Matrix Algebra

Geometric Algorithms
Lecture 9

CAS CS 132

Objectives

1. (From last time) Connect questions about matrix equations and linear transformations
2. Motivate matrix multiplication
3. Define matrix multiplication
4. Look at the algebra of matrix multiplication

Keywords

one-to-one transformation

onto transformation

matrix multiplication

row-column rule

matrix addition and scaling

non-commutativity

Recap

Recall: Matrices as Transformations

Matrices allow us to *transform* vectors.

The transformed vector lies in the span of its columns.

$$\mathbf{x} \mapsto A\mathbf{x}$$

map a vector $\mathbf{x}$ to the vector $A\mathbf{v}$

Recall: Motivating Questions

What kind of functions can we define in this way?

How do we interpret what the transformation does to a set of vectors?

How does this relate back to matrix equations?

Recall: A New Interpretation of the Matrix Equation

$A\mathbf{x} = \mathbf{b}$? $\equiv$ is there a vector which A transforms into $\mathbf{b}$?

Solve $A\mathbf{x} = \mathbf{b}$ $\equiv$ find a vector which A transforms into $\mathbf{b}$

Recall: A New Interpretation of the Matrix Equation

$A\mathbf{x} = \mathbf{b}$? $\equiv$ is there a vector which A transforms into $\mathbf{b}$?

Solve $A\mathbf{x} = \mathbf{b}$ $\equiv$ find a vector which A transforms into $\mathbf{b}$

What about other questions?

One-to-One and Onto Transformations

Other Questions Like...

Does $A\mathbf{x} = \mathbf{b}$ have a solution for any choice of $\mathbf{b}$?

Does $A\mathbf{x} = \mathbf{0}$ have a unique solution?

Other Questions Like...

Do the columns of A have full span?

Are the columns of A linearly independent?

Other Questions Like...

Does $A\mathbf{x} = \mathbf{b}$ have at least one solution for any choice of $\mathbf{b}$?

Does $A\mathbf{x} = \mathbf{b}$ have at most one solution for any choice of $\mathbf{b}$?

Other Questions Like...

Does $A\mathbf{x} = \mathbf{b}$ have at least one solution for any choice of $\mathbf{b}$?

Does $A\mathbf{x} = \mathbf{b}$ have at most one solution for any choice of $\mathbf{b}$?

Wait, what's going on with this second one?

A New Perspective on Linear Independence

$\Longleftarrow$

$A\mathbf{x} = \mathbf{0}$ has a unique solution $\equiv$ $A\mathbf{x} = \mathbf{b}$ has at most one solution for any choice of $\mathbf{b}$

$A\vec{0} = \vec{0}$

$A\vec{x} = \vec{0}$ has at most 1 solution

why?: Choose $\vec{b}$

Suppose $A\vec{x} = \vec{0}$ has unique sol.

Suppose $\vec{u}, \vec{v}$ s.t. $A\vec{u} = \vec{b}$ $A\vec{v} = \vec{b}$

$A(\vec{u} - \vec{v}) = A\vec{u} - A\vec{v} = \vec{b} - \vec{b} = \vec{0}$

it must be $\vec{u} - \vec{v} = \vec{0}$

$\vec{u} = \vec{v}$

Onto Transformations

Onto Transformations

Definition. A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is ***onto*** if any vector $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at least one vector $\mathbf{v}$ in $\mathbb{R}^n$ (where $T(\mathbf{v}) = \mathbf{b}$).

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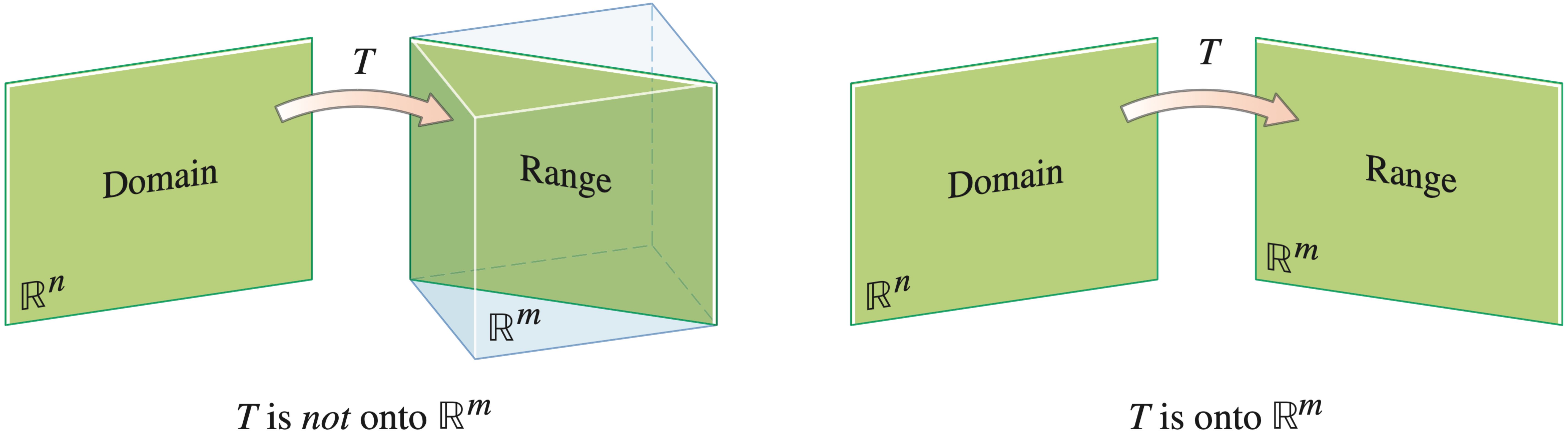

image source: Linear Algebra and its Applications. Lay, Lay, and McDonald

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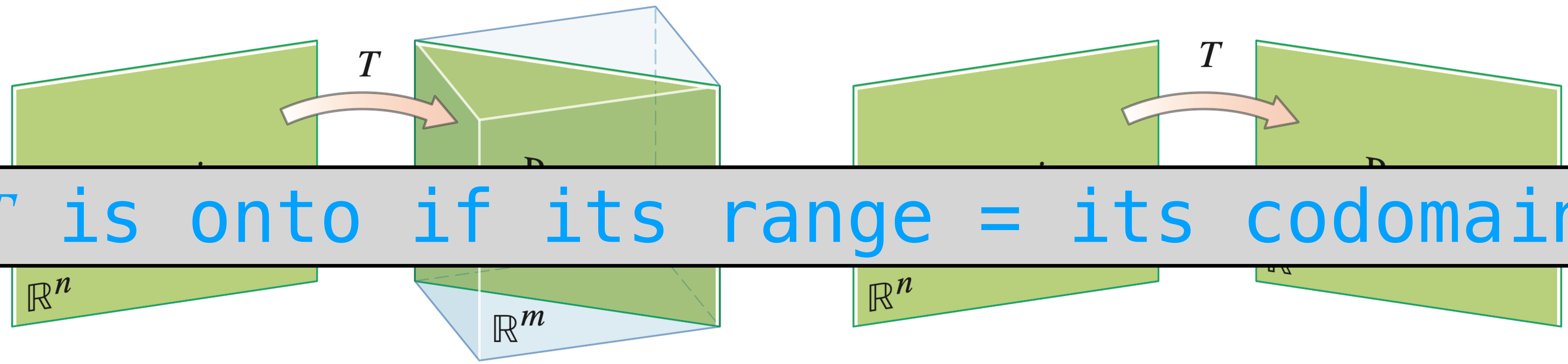

T is onto if its range = its codomain

T is *not* onto $\mathbb{R}^m$ T is onto $\mathbb{R}^m$

image source: Linear Algebra and its Applications. Lay, Lay, and McDonald

One-to-one Transformations

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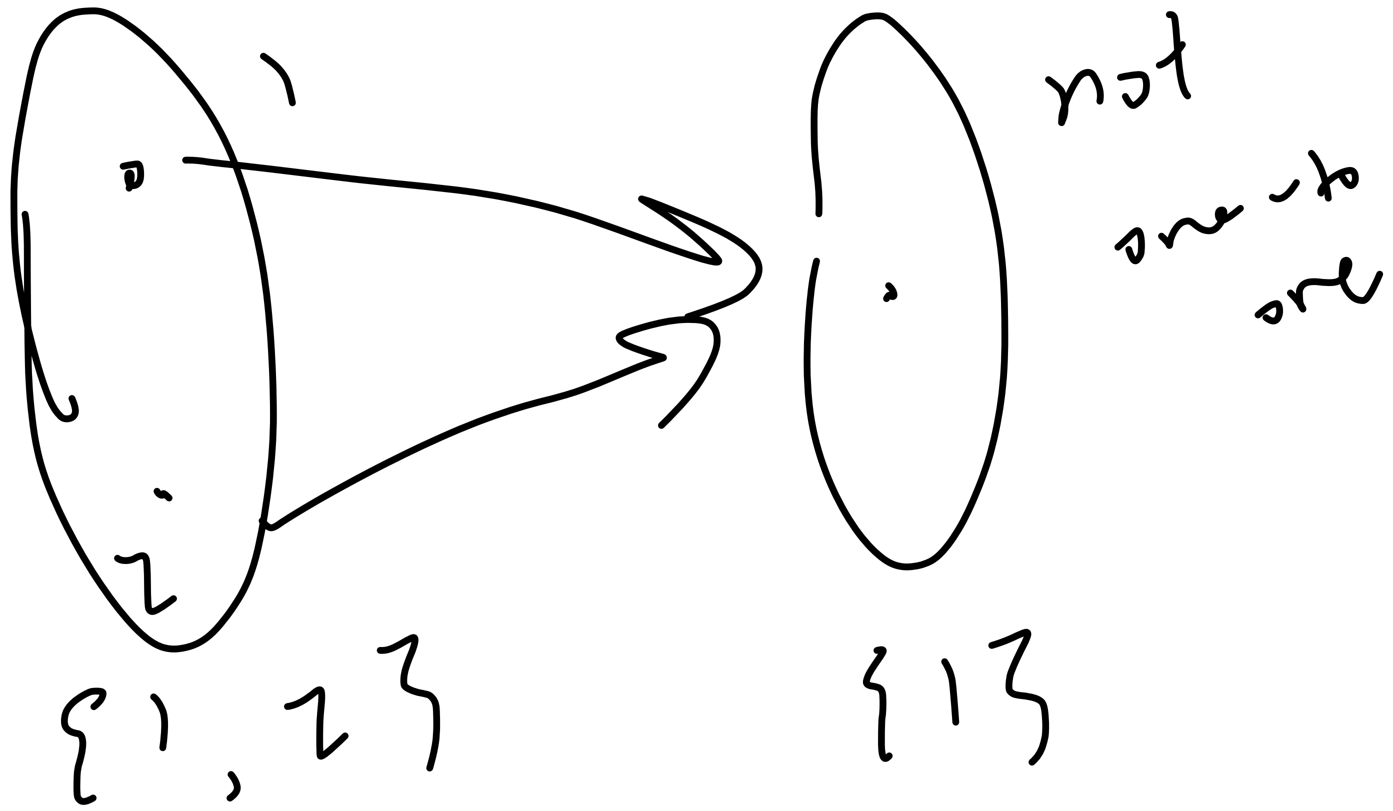
Definition. A transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is ***one-to-one*** if any vector $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at most one vector $\mathbf{v}$ in $\mathbb{R}^n$ (where $T(\mathbf{v}) = \mathbf{b}$).

One-to-one Transformations



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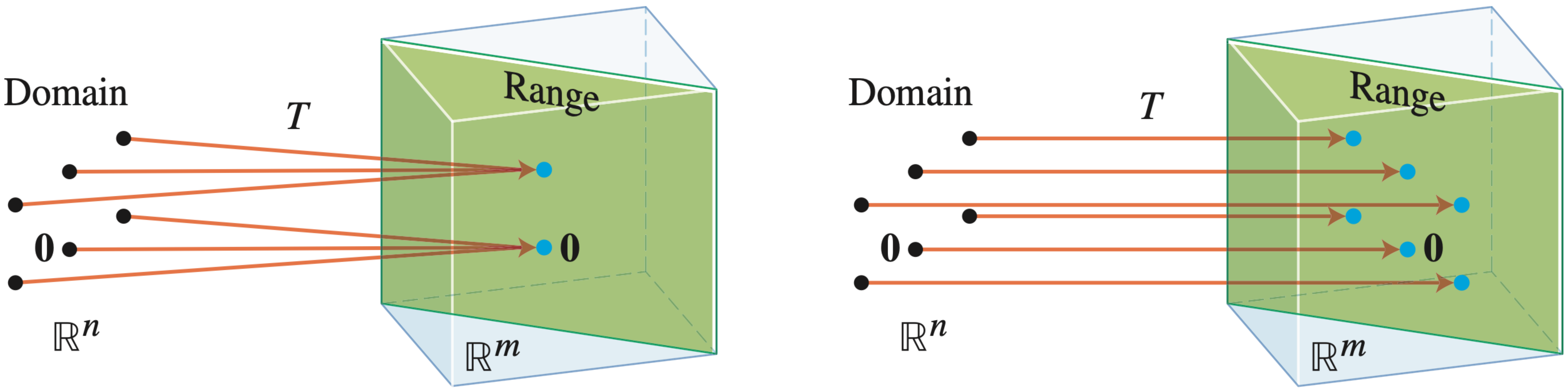

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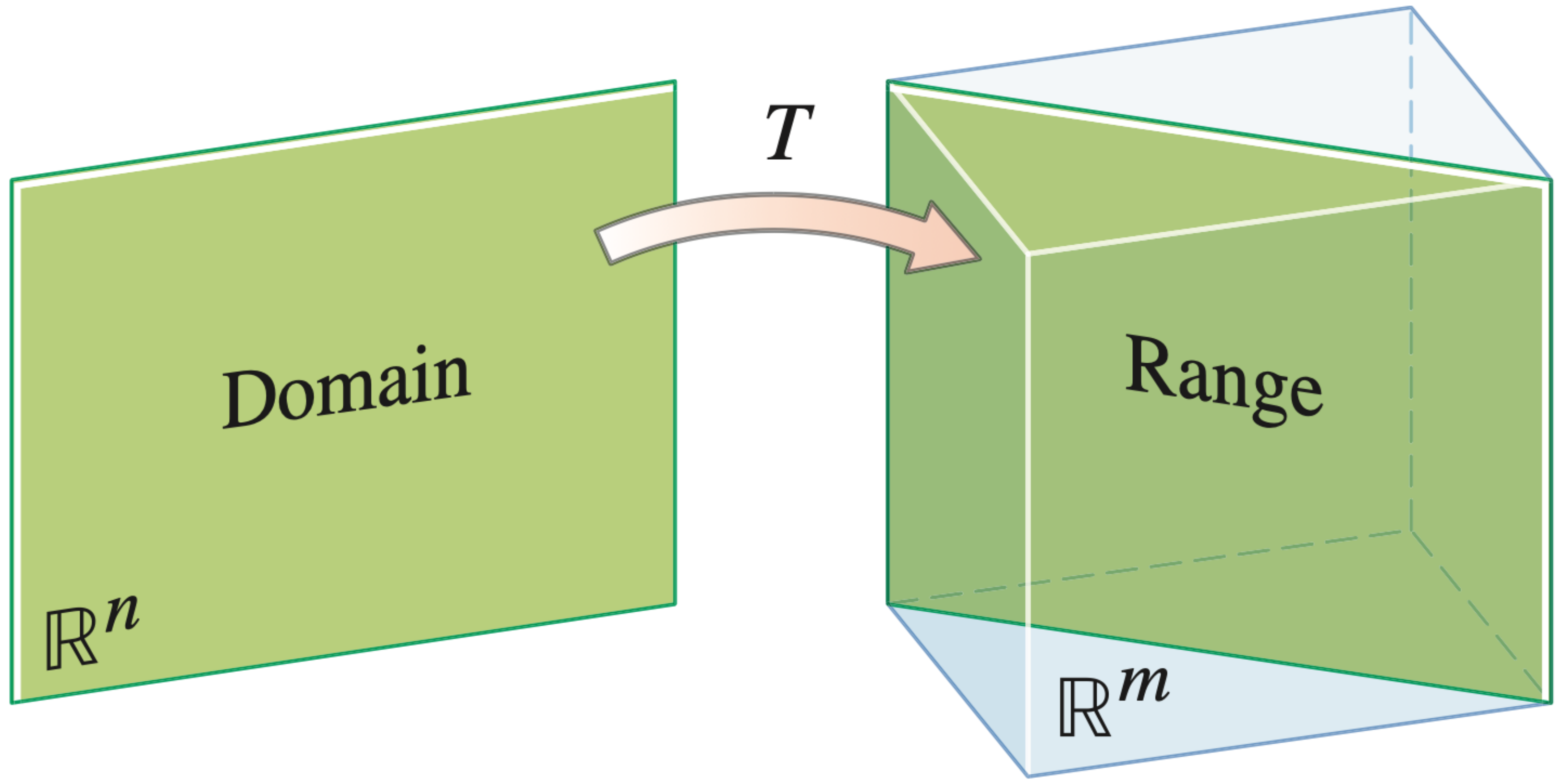
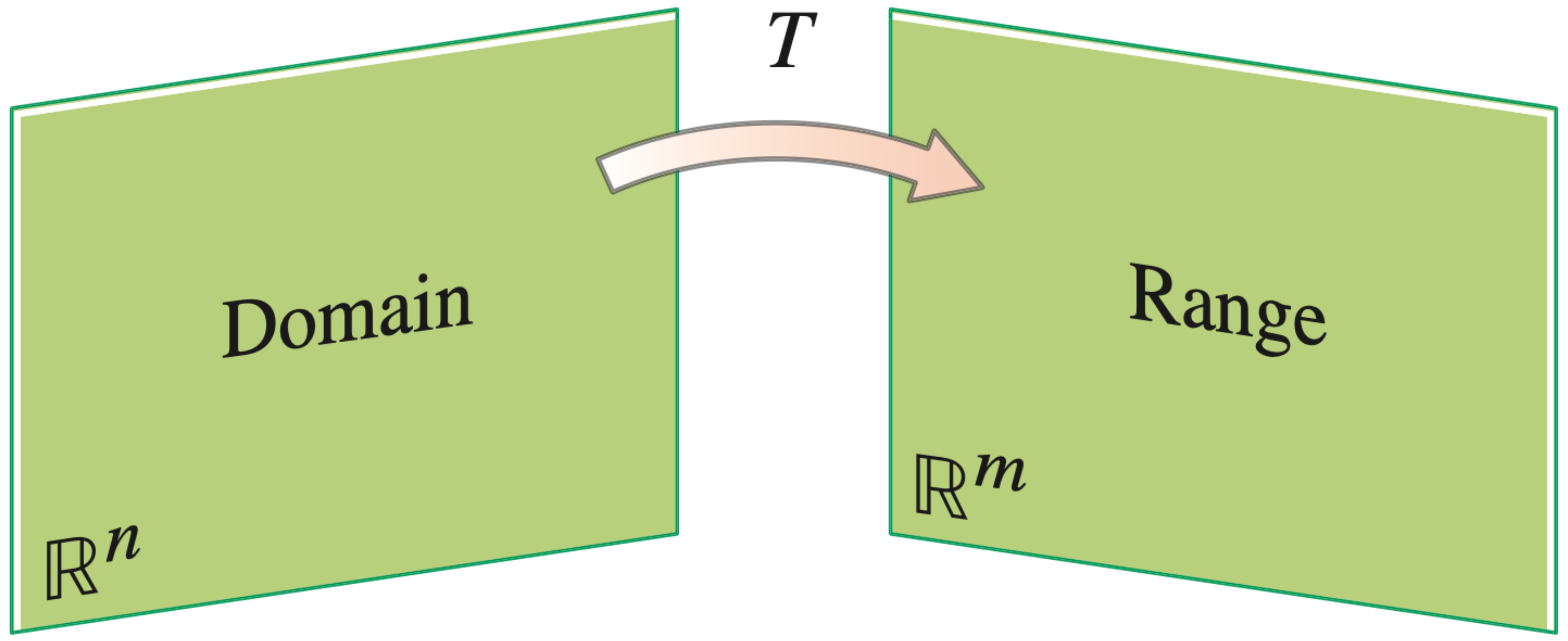
Comparing Pictures



T is *not* onto $\mathbb{R}^m$



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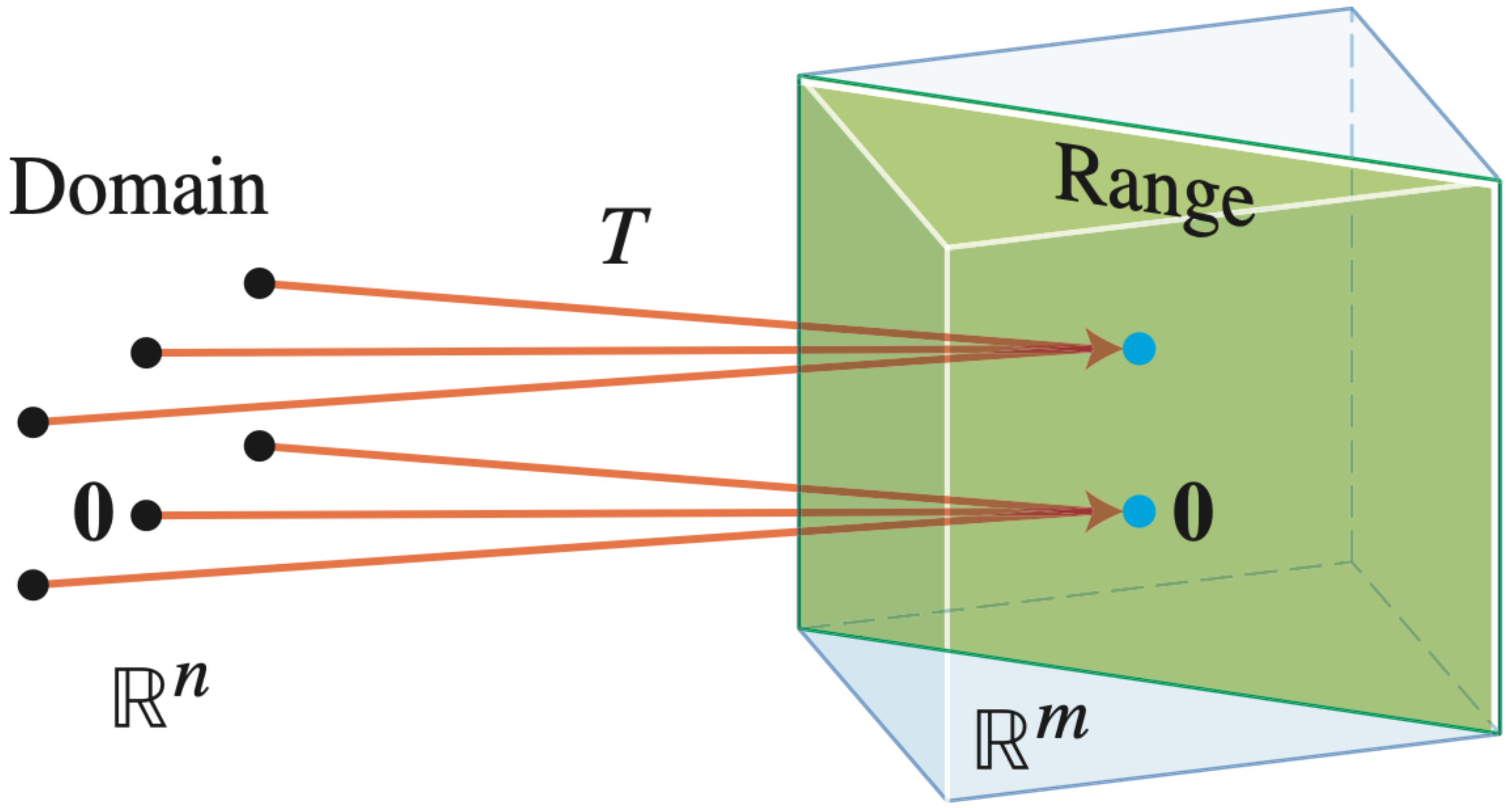

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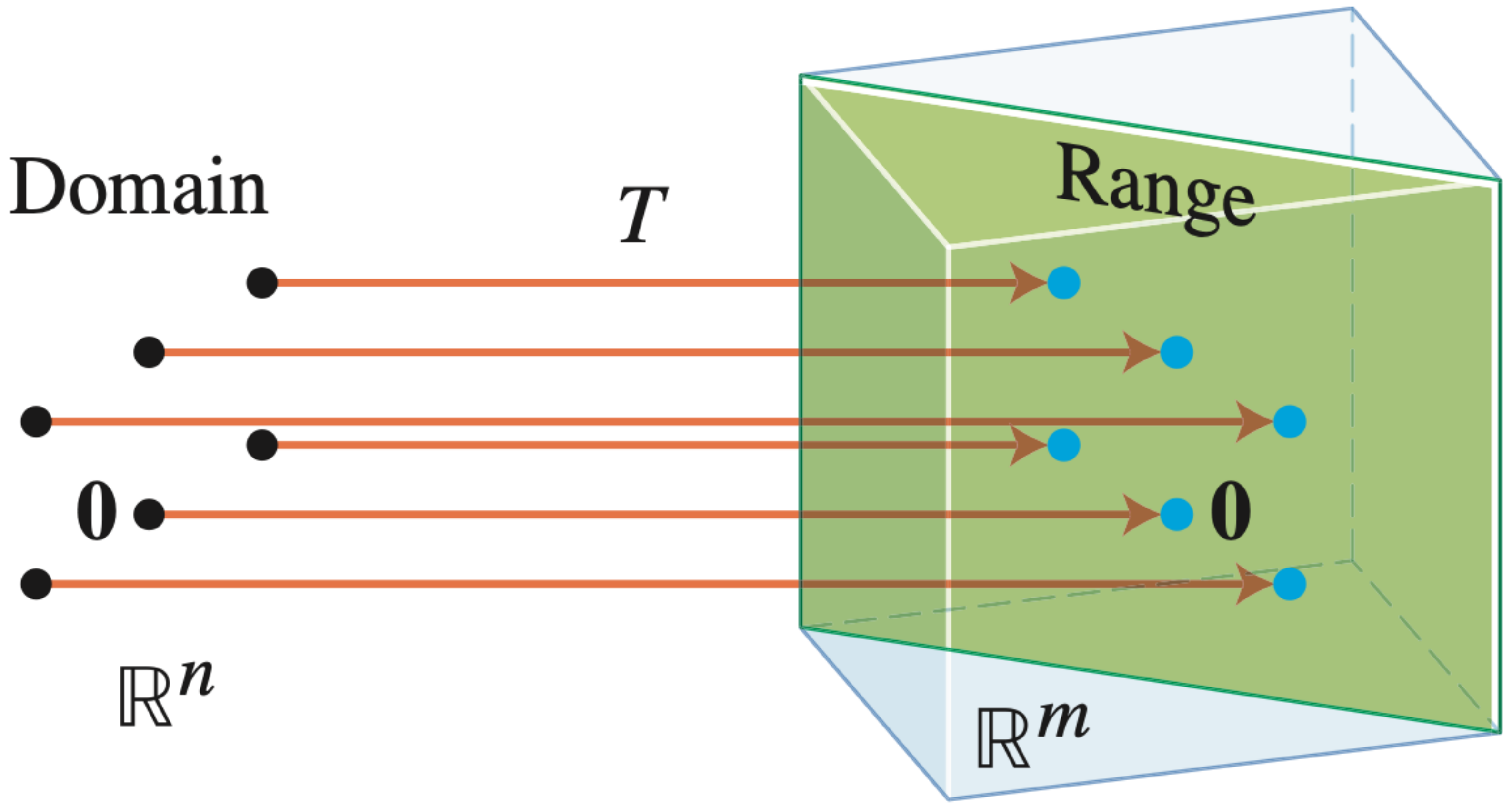

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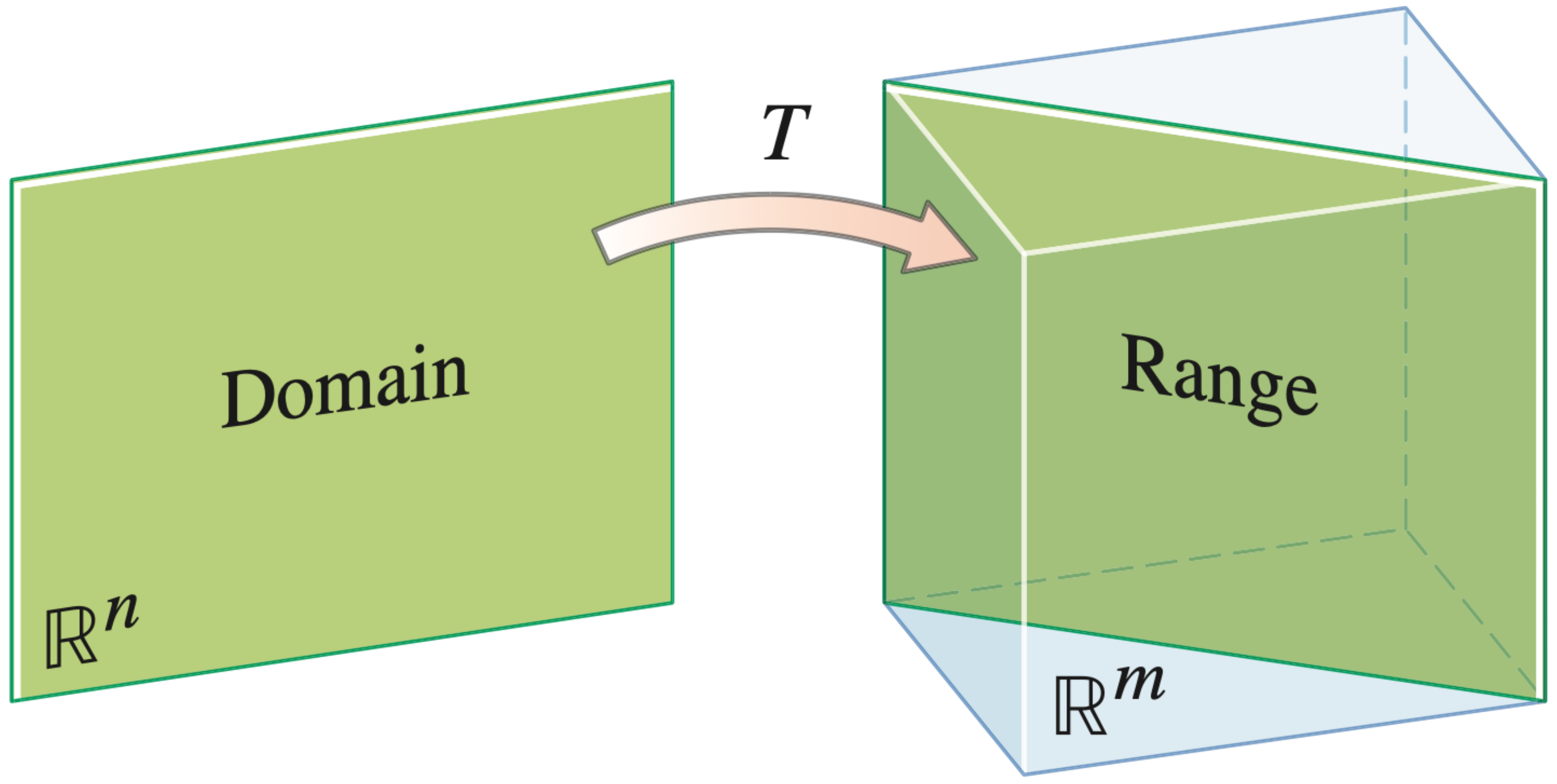

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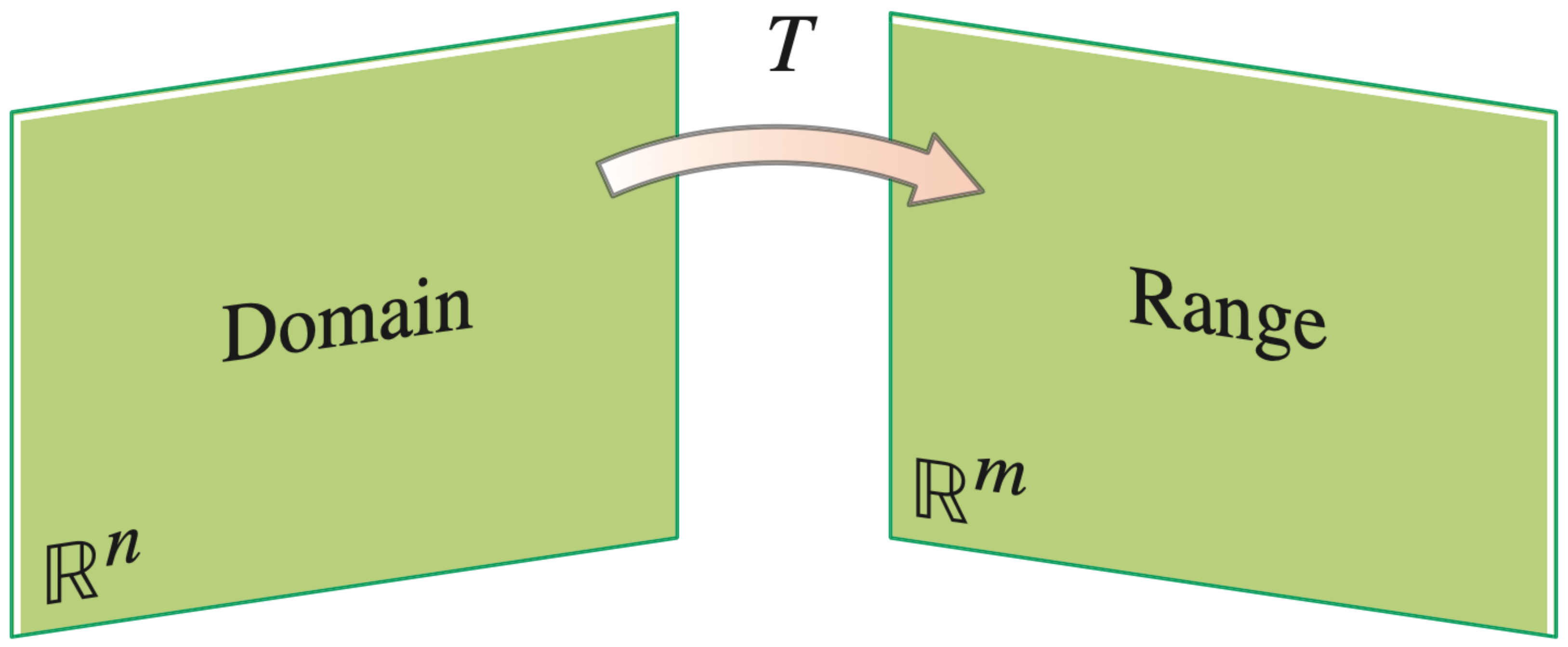

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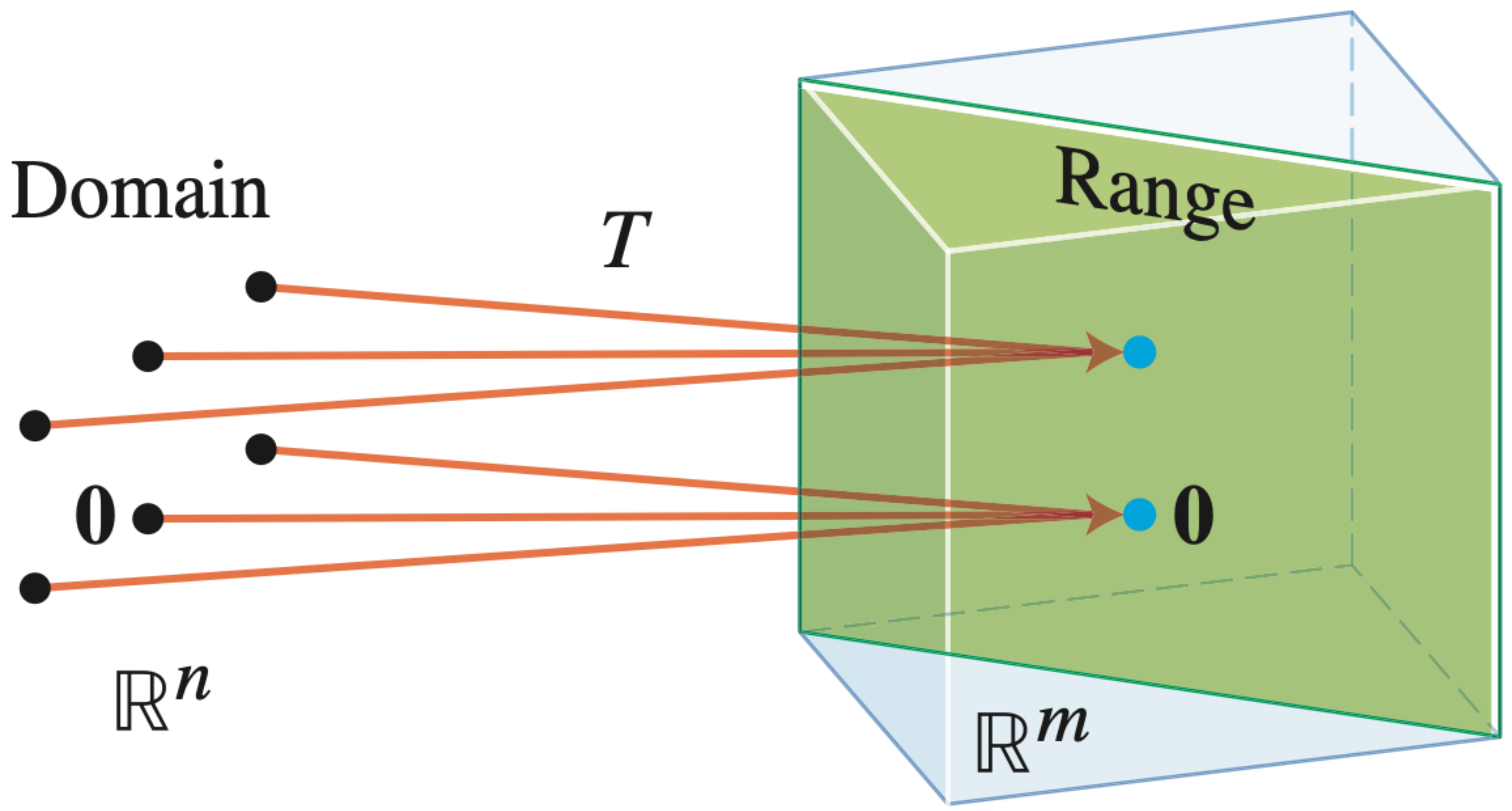

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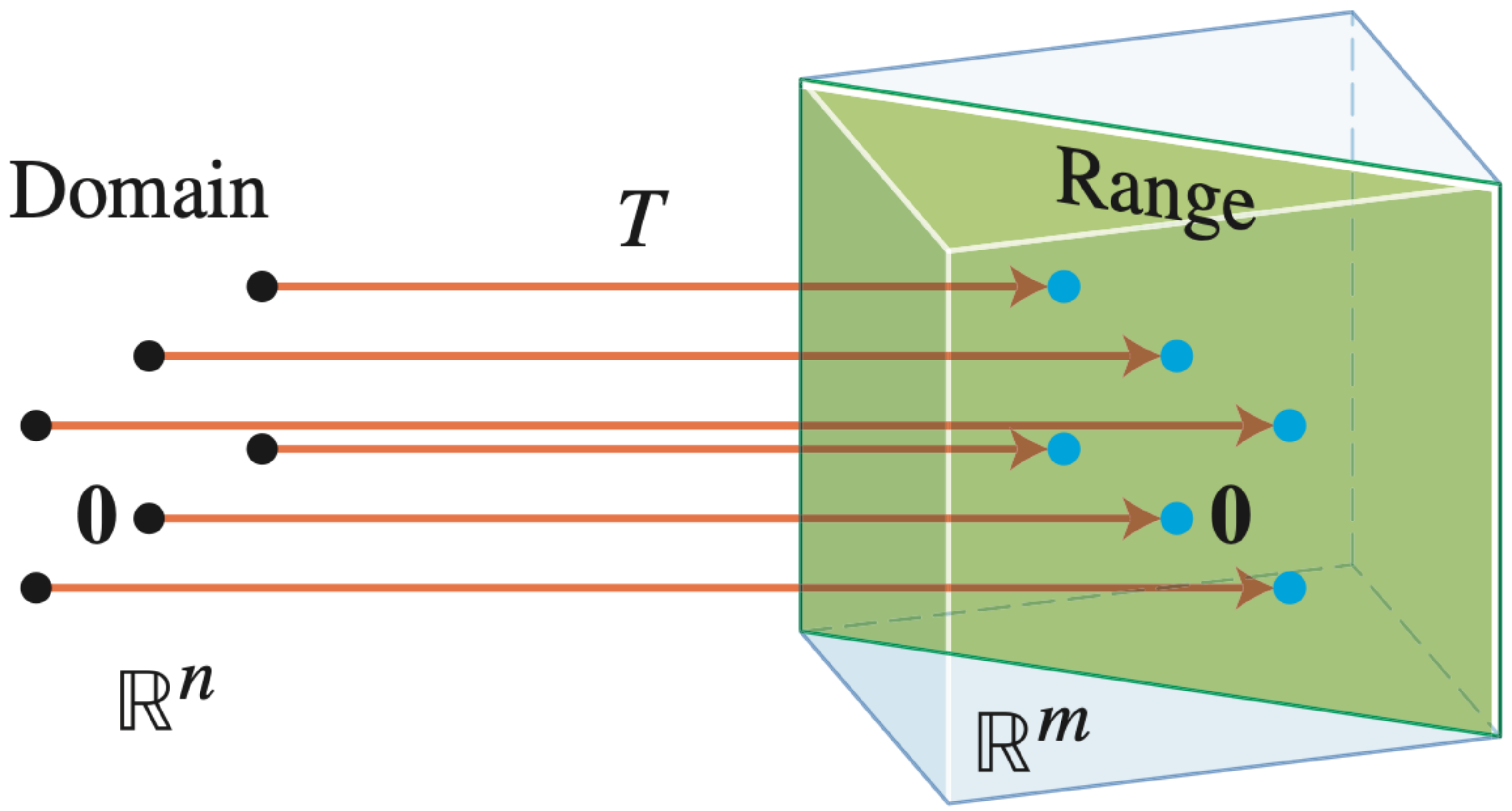

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Taking Stock: Onto

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Theorem. The following are logically equivalent for the linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ implemented by the matrix A.

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- » $A\mathbf{x} = \mathbf{0}$ has only the trivial solution

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- » The columns of A are linearly independent
- » A has a pivot position in every <u>column</u>

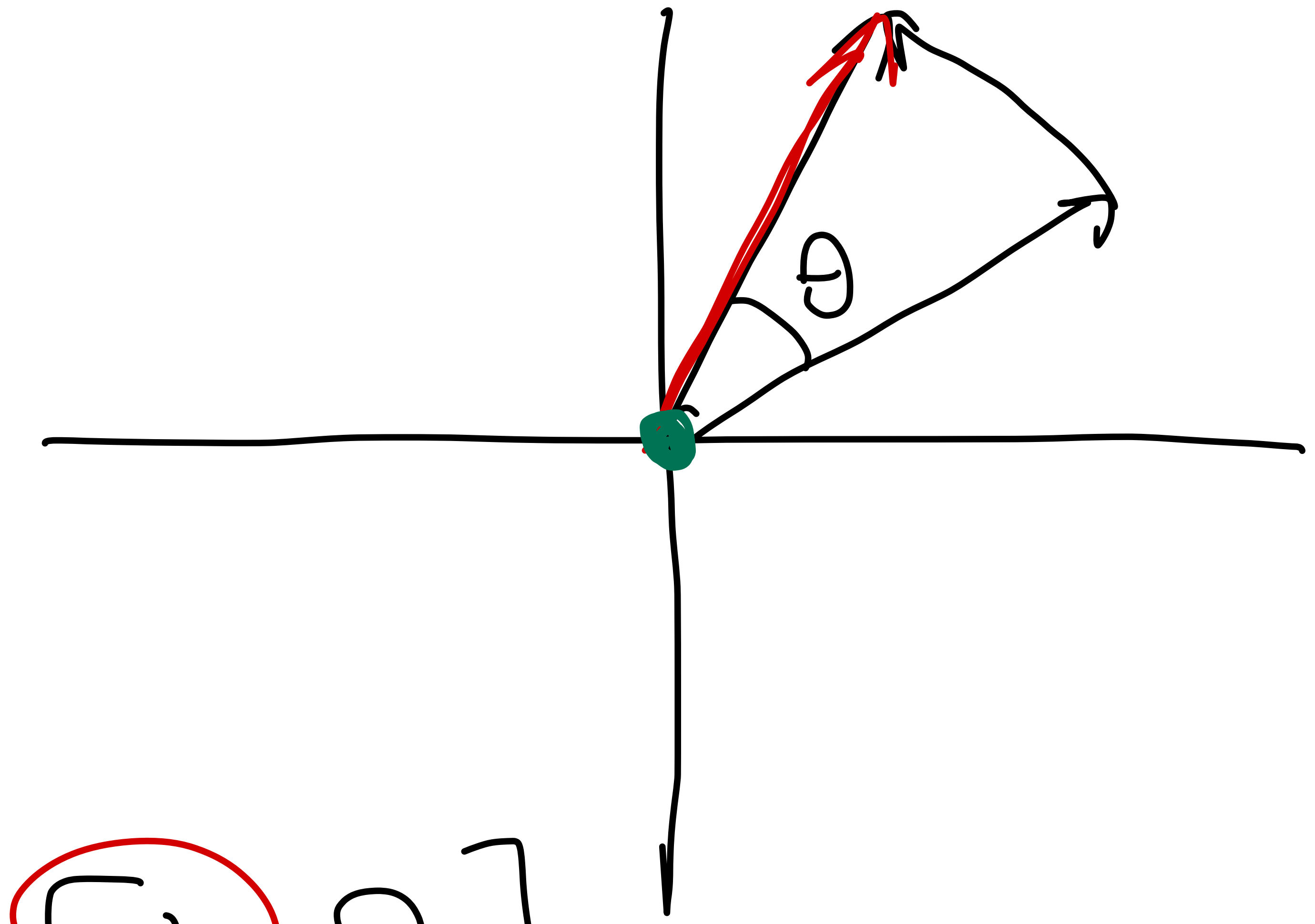
How To: One-to-One and Onto

Question. Show that the linear transformation T is one-to-one/onto.

Solution. (one approach) Find the matrix which implements T and see if it has a pivot in every column/row.

Warning: this is not the only way. Always try to think if you can solve it using *any* of the perspectives

Example: both 1-1 and onto



Rotation about the origin:

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

why?:

1-1: $A\vec{x} = \vec{0}$ has a unique solution $\Rightarrow$ has 1 pivot per column

onto: 1 pivot per row

Example: 1-1, not onto

Lifting:

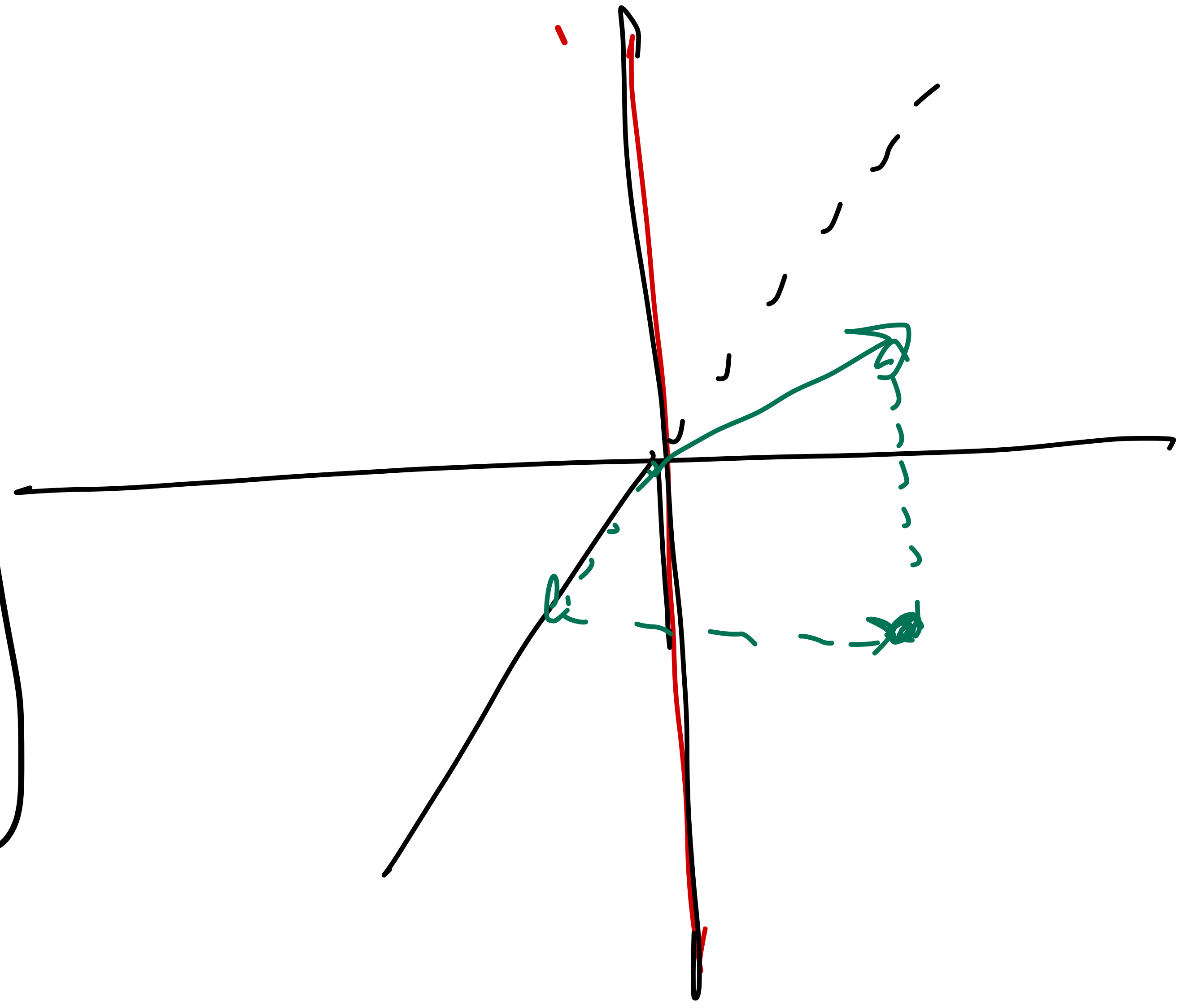
$\mathbb{R}^2 \qquad \mathbb{R}^3$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ x_2 \\ x_1 + x_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

why?:

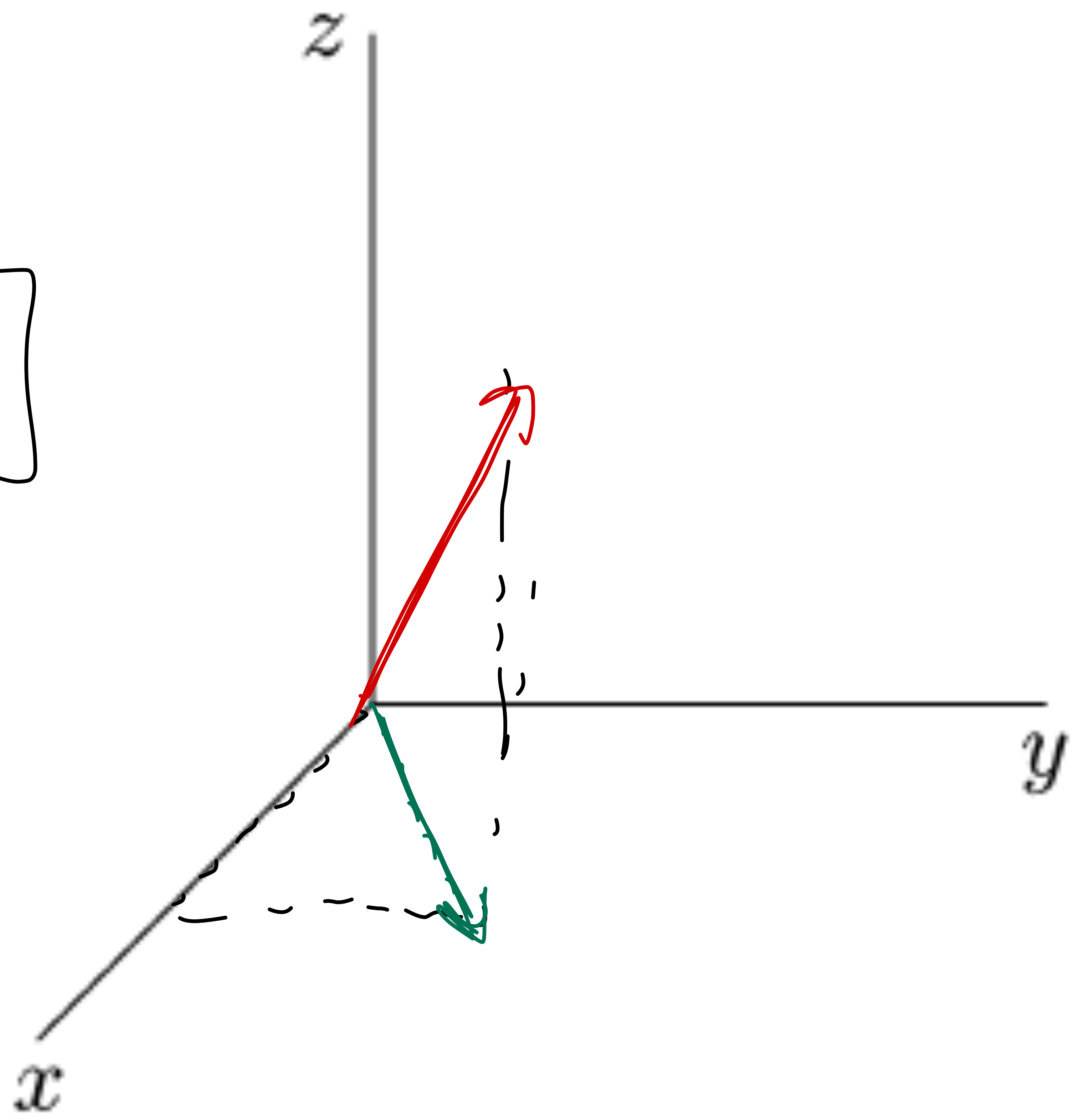
$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$



Example: onto, not 1-1

Projection from $\mathbb{R}^3$ to $\mathbb{R}^2$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

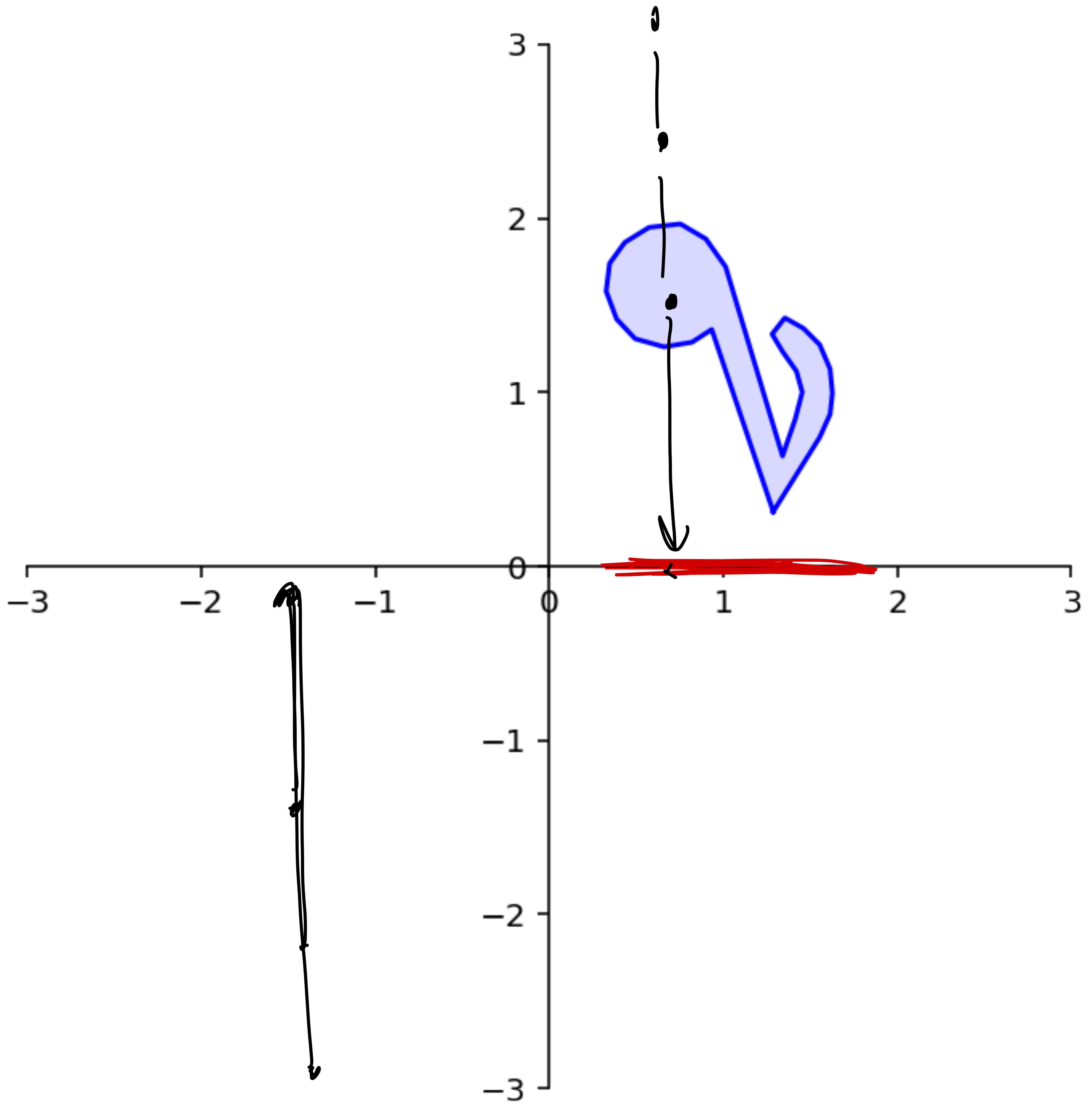


why?:

Example: not 1-1, not onto

Projection onto the x_1-axis:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

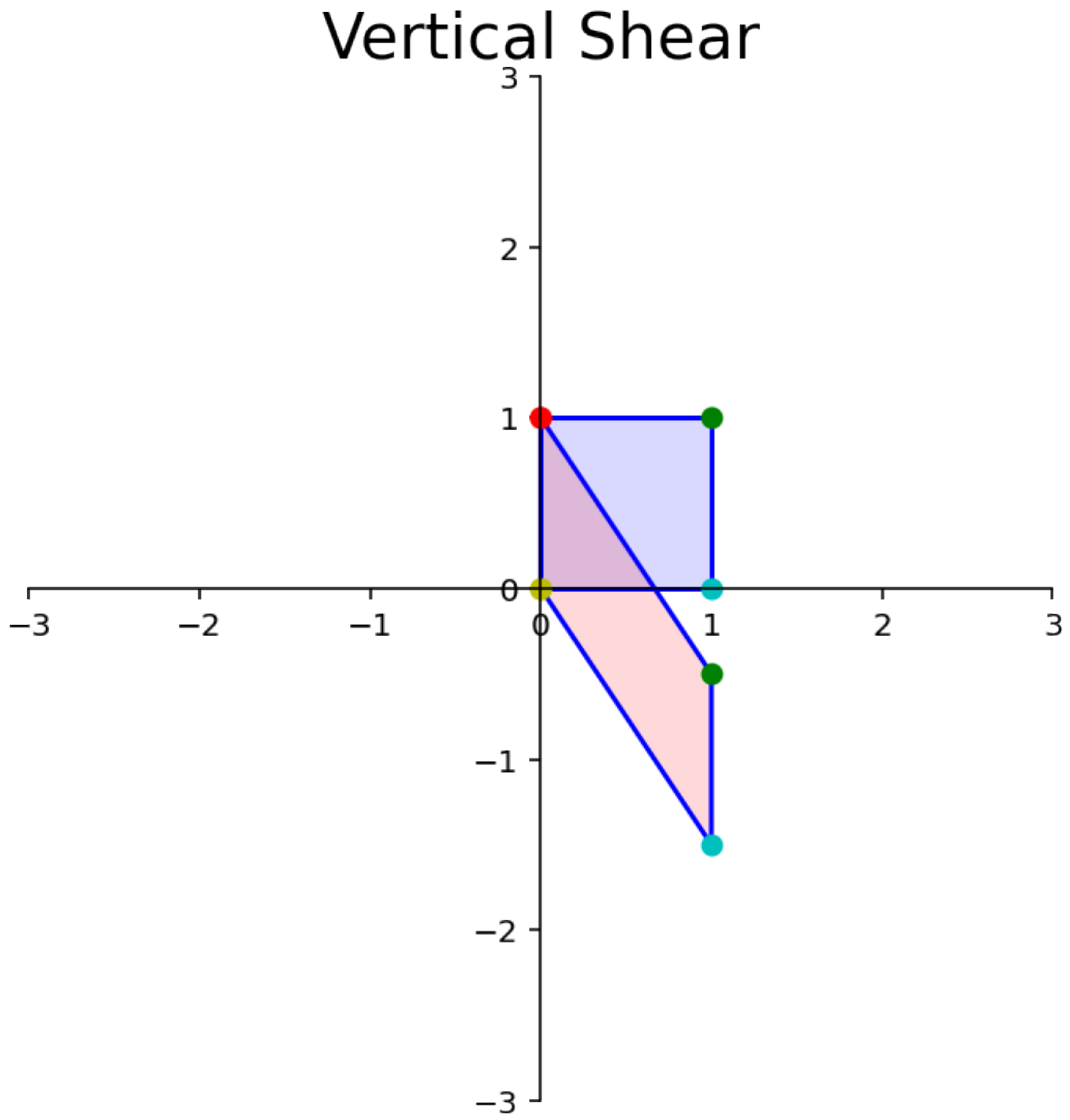


why?:

ran(T) = x_1-axis

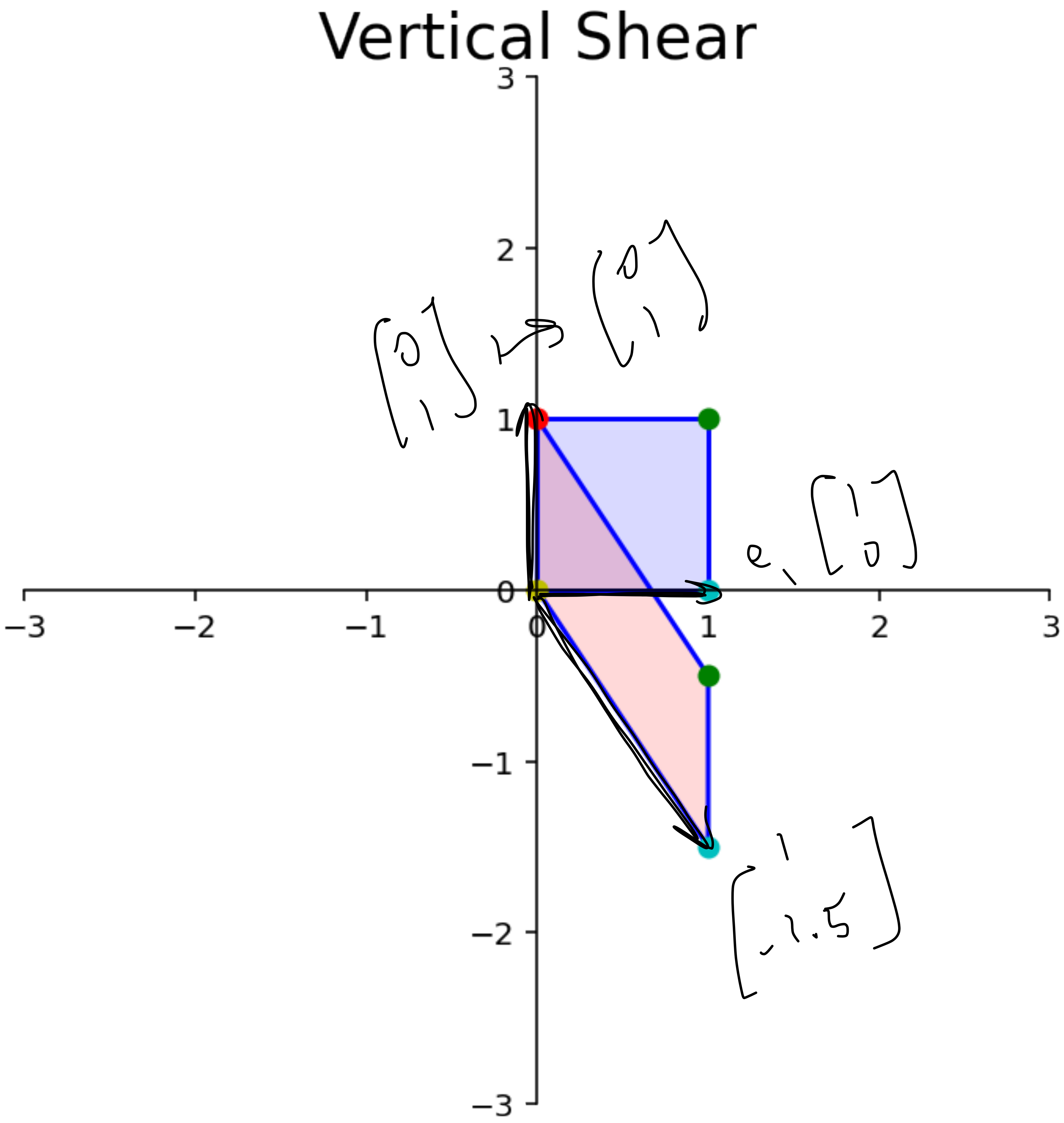
Question

Is vertical shearing a 1–1 transformation? Justify your answer.



Answer: Yes

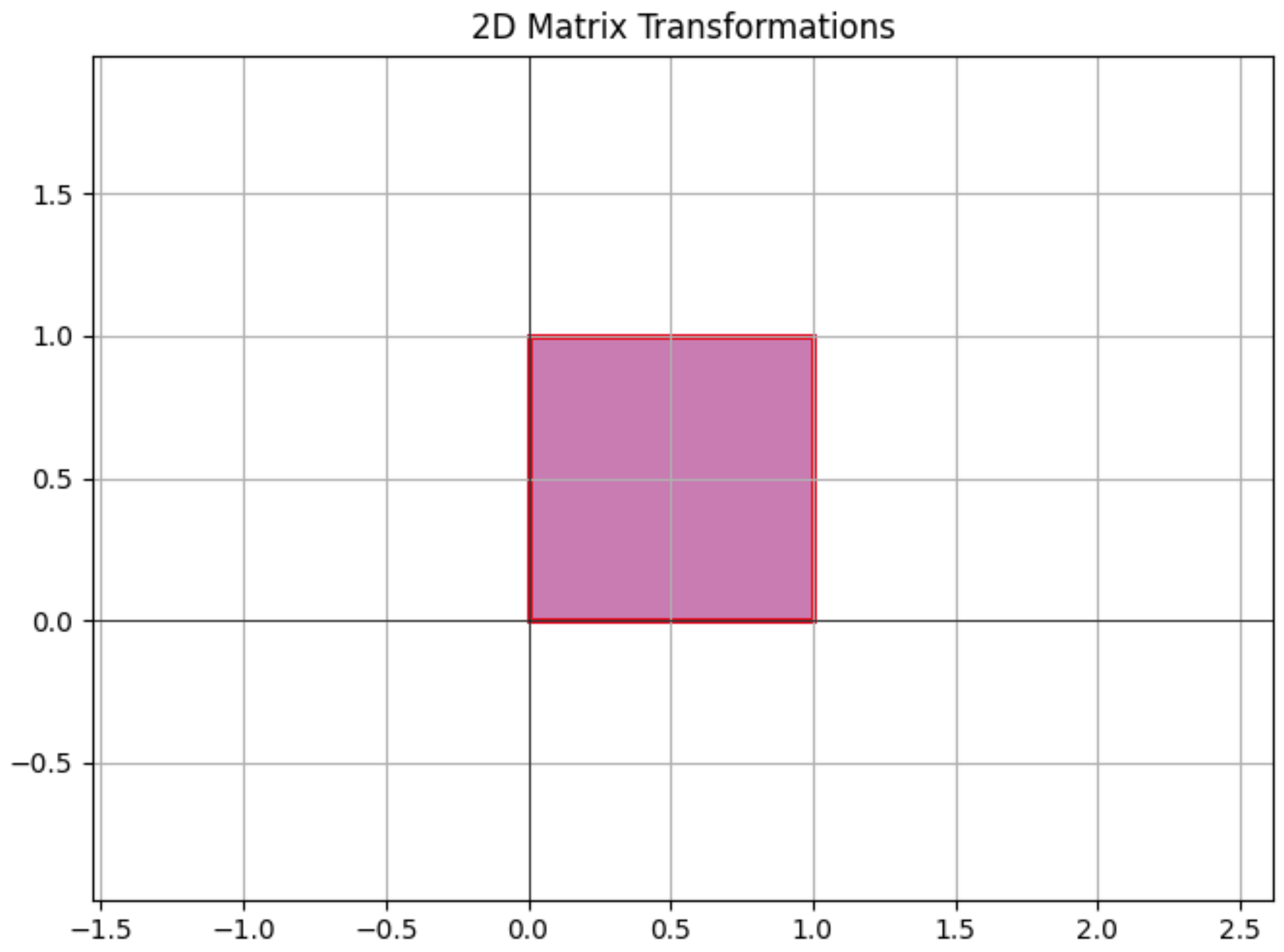
$$\begin{bmatrix} 1 & 0 \\ -1.5 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

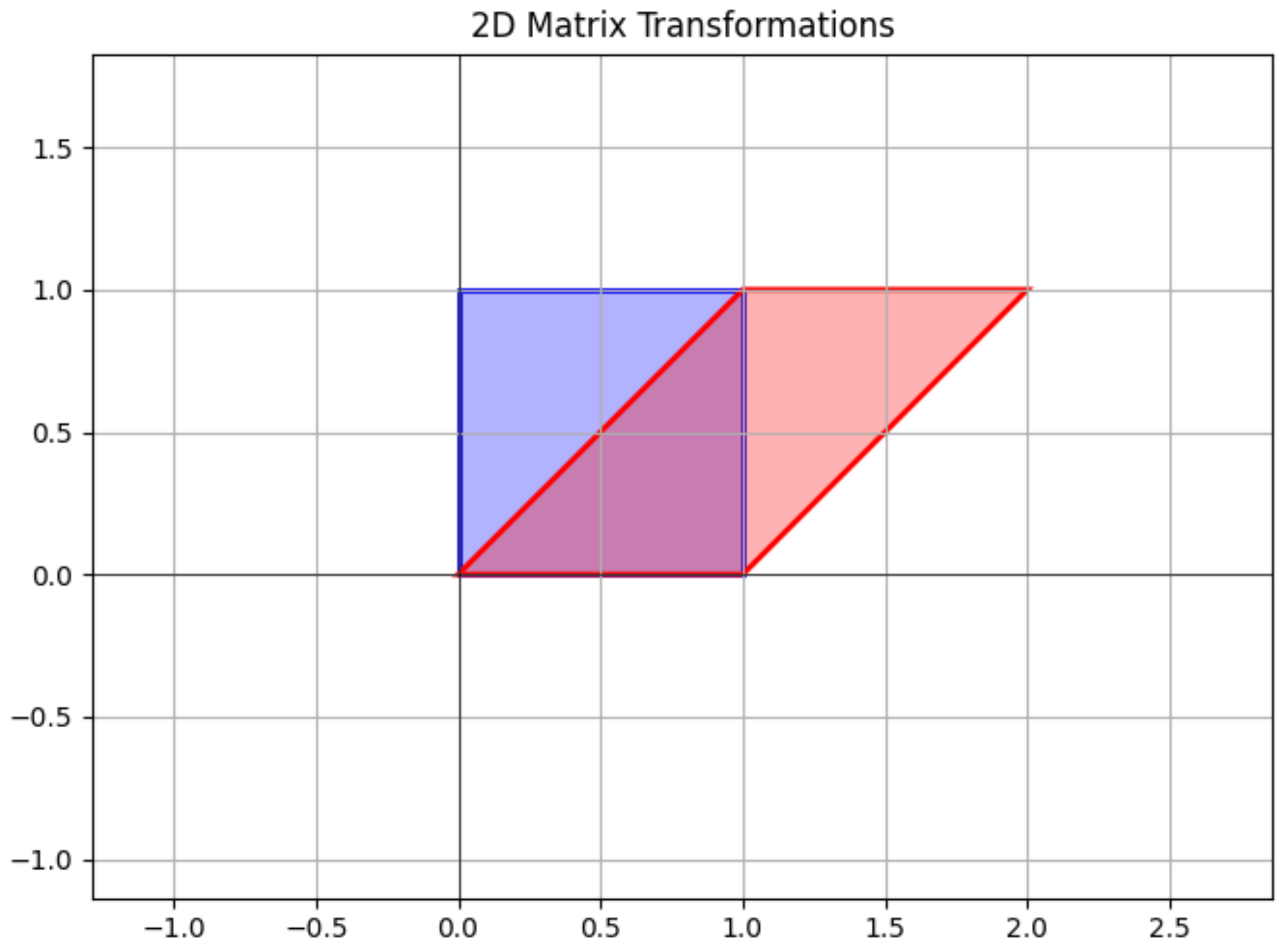


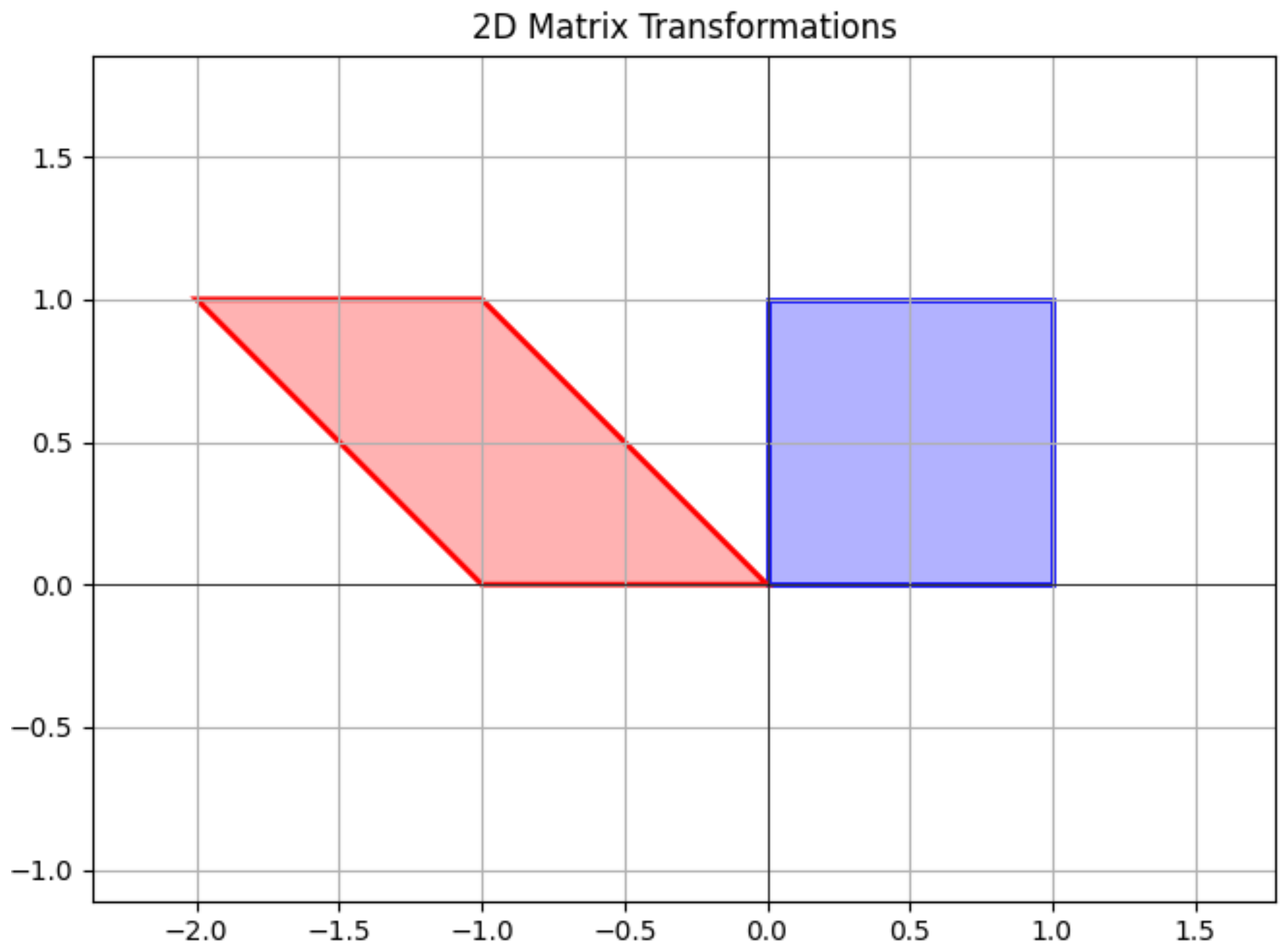
(moving on)

Composing Linear Transformations

Shearing and Reflecting (Geometrically)







shear

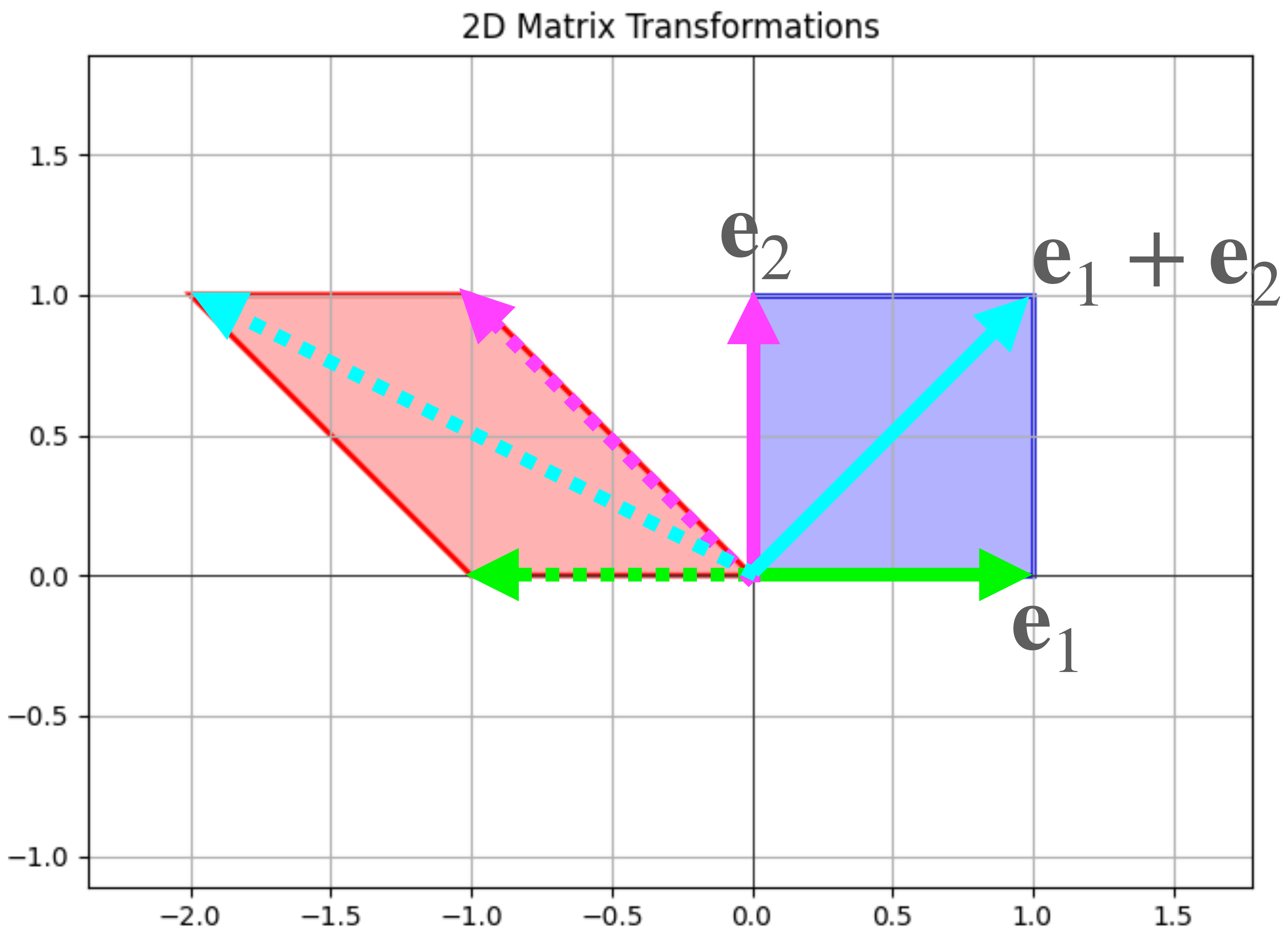
reflect

Shearing and Reflecting Matrix

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$



Shearing and Reflecting (Algebraically)

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right)$$

`reflect` `shear`

First multiply by shear matrix, then multiply by reflection matrix

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

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Shearing and Reflecting (Algebraically)

$$\underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{reflect}} \left(\underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{\text{shear}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right)$$

First multiply by shear matrix, then multiply by reflection matrix

This gives us the same transformation.

Shearing and Reflecting

$$\begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{x} \right)$$

The Key Fact

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Fact. The composition of two linear transformation is a linear transformation.

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Verify: $T(S(a\vec{u}+b\vec{v})) = aT(S(\vec{u})) + b(T(S(\vec{v})))$

The Key Fact

Fact. The composition of two linear transformation is a linear transformation.

Verify:

This means the composition of two matrix transformation can be represented as a *single* matrix.

The Key Question

Given two linear transformations, how to we compute the matrix which implements their composition?

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Matrix Multiplication

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Shearing and Reflecting

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) =$$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \left(x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) =$$

$$x_1 \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\vec{b}_1} + x_2 \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\vec{b}_2} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 \end{bmatrix} \vec{x}$$

General Composition (2D)

$$A\left(\begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = \begin{bmatrix}A\vec{b}_1 & A\vec{b}_2\end{bmatrix}\vec{x}$$

$$A(x_1\vec{b}_1 + x_2\vec{b}_2) =$$

$$x_1 A\vec{b}_1 + x_2 A\vec{b}_2 =$$

Matrix Multiplication

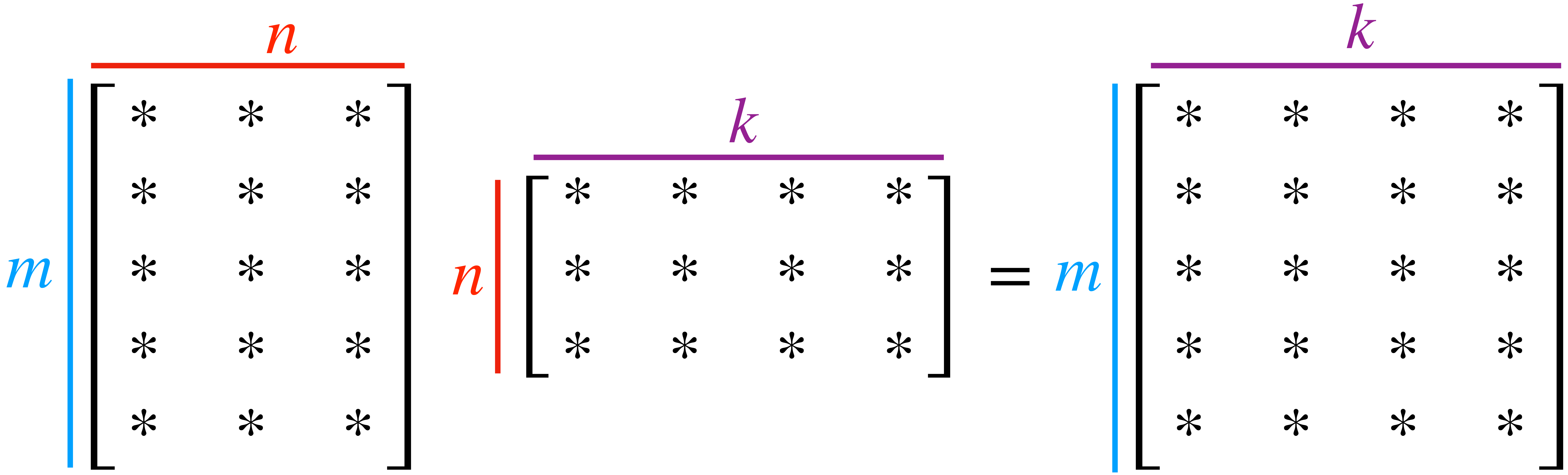
Definition. For a $m \times n$ matrix A and a $n \times p$ matrix B with columns $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p$ the product AB is the $m \times p$ matrix given by

$$AB = A\begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \ldots & \mathbf{b}_p\end{bmatrix} = \begin{bmatrix}A\mathbf{b}_1 & A\mathbf{b}_2 & \ldots & A\mathbf{b}_p\end{bmatrix}$$

Replace each column of B with A multiplied by that column.

Tracking Dimensions

this only works if the number of <u>columns</u> of the left matrix matches the number of <u>rows</u> of the right matrix



$(m \times n)$ $(n \times k)$ $(m \times k)$

Important Note

Even if AB is defined, it may be that BA is <u>not</u> defined

$A\vec{v}$ is defined

$\vec{v}A$ is not

Non-Example

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \left[\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right]$$

2 × 3 ✗ 2 × 2

Non-Example

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \left[\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right]$$

2×3 2×1

These are not defined.

Example

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \left[\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix} \right]$$

2×2 $\quad$ 2×3 $\quad$ 2×2 $\quad$ $\mathbb{R}^2$

The Key Fact (Restated)

For any matrices A and B (such that AB is defined) and any vector $\mathbf{v}$

$$A(B\mathbf{v}) = (AB)\mathbf{v}$$

The matrix implementing the composition is the product of the two underlying matrices.

Row-Column Rule

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$$

Given a $m \times n$ matrix A and a $n \times p$ matrix B, the entry in row i and column j of AB is defined above.

Example

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} -1(1) + 0(0) & (-1)1 + 0(1) \\ \ldots & \ldots \end{bmatrix}$$

Row-Column Rule (Pictorially)

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

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Row-Column Rule (Pictorially)

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

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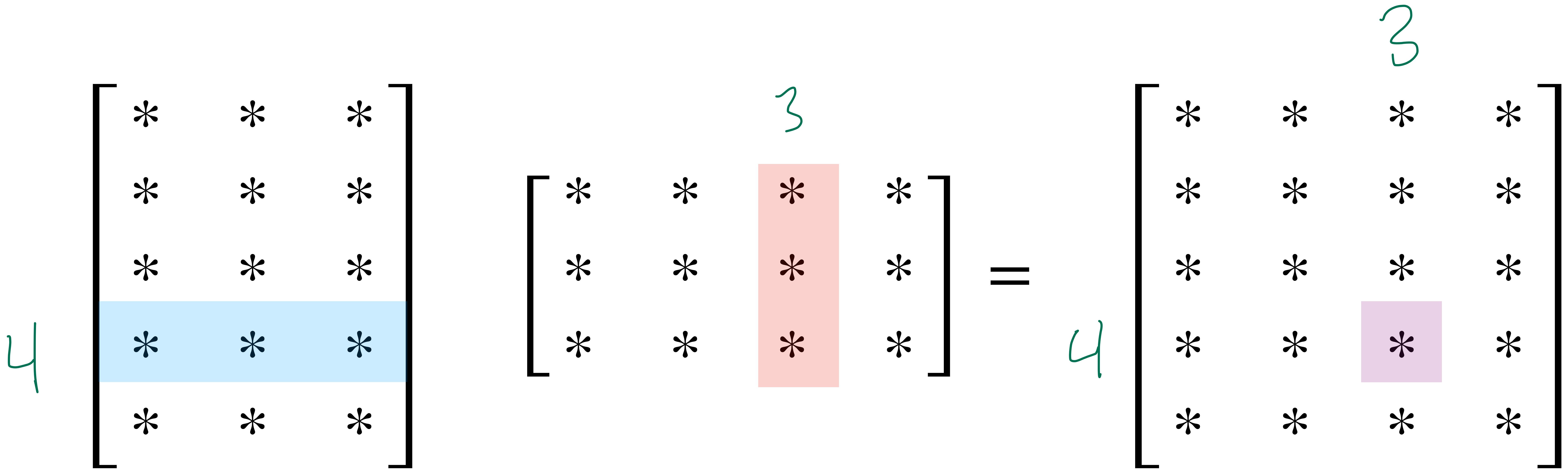

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Question

Compute $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 2 \end{bmatrix}$

short version: What is the entry in the 2nd row and 2nd column?

Answer

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 2 \end{bmatrix}$$

Matrix Operations

Connection with Matrix-Vector Multiplication

Connection with Matrix-Vector Multiplication

```
What about when the right matrix is a single column?
```

Connection with Matrix-Vector Multiplication

`What about when the right matrix is a single column?`

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Connection with Matrix-Vector Multiplication

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$$A[\mathbf{b}_1] = [A\mathbf{b}_1] = A\mathbf{b}_1$$

This is just vector multiplication.

We can think of $[A\mathbf{b}_1 \;\; A\mathbf{b}_2 \;\; \ldots \;\; A\mathbf{b}_p]$ as collection of simultaneous matrix–vector multiplications

Matrix "Interface"

multiplication	what does AB mean when A and B are matrices?
addition	what does $A+B$ mean when A and B are matrices?
scaling	what does cA mean when A is matrix and c is a real number?

Matrix "Interface"

multiplication	what does AB mean when A and B are matrices?
addition	what does $A + B$ mean when A and B are matrices?
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These should be consistent with matrix-vector interface and vector interface

Matrix Addition

$$[\mathbf{a}_1 \quad \dots \quad \mathbf{a}_n] + \left[\mathbf{b}_1 \dots \quad \mathbf{b}_n\right] = \left[(\mathbf{a}_1 + \mathbf{b}_1) \quad \dots \quad (\mathbf{a}_n + \mathbf{b}_n)\right]$$

Addition is done column-wise (or equivalently, element-wise)

e.g. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} (1+2) & (2+3) \\ (3-2) & (4-3) \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 1 & 1 \end{bmatrix}$

Matrix Addition

$m \times n \qquad m \times n \qquad m \times$

$$[\mathbf{a}_1 \quad \dots \quad \mathbf{a}_n] + [\mathbf{b}_1 \dots \quad \mathbf{b}_n] = [(\mathbf{a}_1 + \mathbf{b}_1) \quad \dots \quad (\mathbf{a}_n + \mathbf{b}_n)]$$

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This is exactly the same as vector addition, but for matrices.

Matrix Addition and Scaling

$$c\,[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n] = [c\mathbf{a}_1 \quad c\mathbf{a}_2 \quad \cdots \quad c\mathbf{a}_n]$$

Scaling and adding happen element-wise (or, equivalently, column-wise).

e.g. $2\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2(1) & 2(2) \\ 2(-1) & 2(3) \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -2 & 6 \end{bmatrix}$

Matrix Addition and Scaling

$$c\,[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n] = [c\mathbf{a}_1 \quad c\mathbf{a}_2 \quad \cdots \quad c\mathbf{a}_n]$$

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This is exactly the same as vector scaling, but for matrices.

Algebraic Properties (Addition and Scaling)

In these properties A, B, and C are matrices of the same size and r and s are scalars ($\mathbb{R}$)

$$A + B = B + A$$

$$(A + B) + C = A + (B + C)$$

$$A + 0 = A$$

$$r(A + B) = rA + rB$$

$$(r + s)A = rA + sA$$

$$r(sA) = (rs)A$$

Now we need to know/memorize these.

Algebraic Properties (Addition and Scaling)

In these properties A, B, and C are matrices of the appropriate size so that everything is defined, and r is a scalar

$$A(BC) = (AB)C$$

$$A(B + C) = AB + AC$$

$$(B + C)A = BC + CA$$

$$r(AB) = (rA)B = A(rB)$$

$$I_m A = A = AI_n$$

Now we need to know/memorize these.

Verifying $A(B + C) = AB + AC$

$$A\left(\left[\vec{b}_1 \;\; \vec{b}_2\right] + \left[\vec{c}_1 \;\; \vec{c}_2\right]\right) = AB + AC$$

$$A\left[\vec{b}_1 + \vec{c}_1 \quad \vec{b}_2 + \vec{c}_2\right] =$$

$$\left[A(\vec{b}_1 + \vec{c}_1) \quad A(\vec{b}_2 + \vec{c}_2)\right] =$$

$$\left[A\vec{b}_1 + A\vec{c}_1 \quad A\vec{b}_2 + A\vec{c}_2\right] =$$

$$\left[A\vec{b}_1 \quad A\vec{b}_2\right] + \left[A\vec{c}_1 + A\vec{c}_2\right] =$$

Matrix Multiplication is not Commutative

Important. AB may not be the same as BA

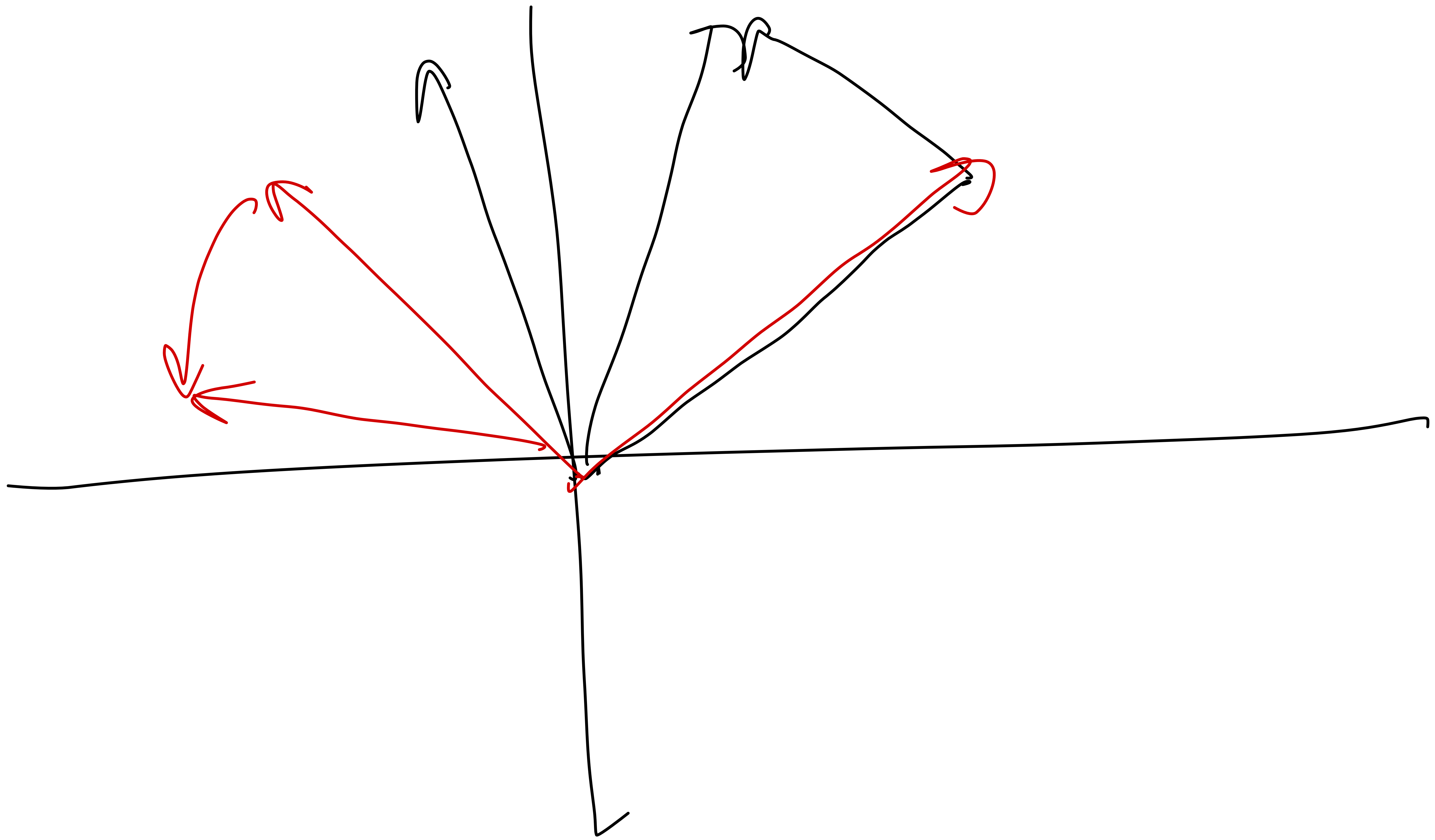
(it may not even be defined)

Question (Conceptual)

Find a pair of 2D linear transformations T_1 and T_2 such that T_1 followed by T_2 is not the same as T_2 followed by T_1.

(also find a pair where they are the same)

Answer: Rotation and Reflection



Computational Aspects of Matrix Multiplication

Matrix Operations in Numpy

Let a and b be 2D numpy arrays and let c be a floating point number.

» a @ b   (matrix multiplication)

» a + b   (matrix addition)

» c * a   (matrix scaling)

We've seen these, we've used them a bit, we'll use them much more.

A Note on Complexity

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$$

Suppose A and B are $n \times n$ matrices.

This operations takes n multiplications and n divisions ($2n$ FLOPS total)

Repeating for each entry gives $\sim 2n^3$ FLOPS

A Note on Parallelization

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$$

The main part of this procedure is highly parallelizable.

A Note on Parallelization

```
a = np.array(...)
b = np.array(...)
prod = np.zeros([a.shape[0], b.shape[1]])
for i in range(a.shape[0]):
    for j in range(b.shape[1]):
        prod[i, j] = np.dot(a[i], b[:,j])
```

The main part of this procedure is highly parallelizable.

One processor per entry gets you to $\sim 2n$ FLOPS

A Note on Libraries

There are a lot of other considerations for doing linear algebra on computers.

Best leave it to experts (or do research in the area).

LAPACK is the state of the art library for matrix operations.

numpy uses LAPACK

Summary

We can reason about matrix equations by reasoning directly about properties of linear transformations.

Matrix multiplication coincides with composition of linear transformations.

There is an algebra of matrices which is consistent with the algebra of vectors.