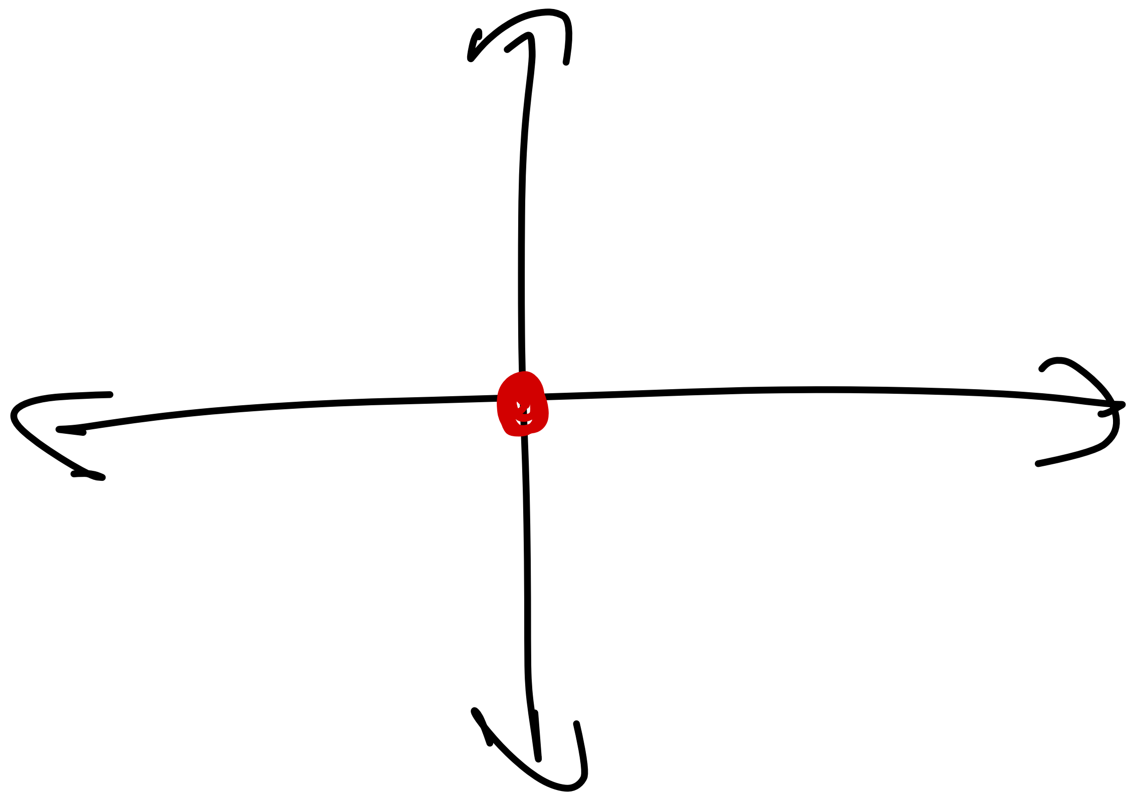
The zero vector is *fixed* by linear transformations.

It can't move anywhere.

Verification

any matrix transformation: $A\vec{0} = \vec{0}$

rotation about the origin:



translation *(non-example)*: $T(\vec{v}) = \vec{v} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad T(\vec{0}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

A Single Condition

$$T(a\mathbf{v} + b\mathbf{u}) = aT(\mathbf{v}) + bT(\mathbf{u})$$

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$= T(a\mathbf{v}) + T(b\mathbf{u})$ (by additivity)

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$T(a\mathbf{v} + b\mathbf{u})$

$= T(a\mathbf{v}) + T(b\mathbf{u})$ (by additivity)

$= aT(\mathbf{v}) + bT(\mathbf{u})$ (by homogeneity for each term)

A Single Condition

Theorem. A transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is linear if and only if for any vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^m$ and any real numbers a and b,

$$T(a\mathbf{u} + b\mathbf{v}) = aT(\mathbf{u}) + bT(\mathbf{v})$$

verify:

$$T(c\vec{u}) = T(c\vec{u} + 0\vec{v}) = cT(\vec{u}) + 0\,T(\vec{v})$$

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Theorem. A transformation $T: \mathbb{R}^m \to \mathbb{R}^n$ is linear if and only if for any vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^m$ and any real numbers a and b,

$$T(a\mathbf{u} + b\mathbf{v}) = aT(\mathbf{u}) + bT(\mathbf{v})$$

verify:

It's often easiest to show this single condition.

Question

$$T(a\mathbf{u} + b\mathbf{v}) = aT(\mathbf{u}) + bT(\mathbf{v})$$

Show that $T(\mathbf{v}) = 5\mathbf{v}$ is linear using the result from the previous slide.

Answer

$$T(\mathbf{v}) = 5\mathbf{v}$$

$$T(a\vec{r} + b\vec{u}) = 5(a\vec{r} + b\vec{u})$$

$$= 5a\vec{r} + 5b\vec{u}$$

$$= a(5\vec{r}) + b(5\vec{u})$$

$$= aT(\vec{r}) + bT(\vec{u})$$

Linear Combinations

$$T(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_n\mathbf{v}_n) = a_1T(\mathbf{v}_1) + a_2T(\mathbf{v}_2) + \ldots + a_nT(\mathbf{v}_n)$$

We can generalize this condition to any linear combination.

Linear Combinations

$$T\left(\sum_{i=1}^{n} a_i \mathbf{v}_i\right) = \sum_{i=1}^{n} a_i T(\mathbf{v}_i)$$

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Linear Combinations

$$T\left(\sum_{i=1}^{n} a_i \mathbf{v}_i\right) = \sum_{i=1}^{n} a_i T(\mathbf{v}_i)$$

We can generalize this condition to any linear combination.

This is the most useful form.

Application: Unit Cost Matrices

A Question for a Business Student

Suppose you have a company that produces two products B and C.

For each product you know how much you spend per dollar on **material** (M), **labor** (L) and **overhead** (O).

$$
\begin{array}{cc} B & C \end{array}
$$

$$
\begin{bmatrix} .45 & .40 \\ .25 & .30 \\ .15 & .15 \end{bmatrix} \begin{array}{c} M \\ L \\ O \end{array}
$$

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How much are you spending, in total on each cost, given that you made s_1 dollars worth of B and s_2 dollars worth of C?

A Question for a Business Student

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How much are you spending, in total on each cost, given that you made s_1 dollars worth of B and s_2 dollars worth of C?

Solution. Use matrix transformations.

As a Matrix Transformation

$$T(\mathbf{x}) = \begin{bmatrix} 0.45 & 0.40 \\ 0.25 & 0.30 \\ 0.15 & 0.25 \end{bmatrix} \mathbf{x}$$

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$$T\left(\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \right) = s_1 \begin{bmatrix} 0.45 \\ 0.25 \\ 0.15 \end{bmatrix} + s_2 \begin{bmatrix} 0.40 \\ 0.30 \\ 0.15 \end{bmatrix} = \begin{bmatrix} \texttt{total material cost} \\ \texttt{total labor cost} \\ \texttt{total overhead cost} \end{bmatrix}$$

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This is much more valuable if we have *a lot* of products and a complex collection of costs.

Moral: Data Manipulation

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This is a very powerful *algorithmic* idea.

(moving on)

Motivating Question

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Matrix of a Linear Transformation

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Theorem. A transformation T is linear if and only if there is a matrix whose corresponding transformation is T (which "implements" T).

Linear transformations are **exactly** matrix transformations.

A Fundamental Concern

Given a linear transformation T, how do we find the matrix A such that

$$T(\mathbf{v}) = A\mathbf{v}?$$

A Thought Experiment

$\mathbb{R}^2 \to \mathbb{R}^2$

Suppose I tell you T is a linear transformation and

$$T\left(\begin{bmatrix}1\\2\end{bmatrix}\right)=\begin{bmatrix}3\\4\end{bmatrix} \qquad T\left(\begin{bmatrix}3\\4\end{bmatrix}\right)=\begin{bmatrix}5\\6\end{bmatrix}$$

Do we know what $T\left(\begin{bmatrix}4\\6\end{bmatrix}\right)$ is?

Answer: Yes

$$T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}3\\4\end{bmatrix} \qquad T\left(\begin{bmatrix}3\\4\end{bmatrix}\right) = \begin{bmatrix}5\\6\end{bmatrix}$$

Because of additivity:

$$T\left(\begin{bmatrix}4\\6\end{bmatrix}\right) = T\left(\begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}3\\4\end{bmatrix}\right) =$$

$$T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) + T\left(\begin{bmatrix}3\\4\end{bmatrix}\right)$$

$$= \begin{bmatrix}3\\4\end{bmatrix} + \begin{bmatrix}5\\6\end{bmatrix} = \begin{bmatrix}8\\10\end{bmatrix}$$

A Thought Experiment

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What about:

$$T\left(\begin{bmatrix}2\\3\end{bmatrix}\right) = T\left(\frac{1}{2}\begin{bmatrix}4\\6\end{bmatrix}\right) = \frac{1}{2}T\left(\begin{bmatrix}4\\6\end{bmatrix}\right)$$

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = T\left(\begin{bmatrix}3\\4\end{bmatrix} - 2\begin{bmatrix}1\\2\end{bmatrix}\right) = \ldots$$

The Takeaway

$$T(\vec{b}) = T\left(\sum_{i=1}^{k} \alpha_i \vec{v}_i\right)$$

$$= \sum_{i=1}^{k} \alpha_i T(\vec{v}_i)$$

Linearity is a **very** strong restriction.

If we know the values of $T : \mathbb{R}^n \to \mathbb{R}^m$ on **any** set of vectors which spans all of $\mathbb{R}^n$, then we know T.

domain?

why?: $\vec{v}_1, \vec{v}_2, \vec{v}_k$ span $\mathbb{R}^n$

$$\vec{b} = \sum_{i=1}^{k} \alpha_i \vec{v}_i$$

Another Thought Experiment (Game)

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(you pick the $\mathbf{v}$'s, and I have to tell the truth)

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But you're only allowed to ask the question:

what is $A\mathbf{v}$?

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This is basically linear algebraic battleship.

Recall: Calculating $A\mathbf{v}$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ \vdots \\ ? \end{bmatrix}$$

$$b_1 = a_{11}v_1 + a_{12}v_2 + \ldots + a_{1n}v_n = \sum_{i=1}^{n} a_{1i}v_i$$

Recall: Matrix-Vector Multiplication

Definition. Given a $(m \times n)$ matrix A with columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$, and a vector $\mathbf{v}$ in $\mathbb{R}^n$, we define

$$A\mathbf{v} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \ldots \quad \mathbf{a}_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + \ldots v_n\mathbf{a}_n$$

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$A\mathbf{v}$ is a linear combination of the columns of A with weights given by $\mathbf{v}$

Isolating a_{11}

$$b_1 = a_{11}v_1 + a_{12}v_2 + \ldots + a_{1n}v_n = \sum_{i=1}^{n} a_{1i}v_i$$

$$\begin{bmatrix} a_{11} & \cdots \\ \vdots & \end{bmatrix} \vec{v} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 1\vec{a}_1 + 0\vec{a}_2 \cdots 0\vec{a}_n = \vec{a}_1$$

Isolating a_{11}

$$b_1 = a_{11}v_1 + a_{12}v_2 + \ldots + a_{1n}v_n = \sum_{i=1}^{n} a_{1i}v_i$$

We actually get the whole column $\mathbf{a}_1$

So its like battleship, but you get to choose one column at a time.

The Takeaway

We can learn the first column of the matrix implementing

T by looking at $T\left(\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right)$

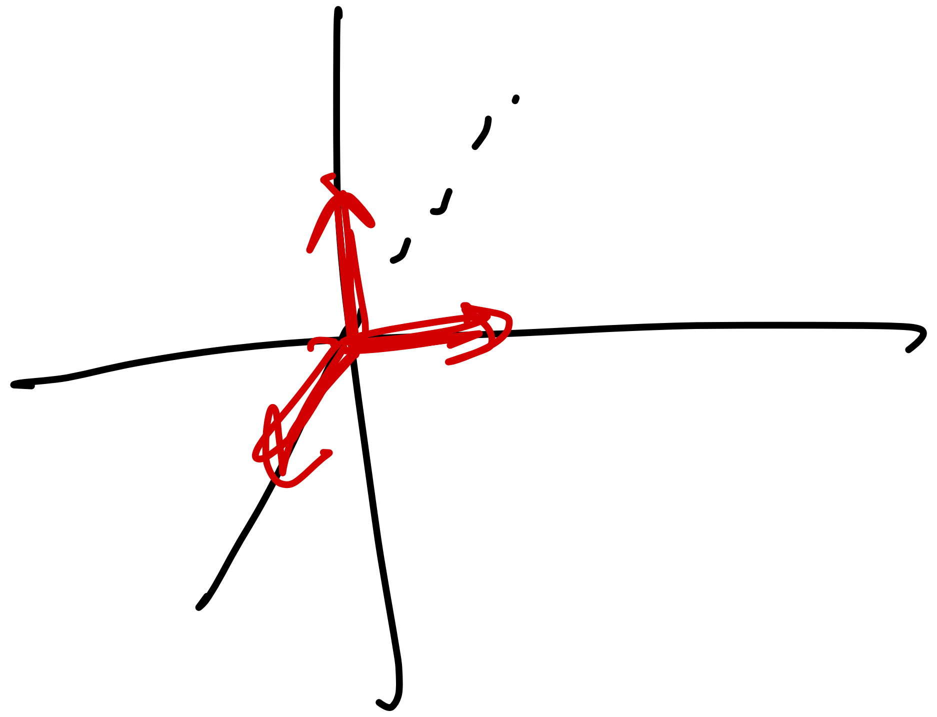
Matrix of a Linear Transformation

Standard Basis

Definition. The ***n*-dimensional standard basis vectors** (or standard coordinate vectors) are the vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ where

$n = 3$

$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$



$$\mathbf{e}_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ \vdots \\ i-1 \\ i \\ i+1 \\ \vdots \\ n-1 \\ n \end{array}$$

Standard Basis

Definition (Alternative). The n–dimensional standard basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are the columns of the $n \times n$ identity matrix.

$$I = [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \cdots \quad \mathbf{e}_n]$$

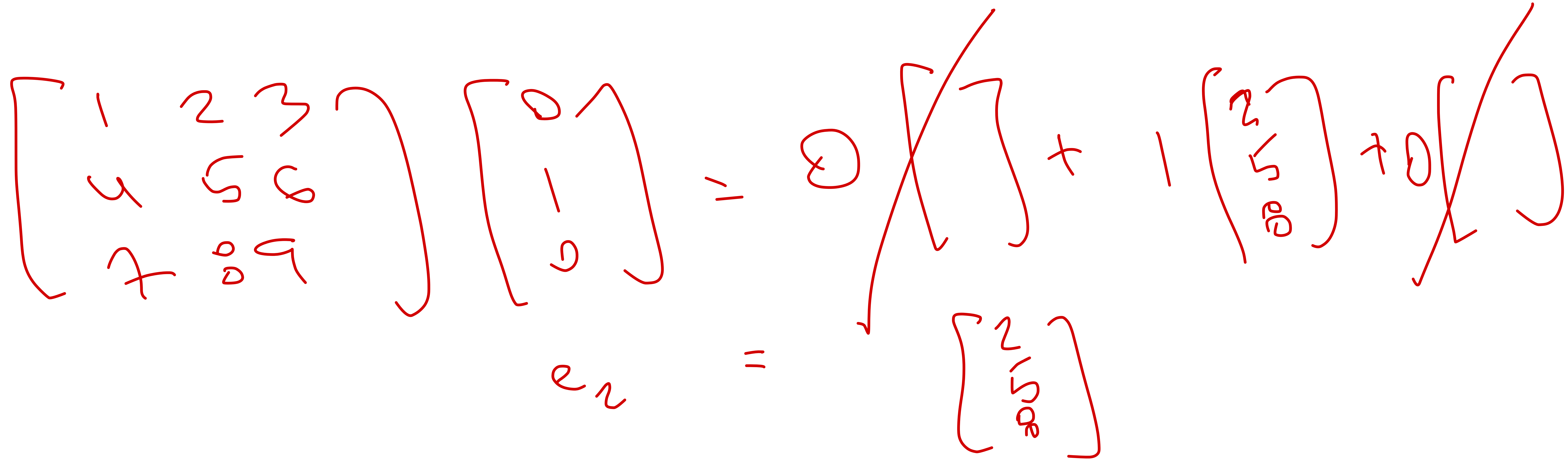
$n = 3$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Standard Basis and the Matrix Equation

The key points: $[\mathbf{a}_1 \ \ \mathbf{a}_2 \ \ \cdots \ \ \mathbf{a}_n]\,\mathbf{e}_i = \mathbf{a}_i$

The standard basis vectors gives us a way to "look into" a matrix.



Standard Basis and Vector Coordinates

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \ldots + v_n\mathbf{e}_n = I\,\vec{r}$$

Column vectors can be viewed as describing how to write a vector as a linear combination of the standard basis.

Example:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Standard Basis and Linear Transformations

Theorem. For any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, the matrix

$$A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \dots & T(\mathbf{e}_n) \end{bmatrix}$$

is the unique matrix such that $T(\mathbf{v}) = A\mathbf{v}$ for all $\mathbf{v}$ in $\mathbb{R}^n$.

More Formally

$$T(\mathbf{v}) =$$

$$= \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \dots & T(\mathbf{e}_n) \end{bmatrix} \mathbf{v}$$

How To: Matrices of Linear Transformations

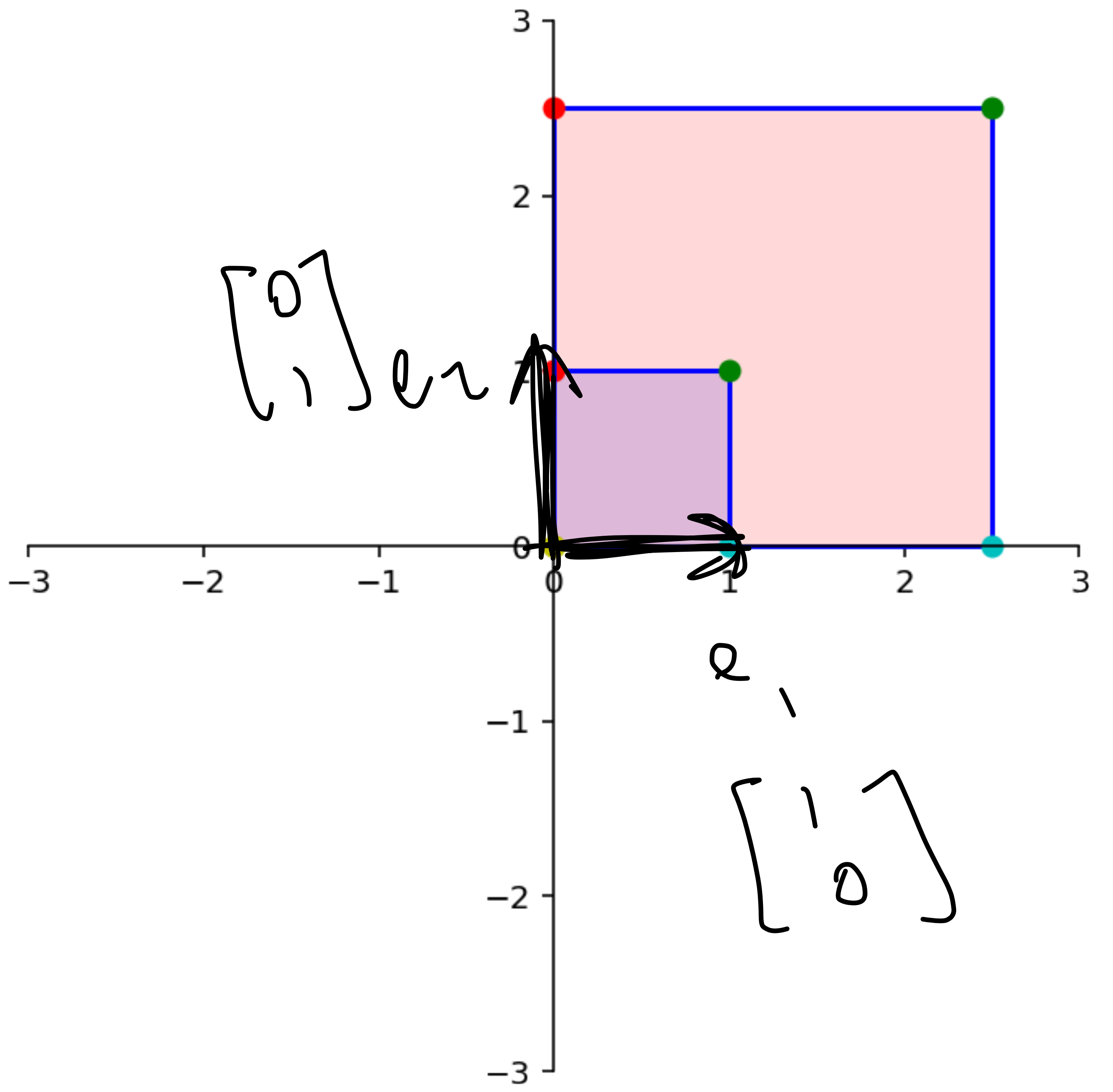
Question. Find the matrix which implements the transformation $T : \mathbb{R}^n \to \mathbb{R}^m$.

Solution. Determine the images of standard basis under T. Then write down

$$\begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \dots & T(\mathbf{e}_n) \end{bmatrix}$$

Question

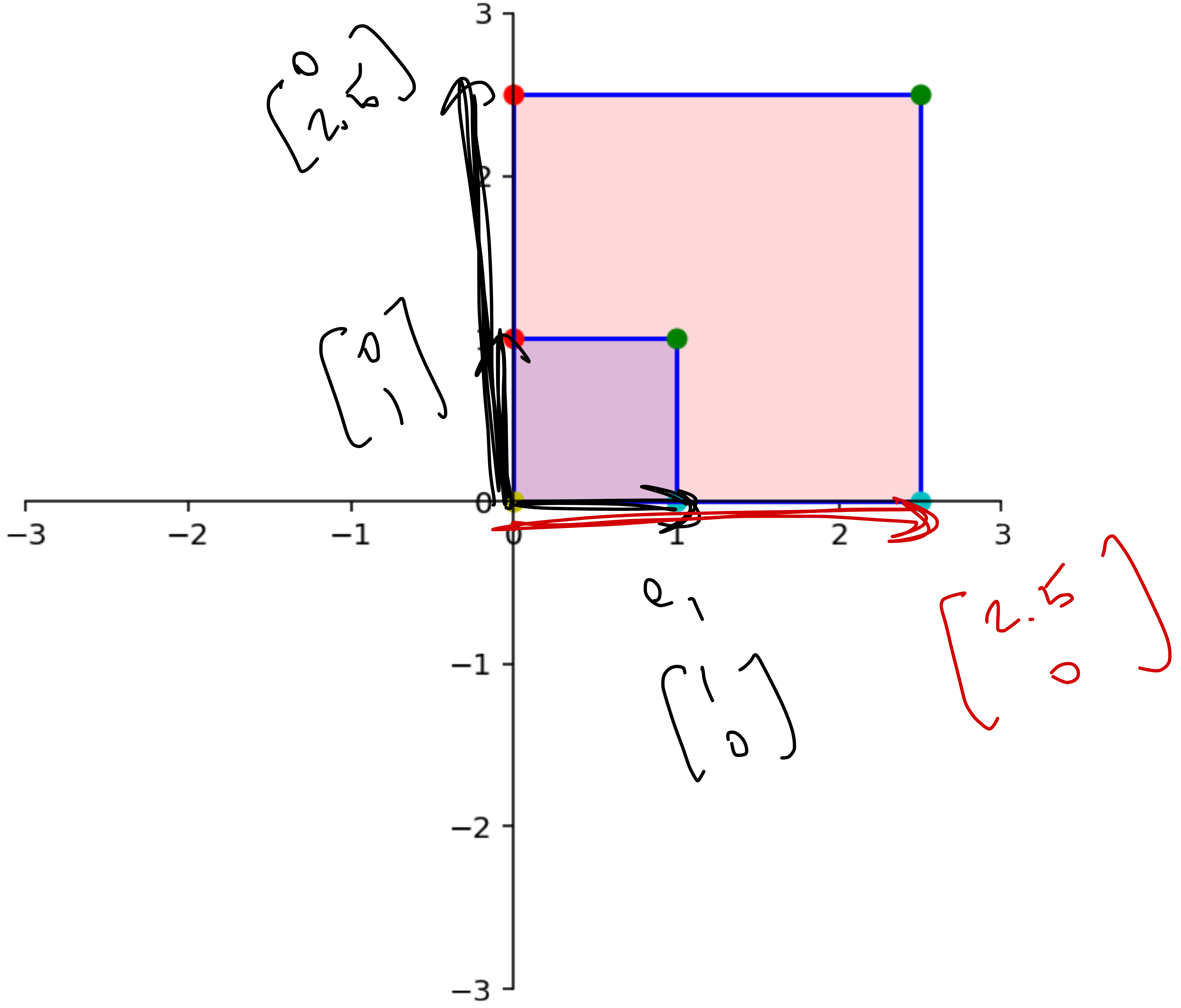
Write down the matrix implementing the following dilation, using this method.



Answer

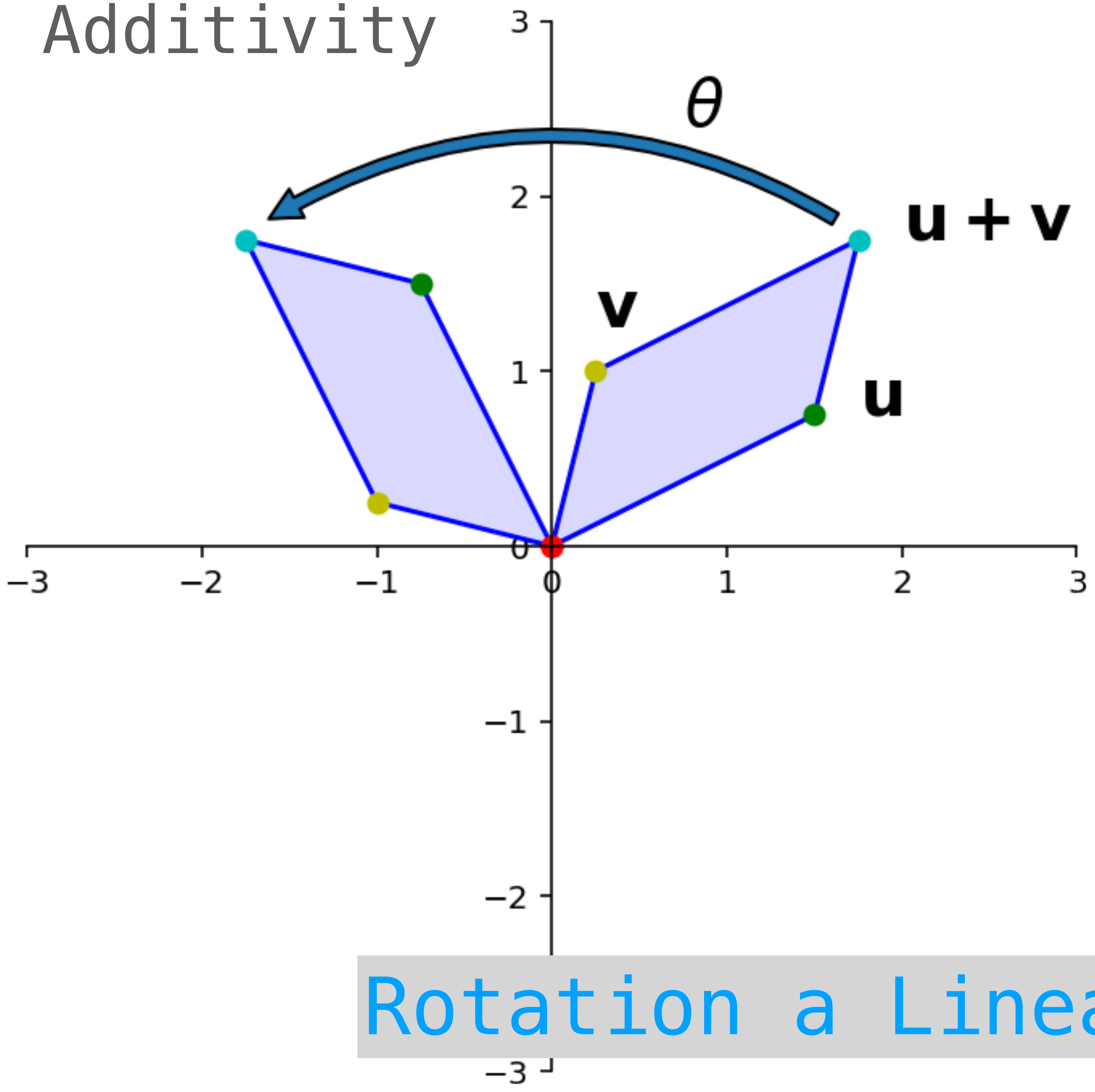
dilation :

$$\begin{bmatrix} 2.5 & 0 \\ 0 & 2.5 \end{bmatrix}$$

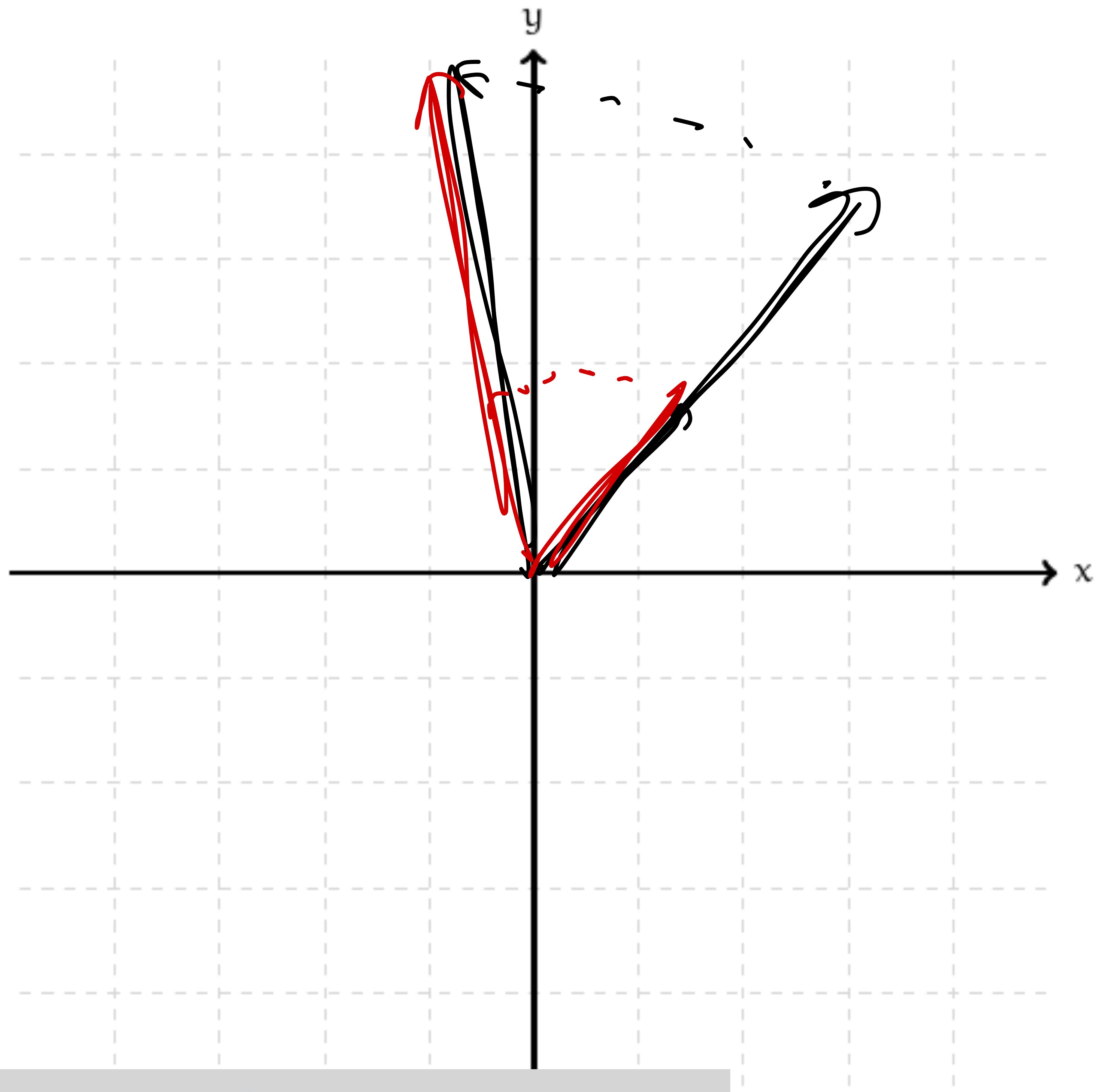


Revisiting Rotation

Additivity



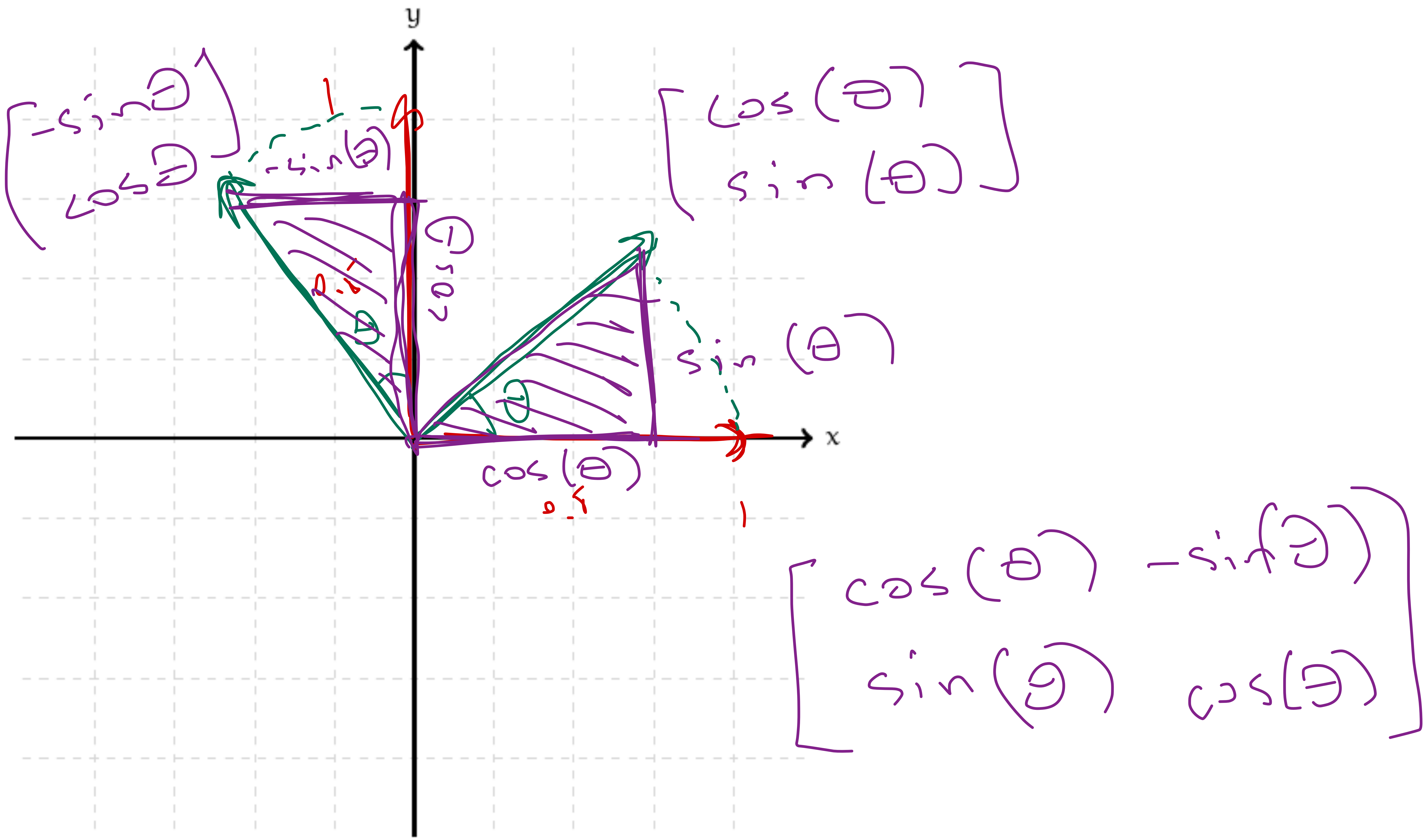
Homogeneity



Rotation a Linear Transformation

Revisiting Rotation

How does rotation affect the standard basis?



Rotation Matrix

Rotation Matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

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Note: This is rotation about the origin.

Rotation Matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Note: This is rotation about the origin.

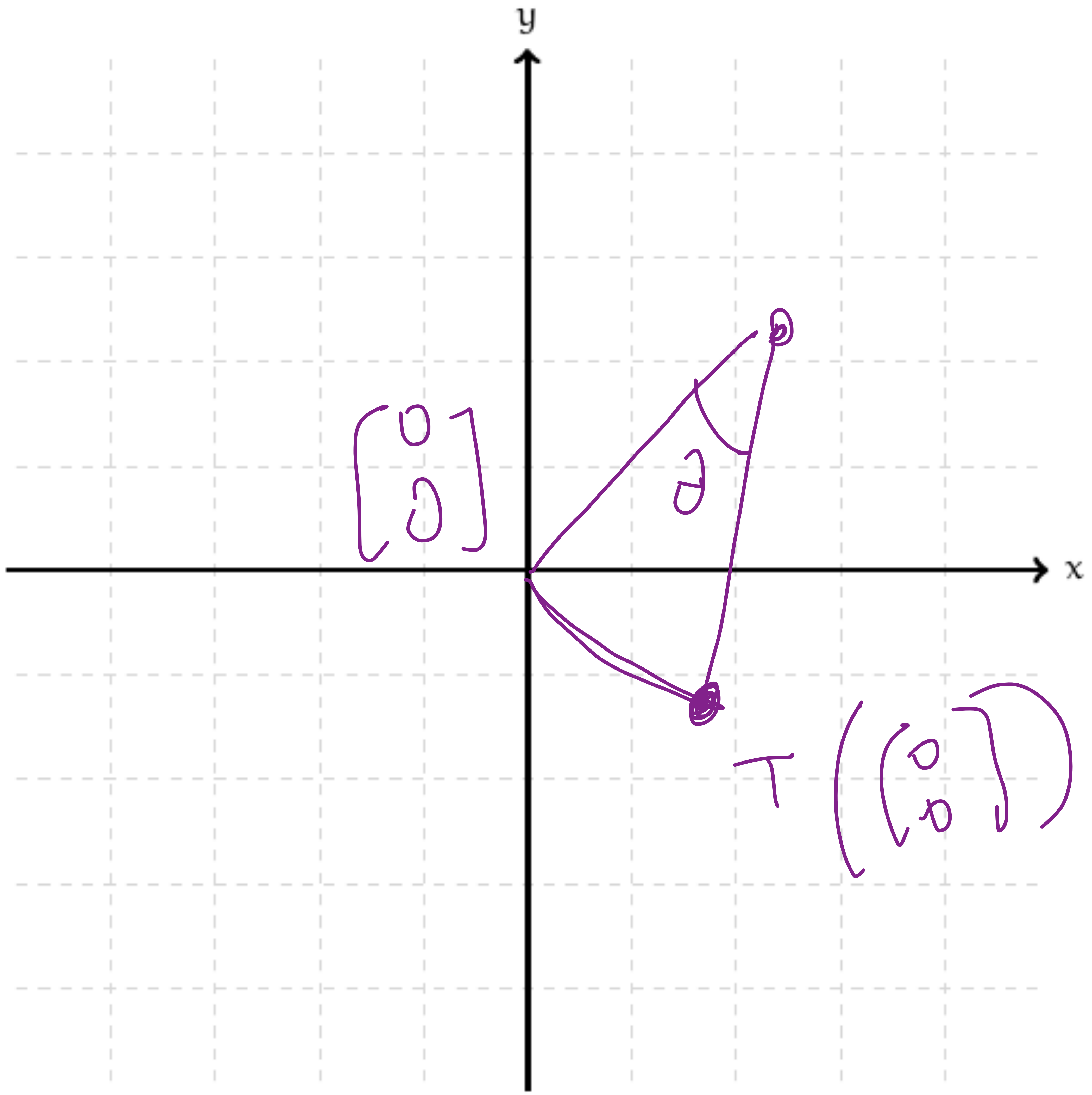
The Takeaway: We can figure out the matrices which implement complex linear transformations by understanding what they do to the standard basis.

Question (Conceptual)

Is rotation about a point other than the origin a linear transformation?

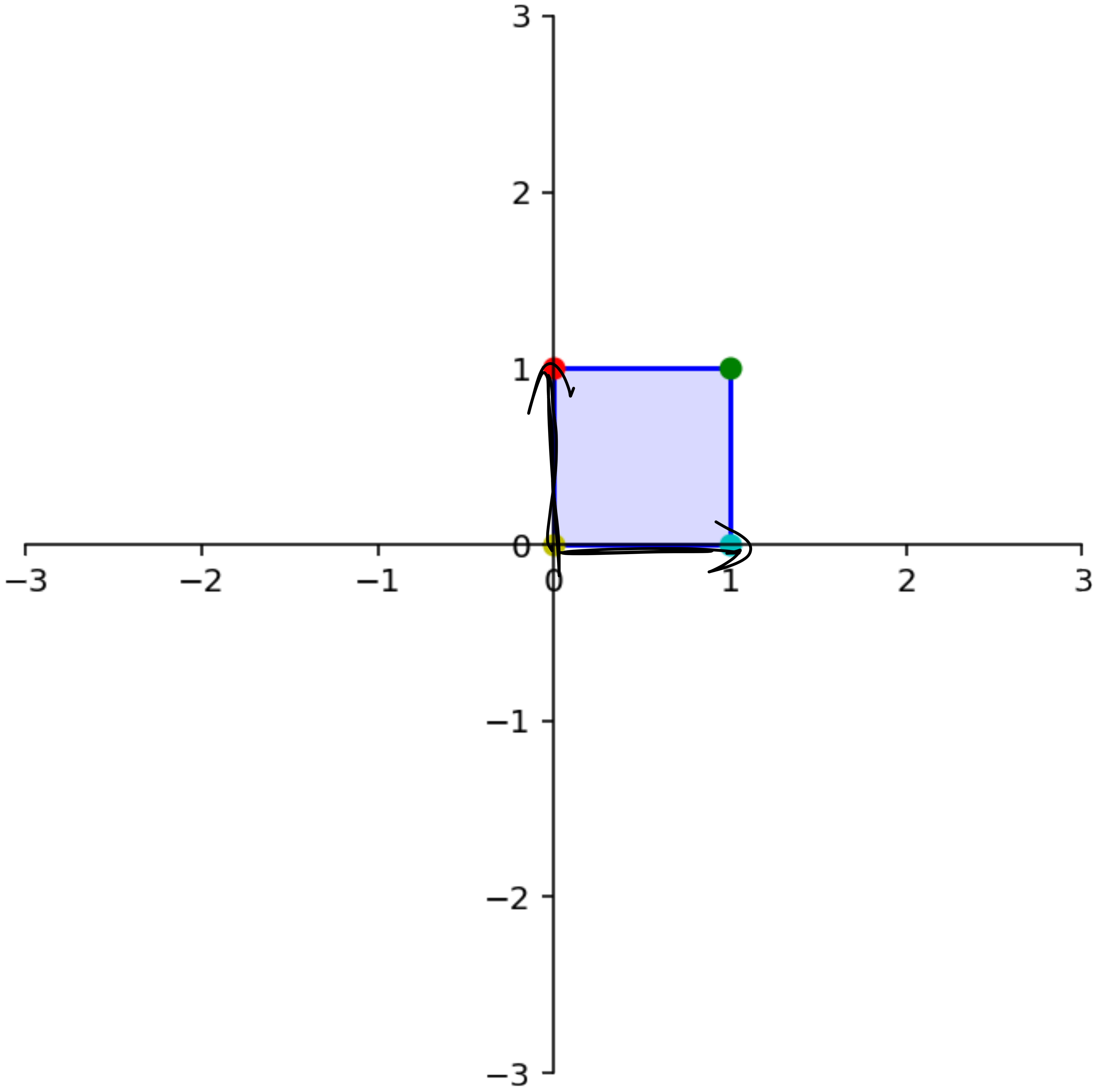
Answer: No

The origin is not fixed by this transformation.



The Unit Square

The ***unit square*** is the set of points in $\mathbb{R}^2$ enclosed by the points $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$.



How To: The Unit Square and Matrices

How To: The Unit Square and Matrices

Question. Find the matrix which implements the linear transformation which is represented geometrically in the following picture.

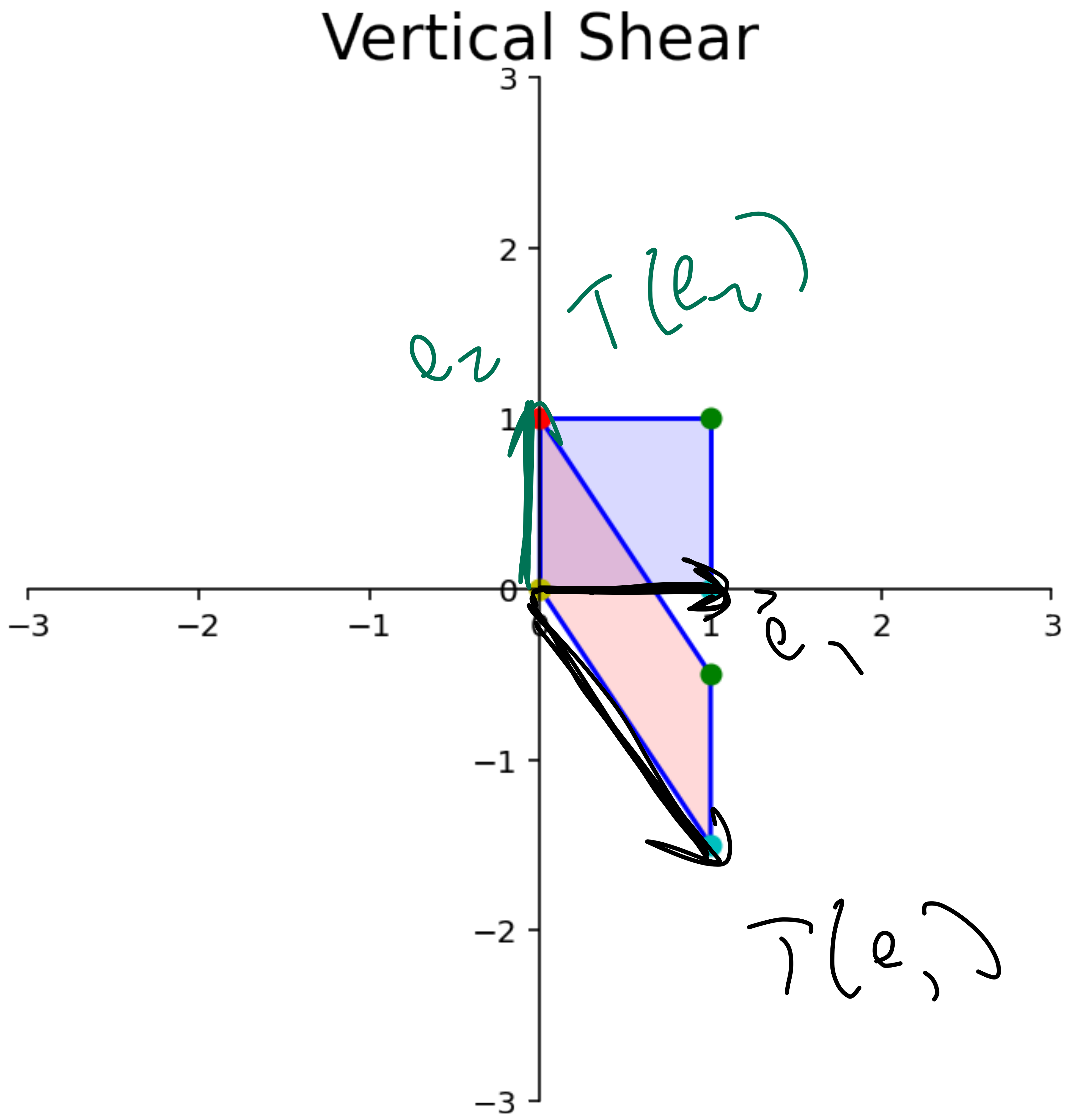
How To: The Unit Square and Matrices

Question. Find the matrix which implements the linear transformation which is represented geometrically in the following picture.

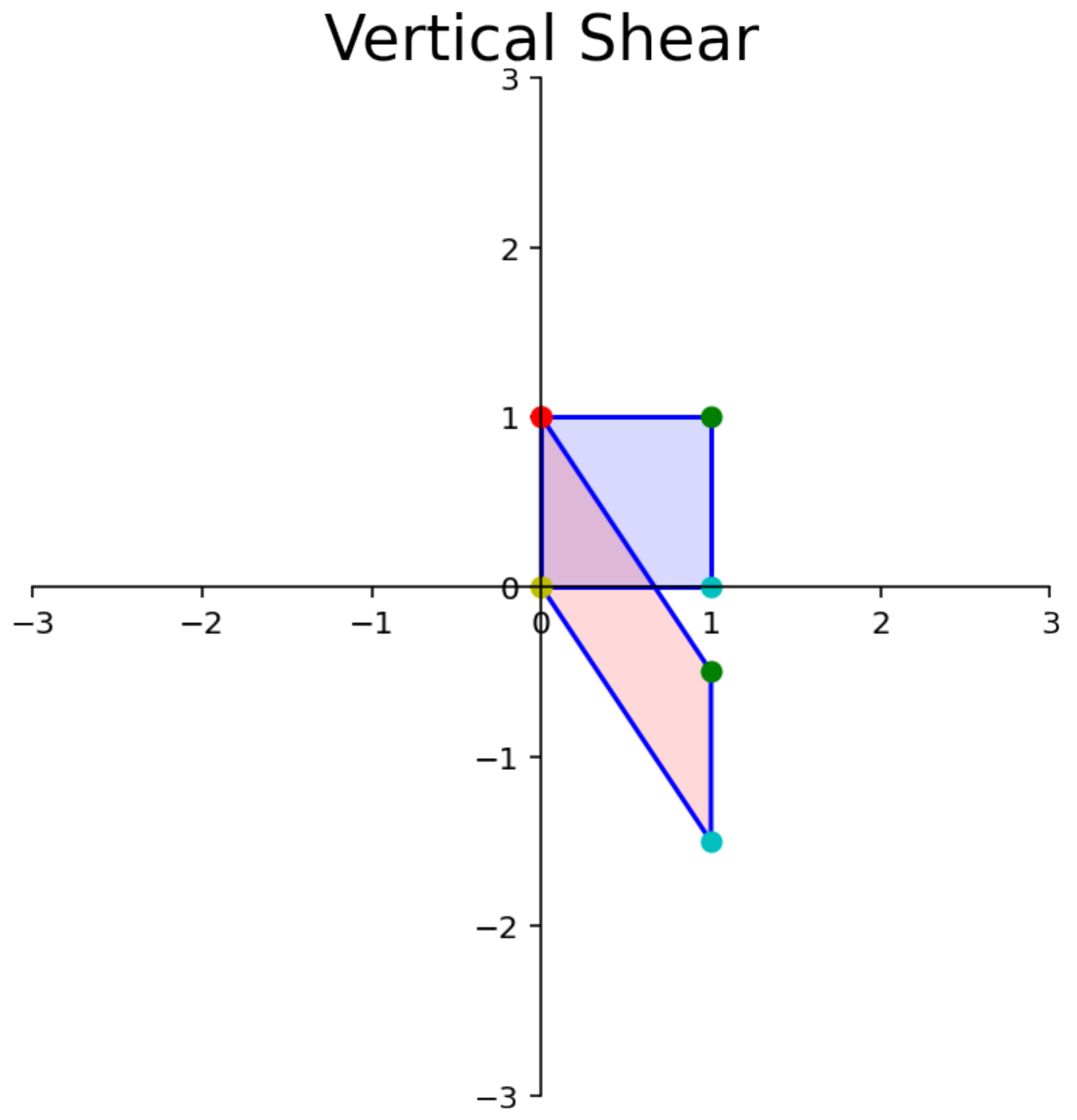
Solution. Find where the standard basis vectors go.

Question

Write down the matrix for the following shearing operation using this method.



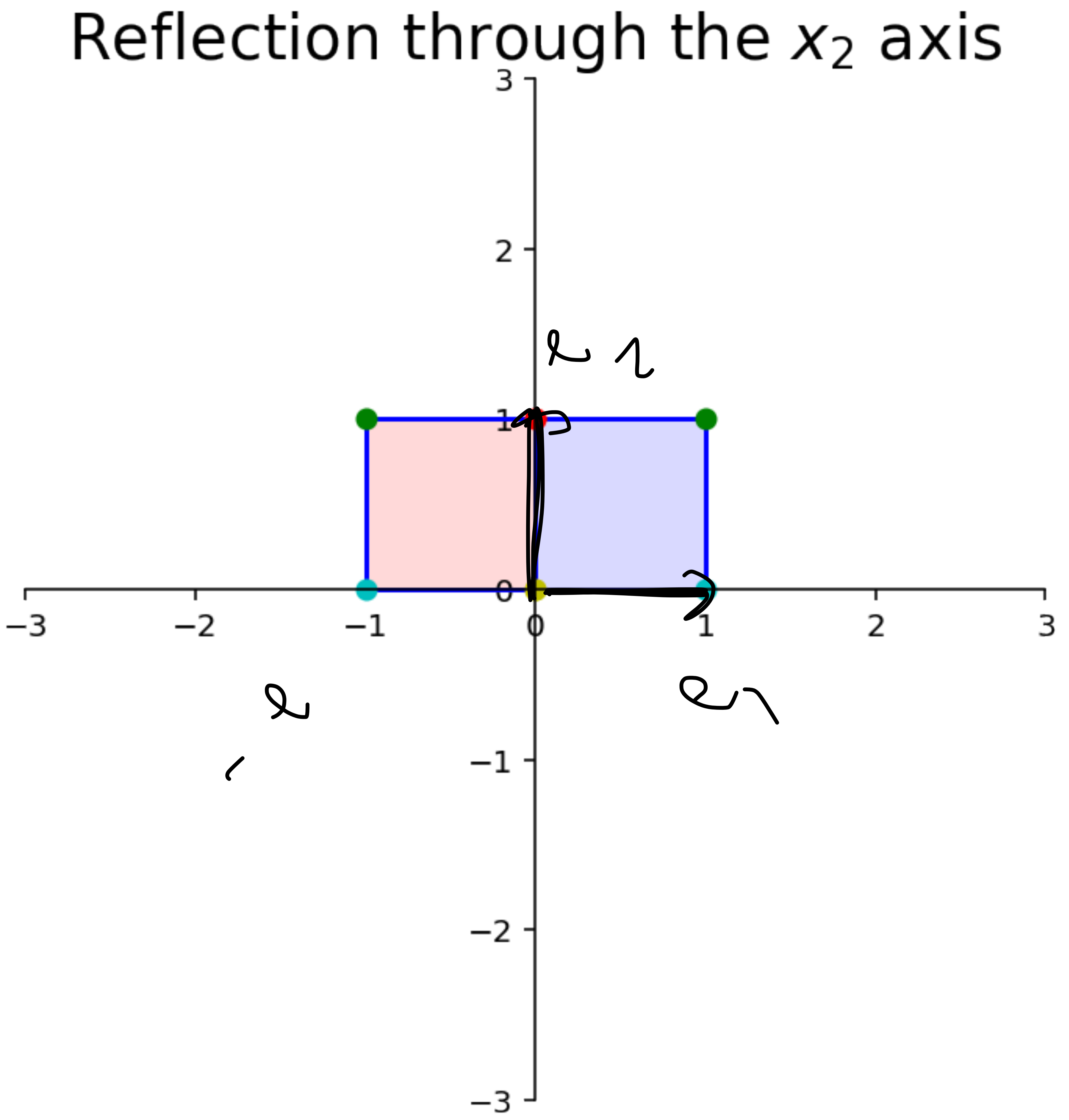
Answer



You need to **know** these matrices, but you don't need to memorize them.

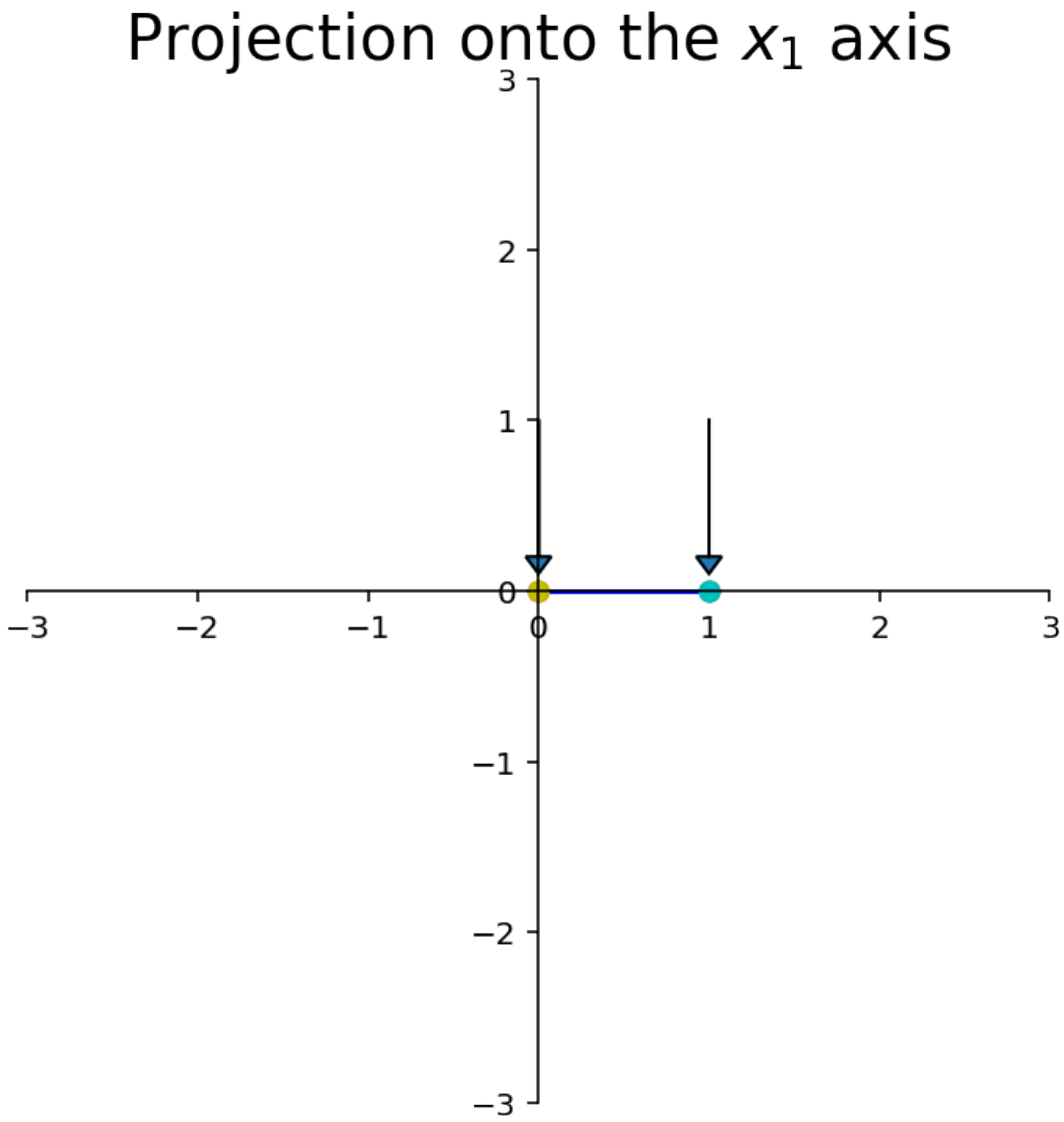
Remember: What does this matrix do to the unit square? Then build the matrix from there.

Reflection through the x_2-axis

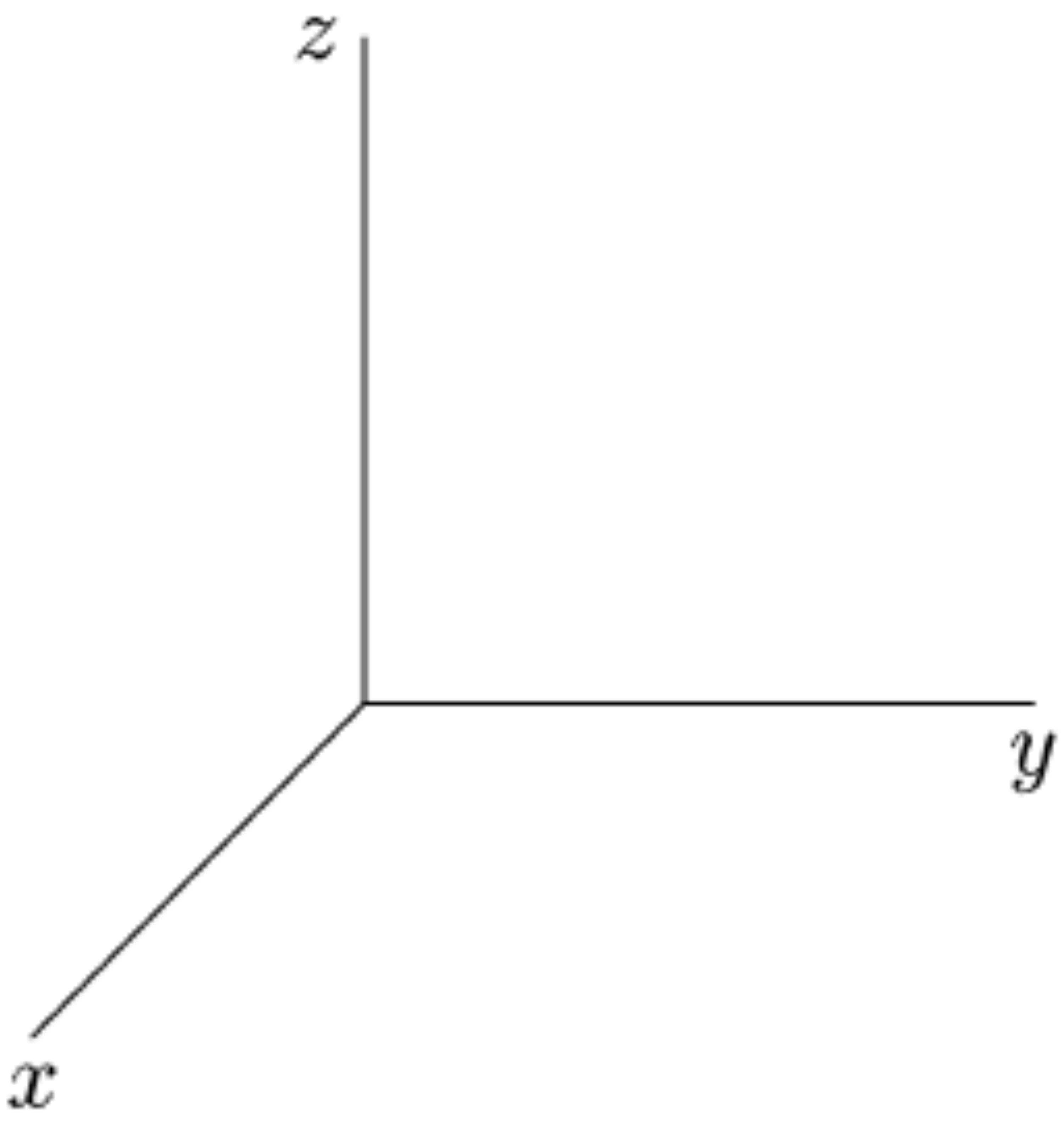


$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Projections



A 3D Example: Rotation about the x_3-Axis (z-Axis)



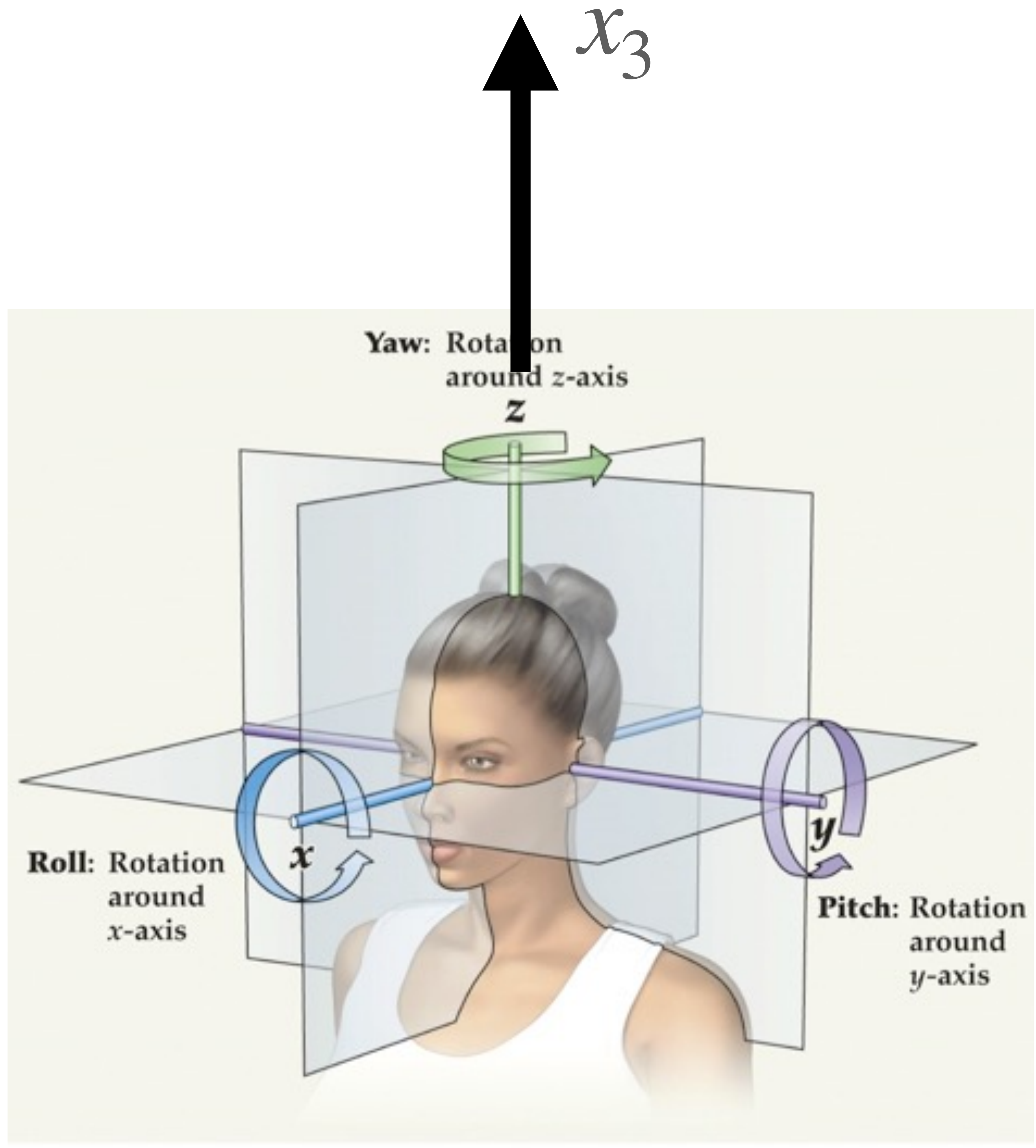


image source

List of Important 2D Linear Transformations

» dilation, contraction
» reflections
» projections
» horizontal/vertical contractions
» horizontal/vertical shearing

Look through the notes for a comprehensive collection of pictures or...

demo

One-to-One and Onto

Recall: Motivating Questions

What kind of functions can we define in this way?

How do we interpret what the transformation does to a set of vectors?

How does this relate back to matrix equations?

Recall: A New Interpretation of the Matrix Equation

$A\mathbf{x} = \mathbf{b}$?	$\equiv$	is there a vector which A transforms into $\mathbf{b}$?
Solve $A\mathbf{x} = \mathbf{b}$	$\equiv$	find a vector which A transforms into $\mathbf{b}$

Recall: A New Interpretation of the Matrix Equation

$A\mathbf{x} = \mathbf{b}$? $\equiv$ is there a vector which A transforms into $\mathbf{b}$?

Solve $A\mathbf{x} = \mathbf{b}$ $\equiv$ find a vector which A transforms into $\mathbf{b}$

What about other questions?

Other Questions Like...

Does $A\mathbf{x} = \mathbf{b}$ have a solution for any choice of $\mathbf{b}$?

Does $A\mathbf{x} = \mathbf{0}$ have a unique solution?

Other Questions Like...

Does $A\mathbf{x} = \mathbf{b}$ have at least one solution for any choice of $\mathbf{b}$?

Does $A\mathbf{x} = \mathbf{b}$ have at most one solution for any choice of $\mathbf{b}$?

Wait

$A\mathbf{x} = \mathbf{0}$ has a unique solution $\equiv$ $A\mathbf{x} = \mathbf{b}$ has at most one solution

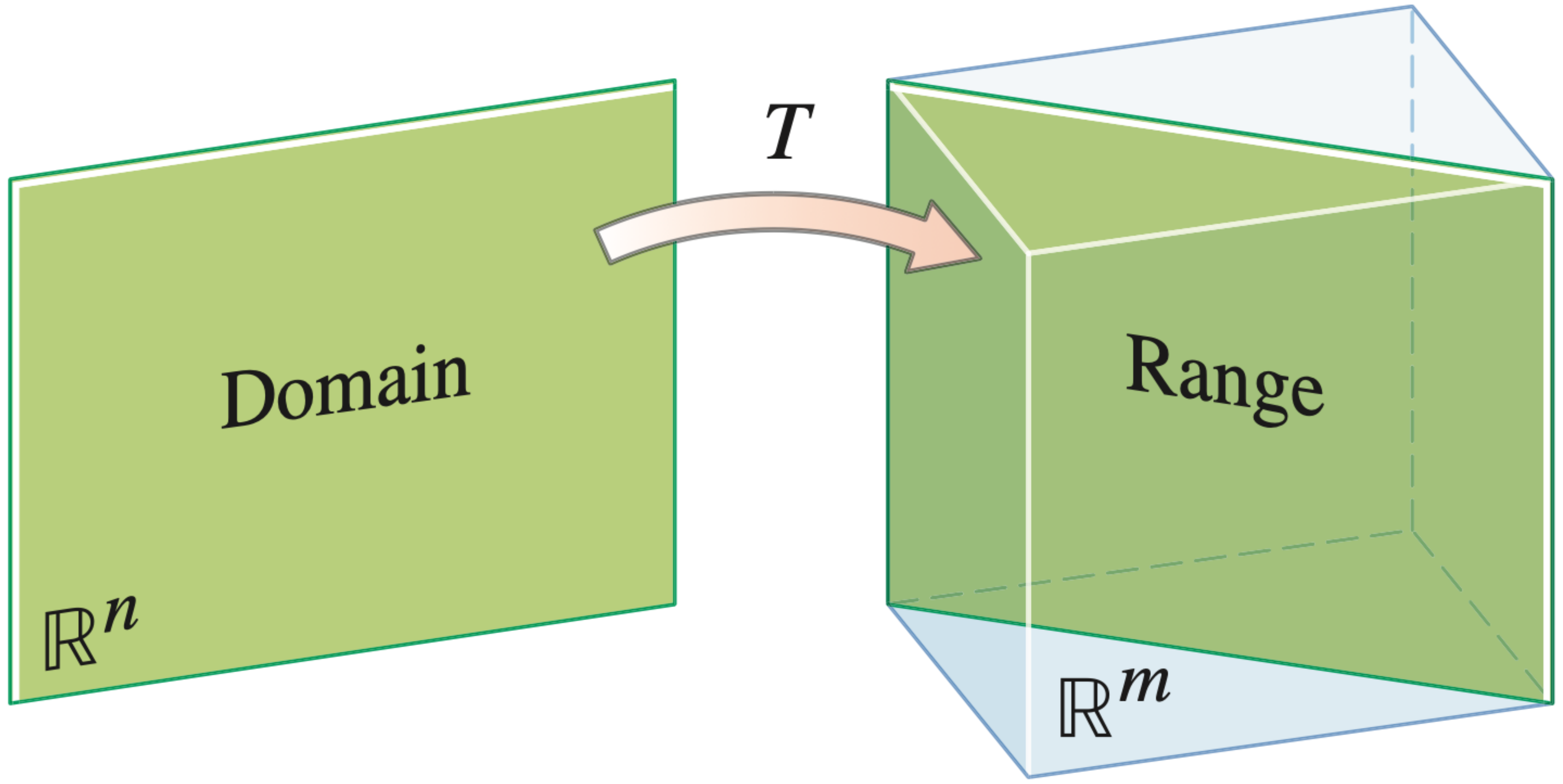
why?:

Onto and One-to-One

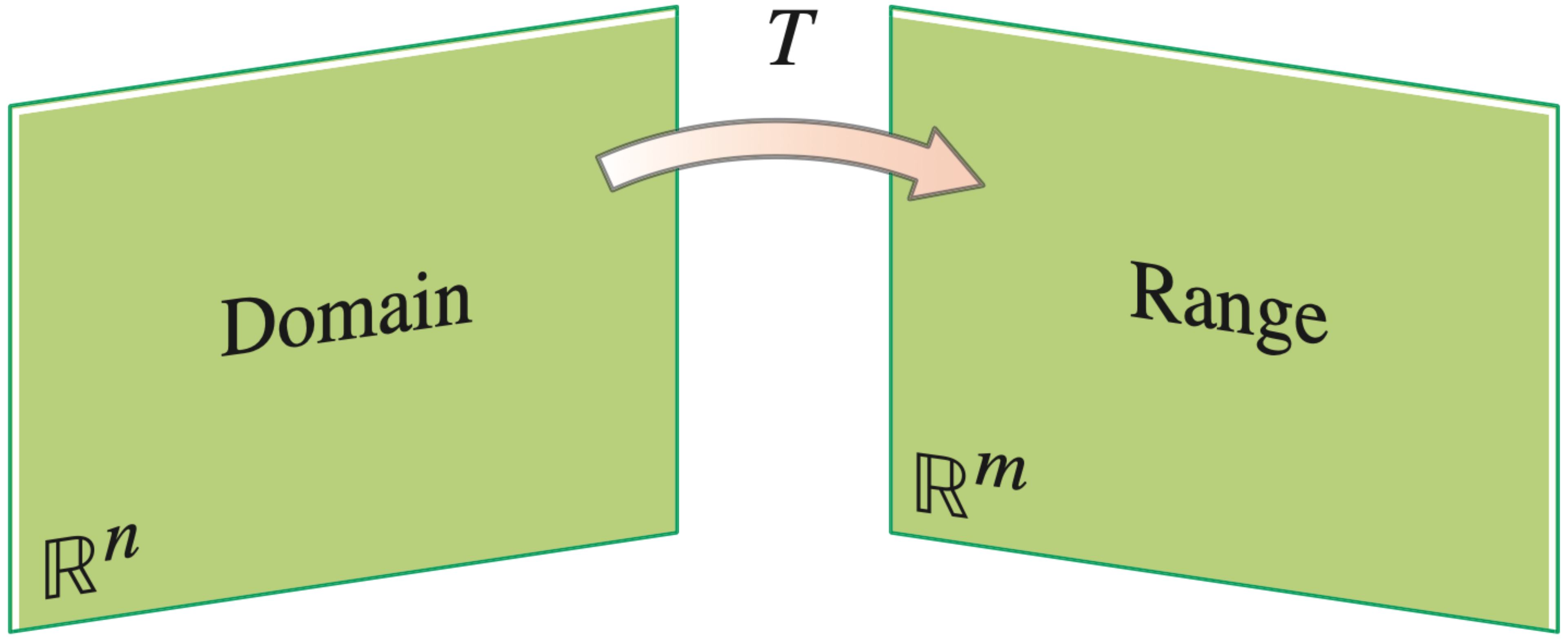
Definition. A transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is ***onto*** if any vector $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at least one vector $\mathbf{v}$ in $\mathbb{R}^n$ (where $T(\mathbf{v}) = \mathbf{b}$).

Definition. A transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is ***one-to-one*** if any vector $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at most one vector $\mathbf{v}$ in $\mathbb{R}^n$ (where $T(\mathbf{v}) = \mathbf{b}$).

Onto (Pictorially)



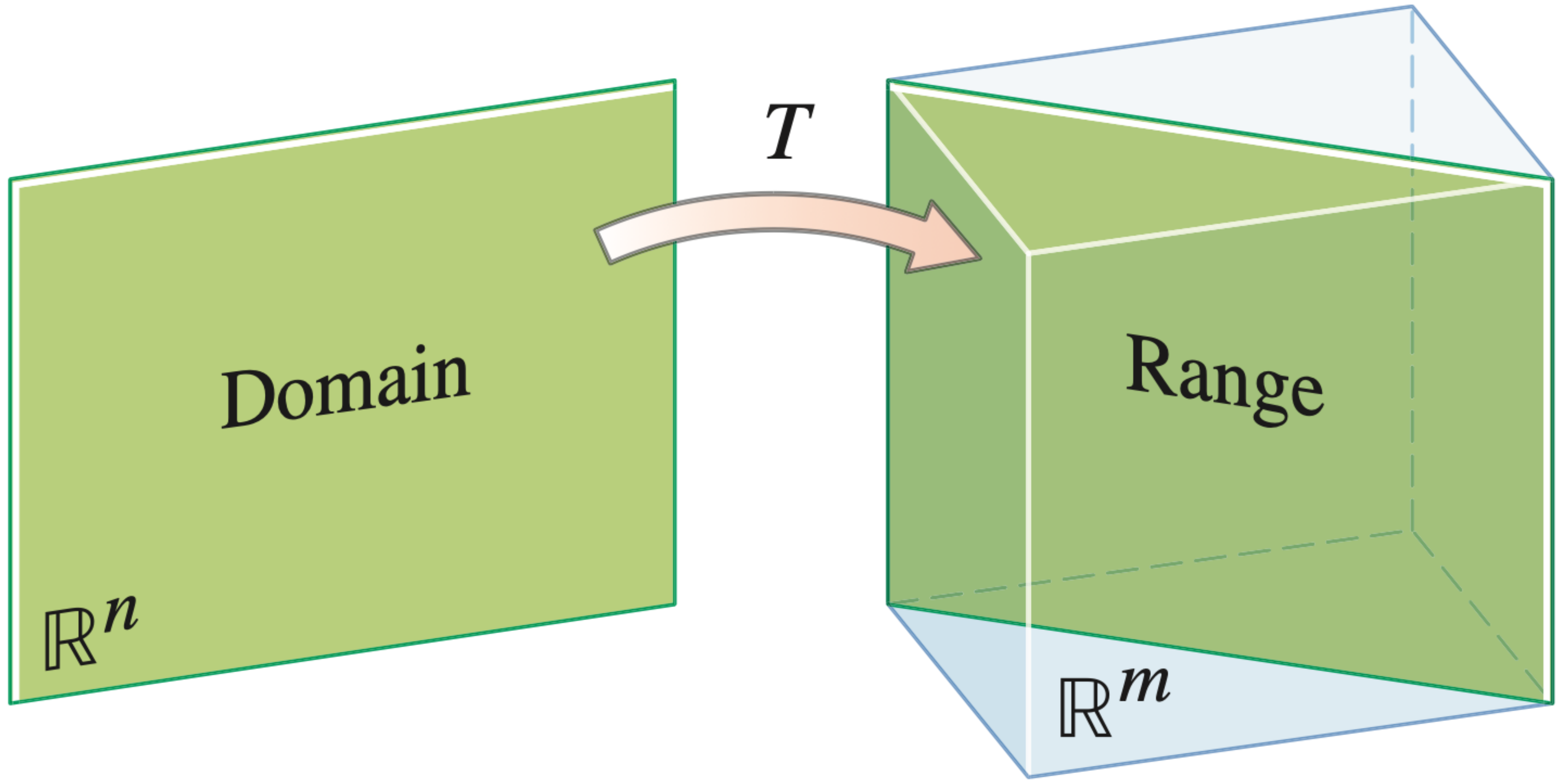
T is *not* onto $\mathbb{R}^m$



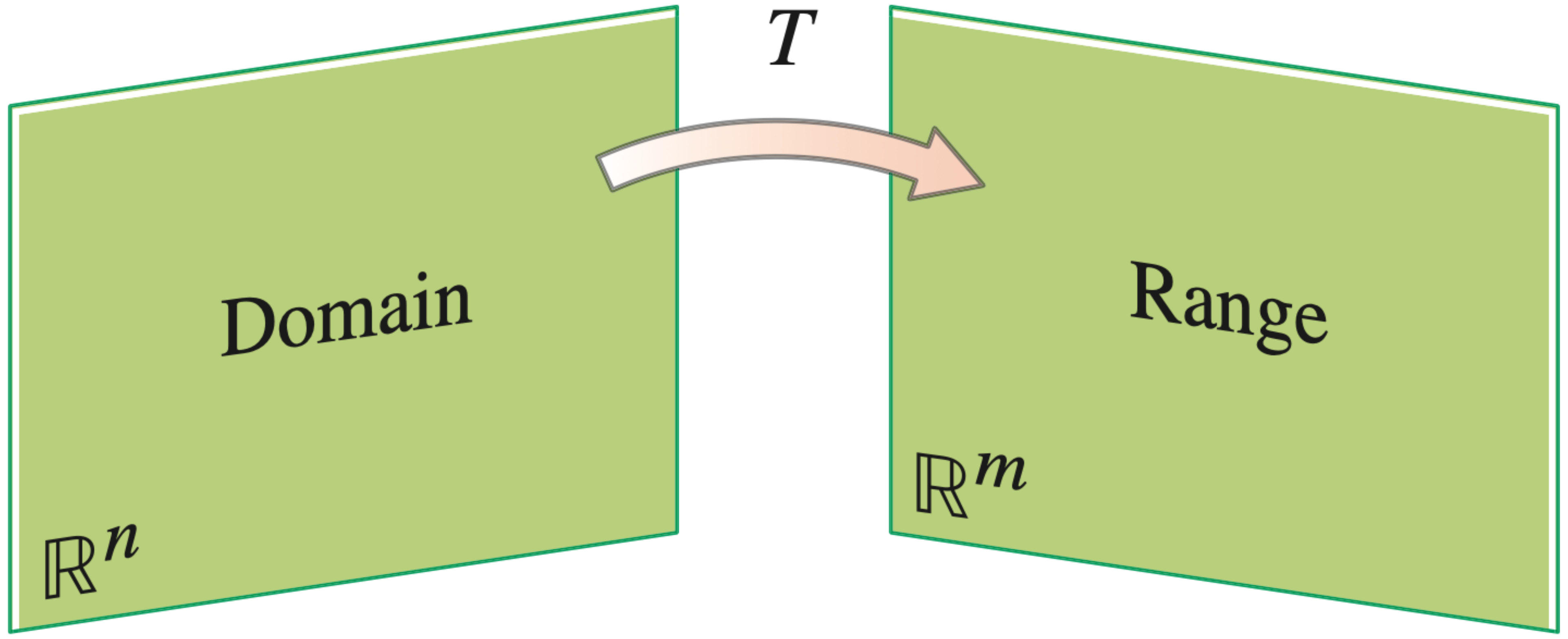
T is onto $\mathbb{R}^m$

image source: Linear Algebra and its Applications. Lay, Lay, and McDonald

Onto (Pictorially)



T is *not* onto $\mathbb{R}^m$

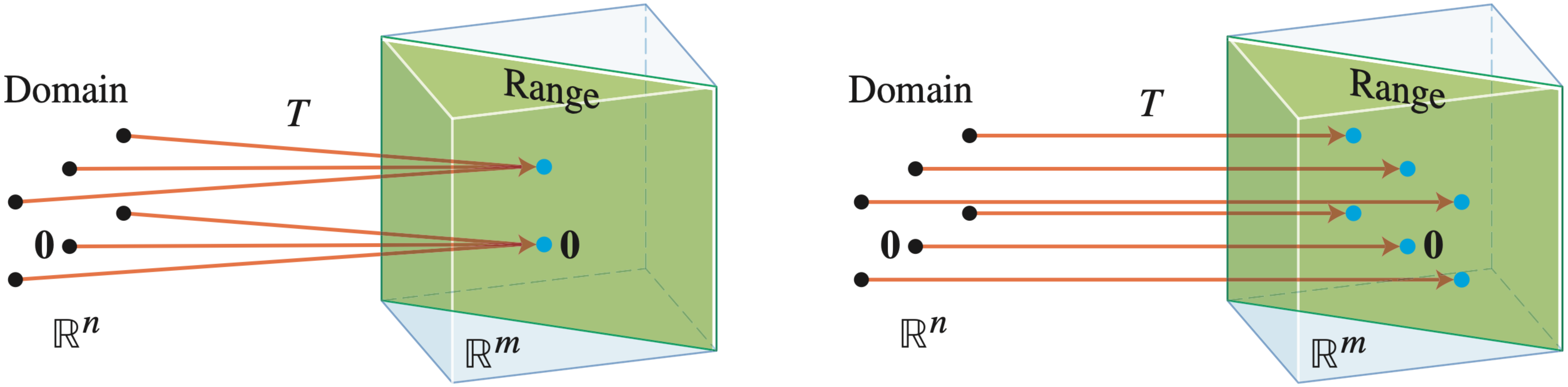


T is onto $\mathbb{R}^m$

T is onto if its range = its codomain

image source: Linear Algebra and its Applications. Lay, Lay, and McDonald

One-to-One (Pictorially)



T is *not* one-to-one

T is one-to-one

image source: Linear Algebra and its Applications. Lay, Lay, and McDonald

Taking Stock: Onto

Theorem. The following are logically equivalent for the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ implemented by the matrix A.

» T is onto
» $A\mathbf{x} = \mathbf{b}$ has a solution for any choice of $\mathbf{b}$
» $\text{range}(T) = \text{codomain}(T)$
» the columns of A span $\mathbb{R}^m$
» A has a pivot position in every <u>row</u>

Taking Stock: One-to-One

Theorem. The following are logically equivalent for the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ implemented by the matrix A.

» T is one-to-one
» $A\mathbf{x} = \mathbf{b}$ has at most one solution for any $\mathbf{b}$
» $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
» The columns of A are linearly independent
» A has a pivot position in every column

How To: One-to-One and Onto

Question. Show that the linear transformation T is one-to-one/onto.

Solution. (one approach) Find the matrix which implements T and see if it has a pivot in every column/row.

Warning: this is not the only way. Always try to think if you can solve it using *any* of the perspectives

Example: both 1-1 and onto

Rotation about the origin:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

why?:

Example: 1-1, not onto

`Lifting:`

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ x_2 \\ x_1 + x_2 \end{bmatrix}$$

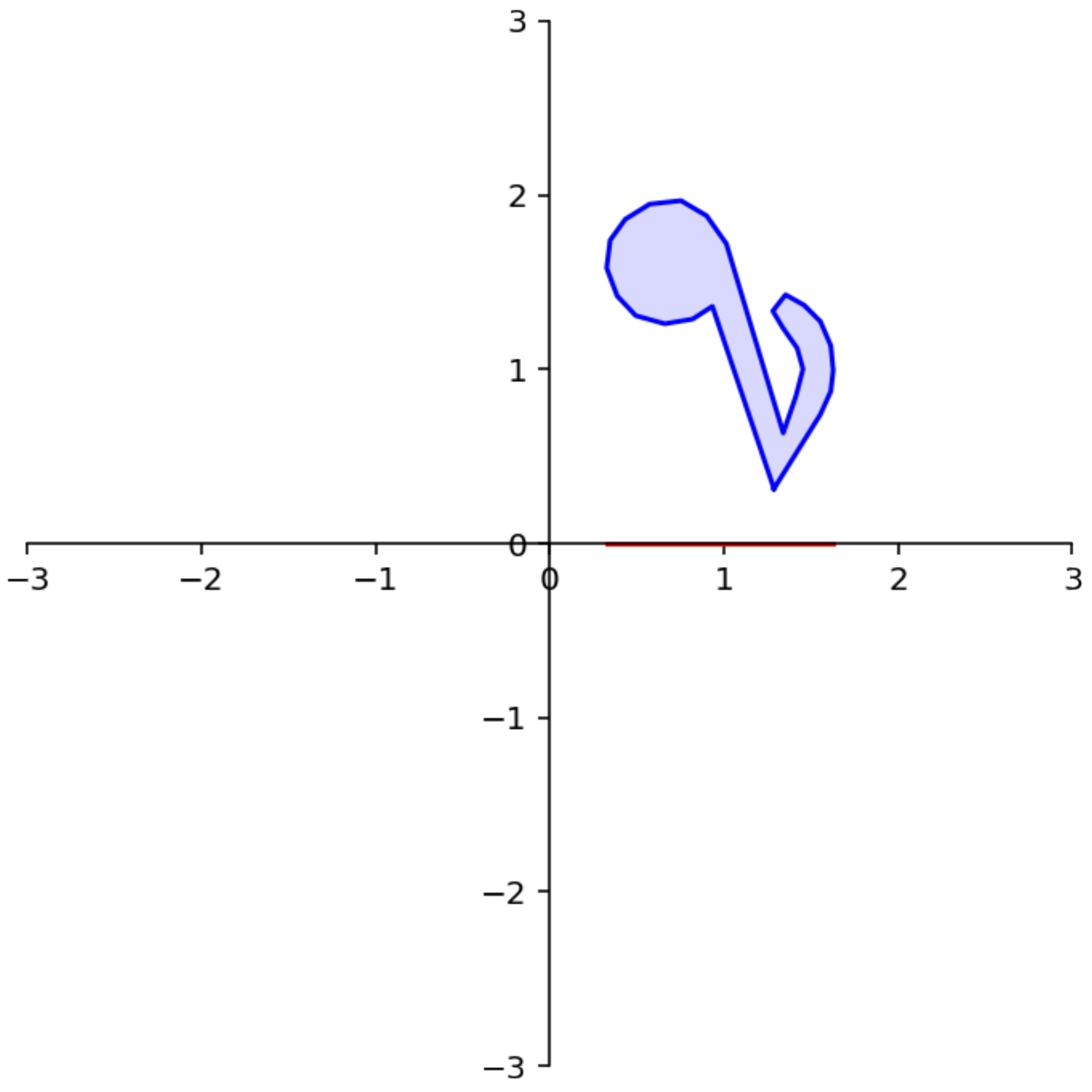
`why?:`

Example: not 1-1, not onto

Projection onto the x_1 axis:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

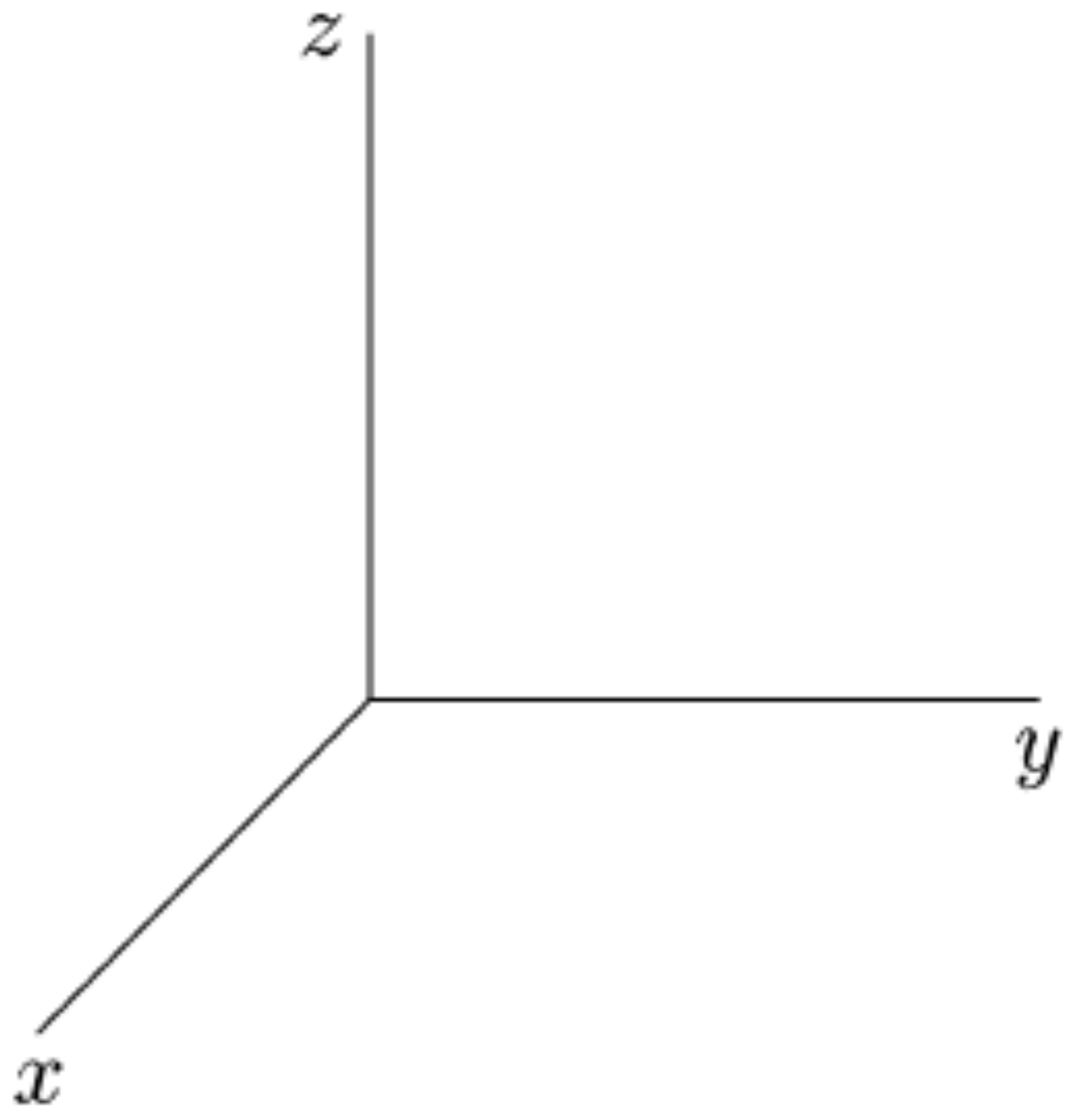
why?:



Example: onto, not 1-1

Projection from $\mathbb{R}^3$ to $\mathbb{R}^2$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



why?:

Summary

Matrix transformations and linear transformations are the same thing.

We can find these matrices by looking at how the transformation behaves on the standard basis.

We can reason about matrix equations by directly reasoning about the linear transformations.