Matrix Inverses

Geometric Algorithms
Lecture 10

CAS CS 132

Objectives

1. Define a few more important matrix operations
2. Motivate and define matrix inverses
3. Application: Adjacency Matrices

Keywords

Matrix Transpose

Inner Product

Matrix Power

Square Matrix

Matrix Inverse

Invertible Transformation

1–1 Correspondence

numpy.linalg.inv

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Invertible Matrix Theorem

Recap Problem

Suppose that A, $B = [\mathbf{b}_1 \ \ \mathbf{b}_2 \ \ \mathbf{b}_3]$ and $C = [\mathbf{c}_1 \ \ \mathbf{c}_2 \ \ \mathbf{c}_3]$ are matrices such that

$$A(B + 5I) = C$$

I : identity matrix

$$\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

Find a solution to the equation $A\mathbf{x} = \mathbf{c}_2$.

hint: your solution should have $\vec{e}_2$ (the standard basis vector)

Answer: $\mathbf{b}_2 + 5\mathbf{e}_2$

$$A(B + 5I) = A\left(\begin{bmatrix}\vec{b}_1 & \vec{b}_2 & \vec{b}_3\end{bmatrix} + 5\begin{bmatrix}\vec{e}_1 & \vec{e}_2 & \vec{e}_3\end{bmatrix}\right)$$

$$= A\begin{bmatrix}(\vec{b}_1 + 5\vec{e}_1) & (\vec{b}_2 + 5\vec{e}_2) & (\vec{b}_3 + 5\vec{e}_3)\end{bmatrix}$$

$$= \begin{bmatrix}\cdots & A(\vec{b}_2 + 5\vec{e}_2) & \cdots\end{bmatrix}$$

$$= \begin{bmatrix}\vec{c}_1 & \vec{c}_2 & \vec{c}_3\end{bmatrix}$$

$$A(B + 5I) = C$$

$$A(\vec{b}_2 + 5\vec{e}_2) = \vec{c}_2$$

solution

More Matrix Operations

Transpose (Pictorially)

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \\ 5 & 5 & 5 & 5 \end{bmatrix}$$

Transpose (Pictorially)

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Transpose

Definition. For a $m \times n$ matrix A, the **transpose of** A, written A^T, is the $n \times m$ matrix such that

$$(A^T)_{ij} = A_{ji}$$

Example.

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

Algebraic Properties (Transpose)

$((A)^T)^T_{ij} = A^T_{ji} = A_{ij}$

$(A^T)^T = A$

$(A + B)^T = A^T + B^T$

$(cA)^T = cA^T$ (where c is a scalar)

$(AB)^T = B^T A^T$

Algebraic Properties (Transpose)

$(A^T)^T = A$

$(A + B)^T = A^T + B^T$

$(cA)^T = cA^T$ (where c is a scalar)

$(AB)^T = B^T A^T$ Important: the order reverses!

Challenge Problem (Not In-Class)

Show that $(AB)^T = B^T A^T$.

Example: $\left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\right)^T = \begin{pmatrix} 1(1) + 0(1) & 1(1) + 0(0) \\ 1(1) + (1) & 1(1) + 1(0) \end{pmatrix}^T = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}^T = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$

Transposes and Inner Products

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For two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$,
is $\mathbf{u}^T\mathbf{v}$ defined?

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$1 \times n \qquad n \times 1 \qquad 1 \times 1$

$$\begin{bmatrix} u_1 & u_2 & u_3 & u_4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \quad ?$$

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Transposes and Inner Products

Transposes and Inner Products

$$\begin{bmatrix} u_1 & u_2 & u_3 & u_4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4$$

Transposes and Inner Products

$[1 \quad 2 \quad 3]^T$

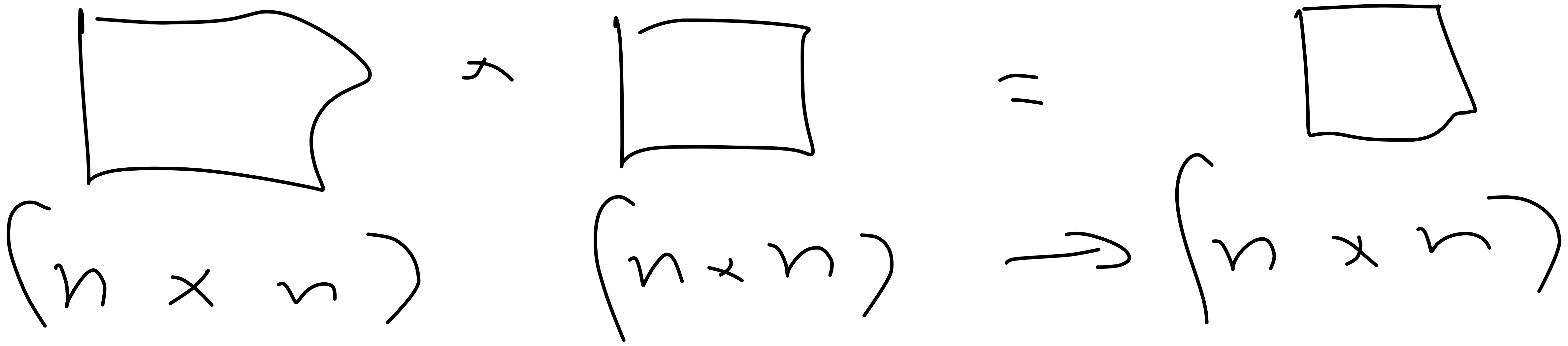
$$[u_1 \quad u_2 \quad u_3 \quad u_4] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$$

Definition. The **inner product** of two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ is

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}$$

Matrix Powers

Matrix Powers



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(we want $A^0 A^k = A^{0+k} = A^k$)

Final Warnings about Matrix Multiplication

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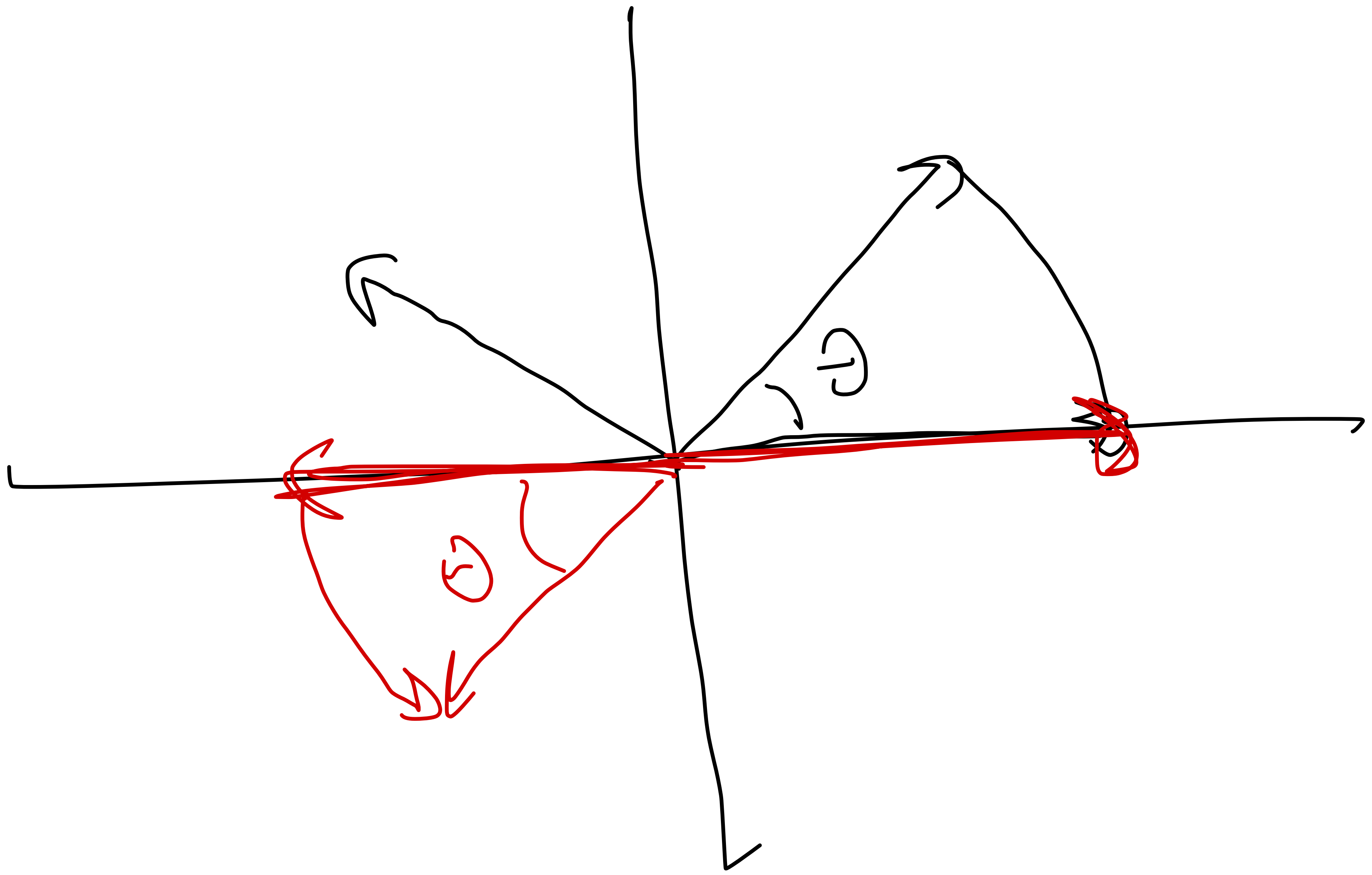
1. AB is not necessarily equal to BA, even if both are defined.

rotation then reflection $\neq$ refl. then rot.



Final Warnings about Matrix Multiplication

1. AB is not necessarily equal to BA, even if both are defined.

2. If $AB = AC$ then it is not necessary that $B = C$.

   $A = O$ $\qquad$ $OB = OC$

   $B \neq C$

Final Warnings about Matrix Multiplication

1. AB is not necessarily equal to BA, even if both are defined.

2. If $AB = AC$ then it is not necessary that $B = C$.

3. If $AB = 0$ (the zero matrix) it is not necessarily the case that $A = 0$ or $B = 0$.

Question

Find two nonzero 2×2 matrices A and B such that $AB = 0$.

Challenge. *Choose A and B such that they have* *all* *nonzero entries.*

Answer

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A [\vec{b}_1 \quad \vec{b}_2] = 0$$

$$A\vec{b}_1 = \vec{0}$$

$$A\vec{b}_2 = \vec{0}$$

$$[A\vec{b}_1 \quad A\vec{b}_2] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

So Far: Matrix Operations

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`transpose` A^T

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`scaling`	cA

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`transpose`	A^T	
`scaling`	cA	
`addition (subtraction)`	$A + B$	$A + (-1)B = A - B$
`multiplication (powers)`	AB	A^k

`What's missing?`

Matrix Inverses

Basic Algebra

$$2x = 10$$

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`How do we solve this equation?`

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Divide on both sides by 2 to get $x = 5$.

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Multiply each side by $\frac{1}{2}$ *a.k.a.* 2^{-1}.

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Multiply each side by $\frac{1}{2}$ *a.k.a.* 2^{-1}.

$\frac{1}{2}$ is the **reciprocal** or **multiplicative inverse** of 2.

Basic Algebra

$$2^{-1}(2x) = 2^{-1}(10)$$

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Basic Algebra

$$1x = 5$$

How do we solve this equation?

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Multiply each side by $\frac{1}{2}$ *a.k.a.* 2^{-1}.

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Basic Algebra

$$x = 5$$

How do we solve this equation?

Divide on both sides by 2 to get $x = 5$.

Multiply each side by $\frac{1}{2}$ *a.k.a.* 2^{-1}.

$\frac{1}{2}$ is the **reciprocal** or **multiplicative inverse** of 2.

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$$A\mathbf{x} = \mathbf{b}$$

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$$A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}$$

How do we solve this equation?

Multiply each side by A^{-1} to get $\mathbf{x} = A^{-1}\mathbf{b}$.

A^{-1} is the **multiplicative inverse** of A

Wouldn't it be nice...

$$I\mathbf{x} = A^{-1}\mathbf{b}$$

How do we solve this equation?

Multiply each side by A^{-1} to get $\mathbf{x} = A^{-1}\mathbf{b}$.

A^{-1} is the **multiplicative inverse** of A

Wouldn't it be nice...

$$\mathbf{x} = A^{-1}\mathbf{b}$$

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A^{-1} is the **multiplicative inverse** of A

Do all matrices have inverses?

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No.

When does a matrix have an inverse?

Square Matrices

Definition. A $m \times n$ matrix A is **square** if $m = n$

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

i.e., it has same number of rows as columns.

Why are square matrices special?

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» that can have inverses.

Dimension Tracking

$$A \quad \mathbf{x} = \mathbf{b}$$

$\mathbf{x} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$

$(m \times n)$ $(n \times 1)$ $(m \times 1)$

Dimension Tracking

$k \times 1$

$k \times m \quad m \times 1$

$$A^{-1}A\ \ \mathbf{x} = A^{-1}\mathbf{b}$$

$(k \times m) \ (m \times n) \ (n \times 1)$

$(m \times 1)$

$k = m$

Dimension Tracking

$$\mathbf{x} = A^{-1}\mathbf{b}$$

$\in \mathbb{R}^n$

$k \times 1 = m \times 1$

$n \times 1$

$n \times 1$

$k = m = n$

Dimension Tracking

$$\mathbf{x} = A^{-1}\mathbf{b}$$

The only way for the dimensions to make sense is if A is square

Matrix Inverses

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$$AB = I_n \text{ and } BA = I_n$$

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$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

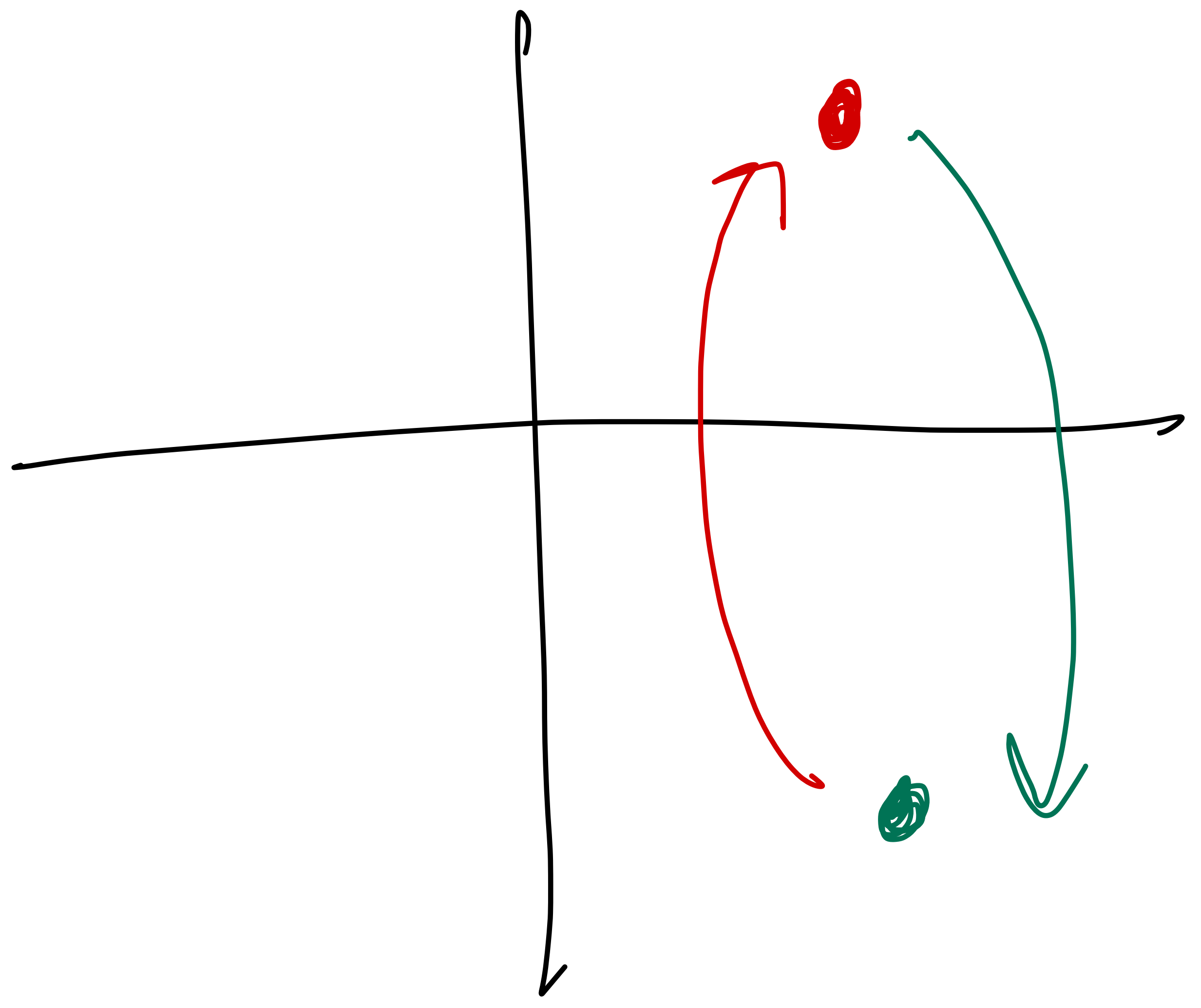
Example. $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

Example: Geometric

Reflection across the x_1-axis in $\mathbb{R}^2$ is it's own inverse.

Verify:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



Example: No inverse

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Verify: $A[\vec{b}_1 \;\; \vec{b}_2 \;\; \vec{b}_3] = [A\vec{b}_1 \;\; A\vec{b}_2 \;\; A\vec{b}_3]$

$$\begin{bmatrix} & & \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \\ 0 \end{bmatrix}$$

Inverses are Unique

Theorem. If B and C are inverses of A, then $B = C$.

Verify: $B = BI = B(AC) = (BA)C = IC = C$

Inverses are Unique

Theorem. If B and C are inverses of A, then $B = C$.

Verify:

If A is invertible, then we write A^{-1} for *the* inverse of A.

Solutions for Invertible Matrix Equations

Theorem. For a $n \times n$ matrix A, if A is invertible then

$$A\mathbf{x} = \mathbf{b}$$

has a <u>unique</u> solution for any choice of $\mathbf{b}$.

Verify:

$x = A^{-1}\vec{b}$ is a solution

Suppose $A\vec{c} = \vec{b}$

$\vec{c} = I\vec{c} = A^{-1}A\vec{c} = A^{-1}\vec{b}$

Unique Solutions

If $A\mathbf{x} = \mathbf{b}$ has a unique solution for any choice of $\mathbf{b}$, then it has

» exactly one solution for any choice of $\mathbf{b}$

Unique Solutions

If $A\mathbf{x} = \mathbf{b}$ has a unique solution for any choice of $\mathbf{b}$, then it has

» at least one solution for any choice of $\mathbf{b}$

» at most one solution for any choice of $\mathbf{b}$

Unique Solutions

If $A\mathbf{x} = \mathbf{b}$ has a unique solution for any choice of $\mathbf{b}$, then it has

» T is onto

» T is one-to-one

where T is implemented by A

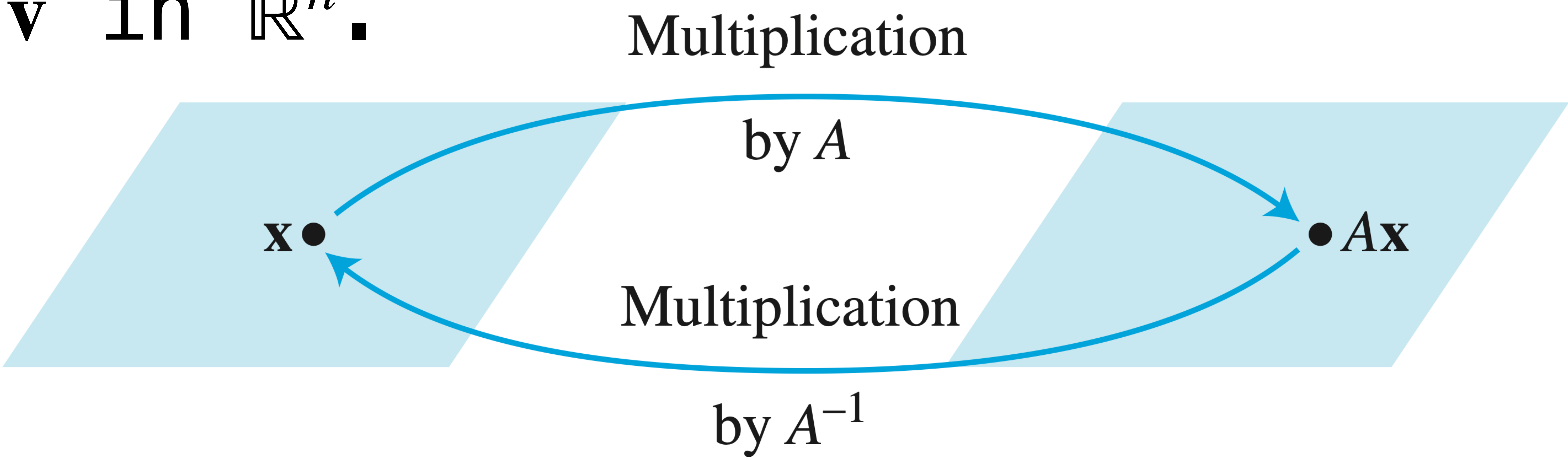
Connection to Transformations

Definition. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is **invertible** if there is a linear transformation S such that

$A^{-1}A\vec{x} = \vec{x}$ $\quad$ $AA^{-1}\vec{x} = \vec{x}$

$$S(T(\mathbf{v})) = \mathbf{v} \text{ and } T(S(\mathbf{v})) = \mathbf{v}$$

for any $\mathbf{v}$ in $\mathbb{R}^n$.



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Theorem. A $n \times n$ matrix A is invertible if and only if the matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$ is invertible.

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A matrix is invertible if it's possible to "undo" its transformation without "losing information".

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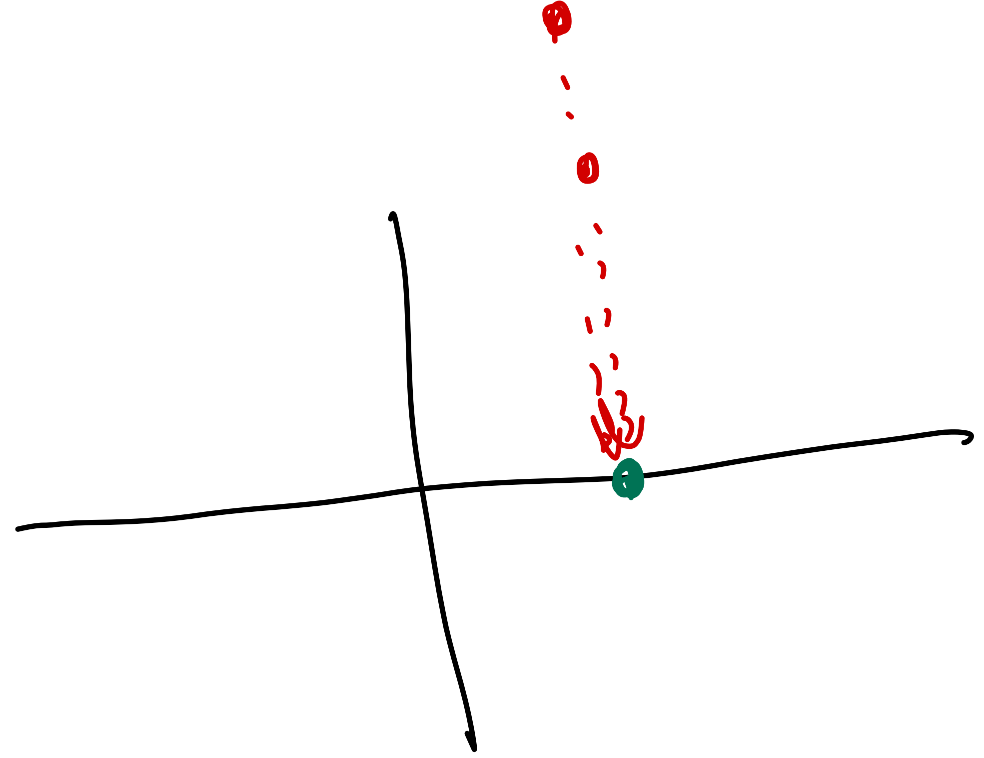

Theorem. A $n \times n$ matrix A is invertible if and only if the matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$ is invertible.

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Non-Example. Projection onto the x_1-axis.

Connection to Transformations

Connection to Transformations

Definition. A transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is a **one-to-one correspondence** (bijection) if any vector $\mathbf{b}$ in $\mathbb{R}^n$ is the image of **exactly** one vector $\mathbf{v}$ in $\mathbb{R}^n$ (where $T(\mathbf{v}) = \mathbf{b}$).

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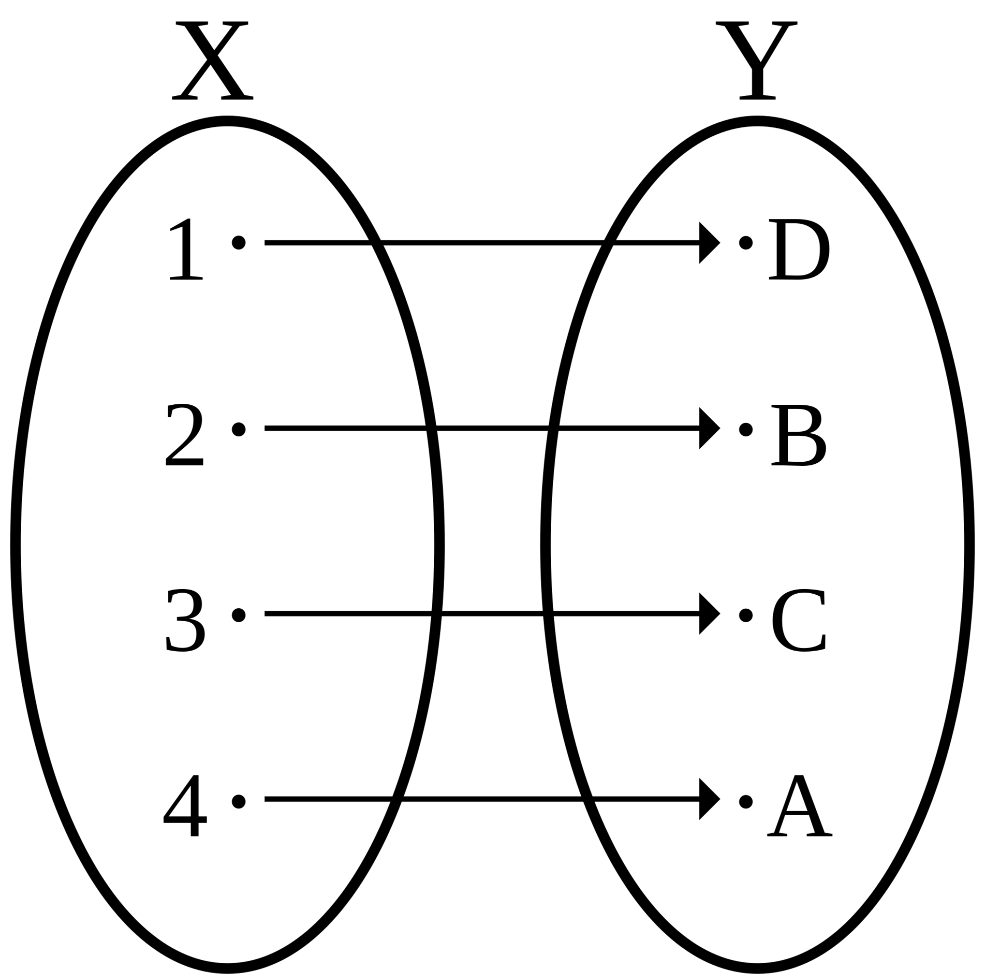
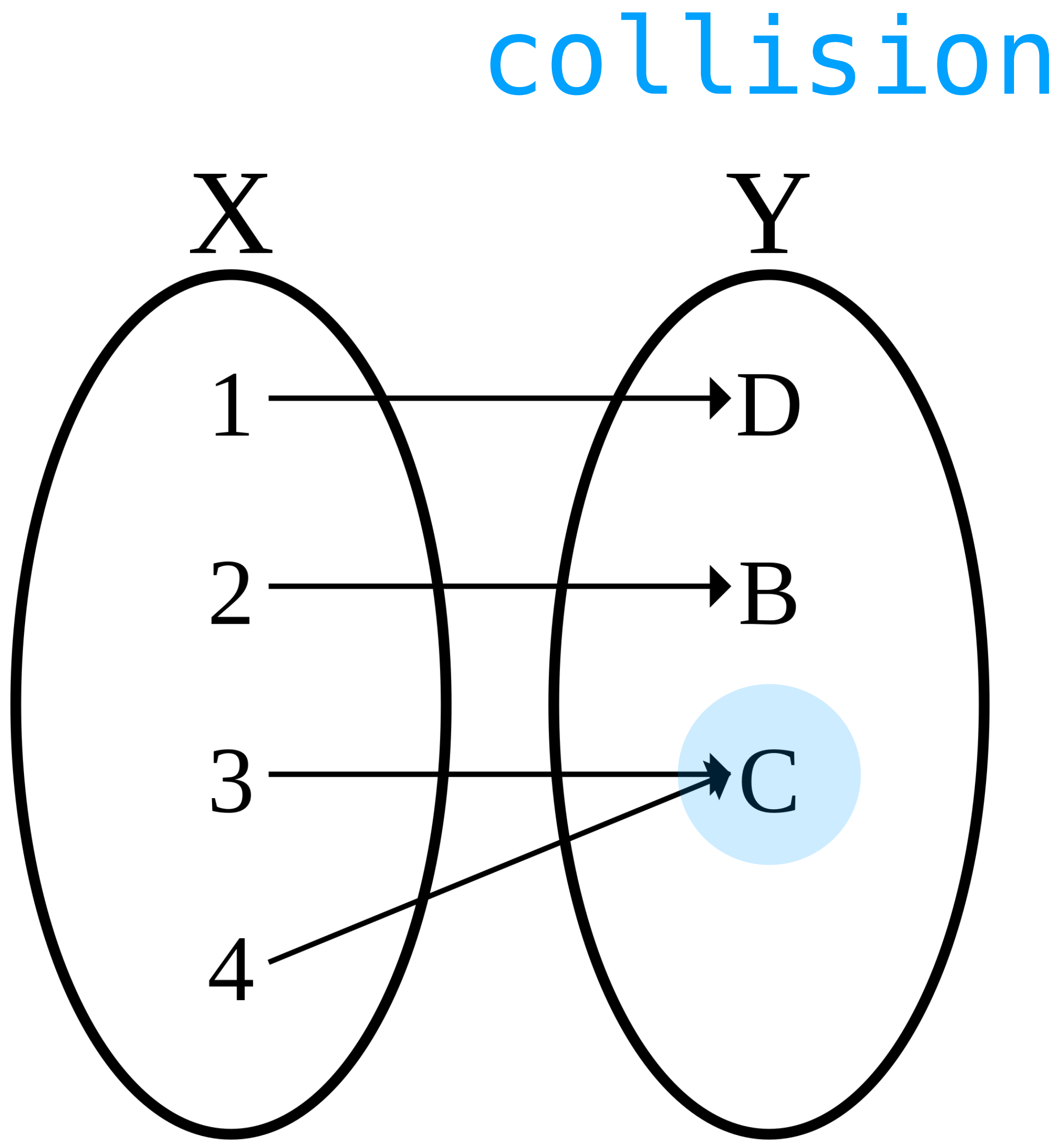
Invertible transformations are 1-1 correspondences.

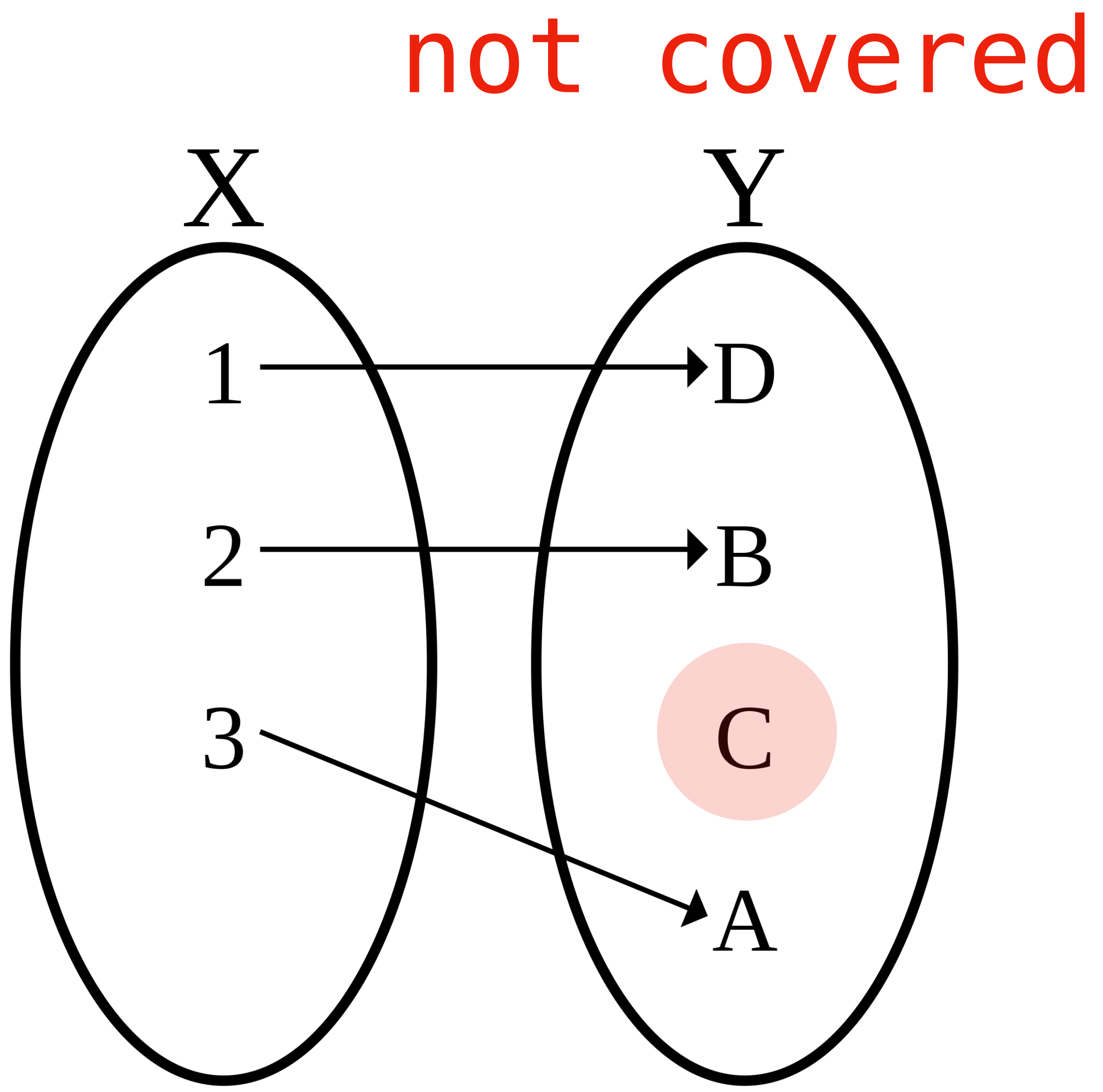
Kinds of Transformations (Pictorially)



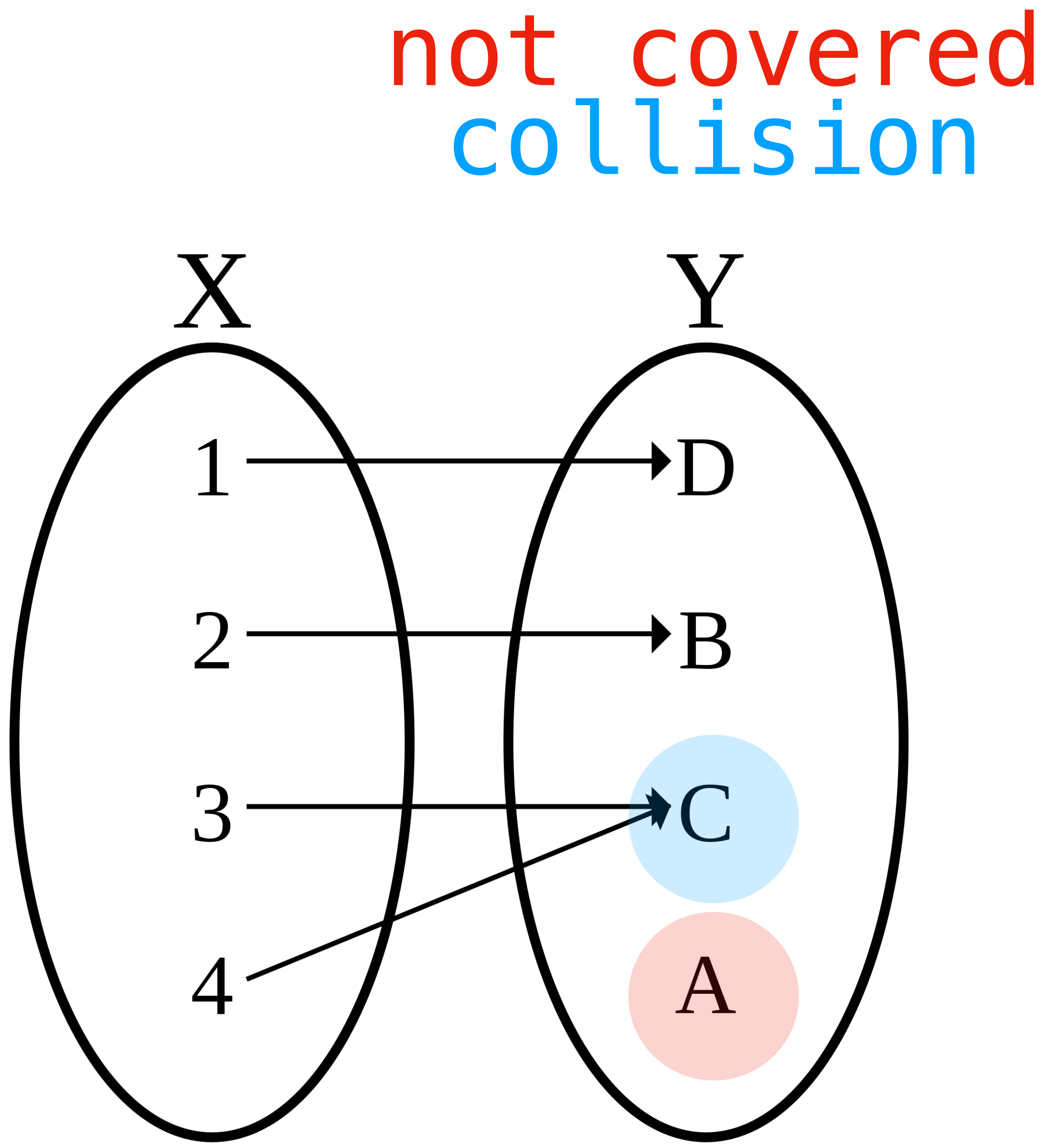
1–1 correspondence



onto, not 1–1



1–1 not onto



not 1–1, not onto

Computing Matrix Inverses

In General

$$A\begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3\end{bmatrix} = I$$

Can we solve for each $\mathbf{b}_i$?:

$$\begin{bmatrix}A\vec{b}_1 & A\vec{b}_2 & A\vec{b}_3\end{bmatrix} = \begin{bmatrix}\vec{e}_1 & \vec{e}_2 & \vec{e}_3\end{bmatrix}$$

$$A\vec{b}_1 = \vec{e}_1$$

How To: Matrix Inverses

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Question. Find the inverse of an invertible $n \times n$ matrix A.

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Solution. Solve the equation $A\mathbf{x} = \mathbf{e}_i$ for every standard basis vector. Put those solutions $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_n$ into a single matrix

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$$[\mathbf{s}_1 \quad \mathbf{s}_2 \quad \cdots \quad \mathbf{s}_n]$$

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Solution. Row reduce the matrix $[A \;\; I]$ to a matrix $[I \;\; B]$. Then B is the inverse of A.

$$\begin{bmatrix} A \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{c|ccc} A & 1 & 0 & 0 \\ & 0 & 1 & 0 \\ & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & A^{-1} \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \end{array}\right]$$

How To: Matrix Inverses

Question. Find the inverse of the $n \times n$ matrix A.

Solution. Row reduce the matrix $[A \quad I]$ to a matrix $[I \quad B]$. Then B is the inverse of A.

This is really the same thing. It's a simultaneous reduction.

How To: Matrix Inverse Computationally

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Warning: this only works if the matrix is invertible.

demo

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(see the notes on linear transformations for more information about determinants)

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No. The determinant is $(-6)(-7) - 14(3) = 42 - 42 = 0$

Algebra of Matrix Inverses

Algebraic Properties (Matrix Inverses)

Theorem. For a $n \times n$ invertible matrix A

$$(A^{-1})^{-1} = A$$

Verify:

Algebraic Properties (Matrix Inverses)

Theorem. For a $n \times n$ invertible matrix A, the matrix A^T is invertible and

$$(A^T)^{-1} = (A^{-1})^T$$

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Algebraic Properties (Matrix Inverses)

Theorem. For a $n \times n$ invertible matrices A and B, the matrix AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

Verify:

Question

Suppose that A is a $n \times n$ invertible matrix such that $A = A^T$ and B is a $m \times n$ matrix.

Simplify the expression $A(BA^{-1})^T$ using the algebraic properties we've seen.

Answer: B^T

$$A(BA^{-1})^T$$

$$A = A^T$$

Invertible Matrix Theorem

High Level

How do we know if a matrix is invertible?

By connecting everything we've said so far.

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14. $\mathbf{x} \mapsto A\mathbf{x}$ is invertible

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Theorem. If A is square, then

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Warning. Remember this only applies square matrices.

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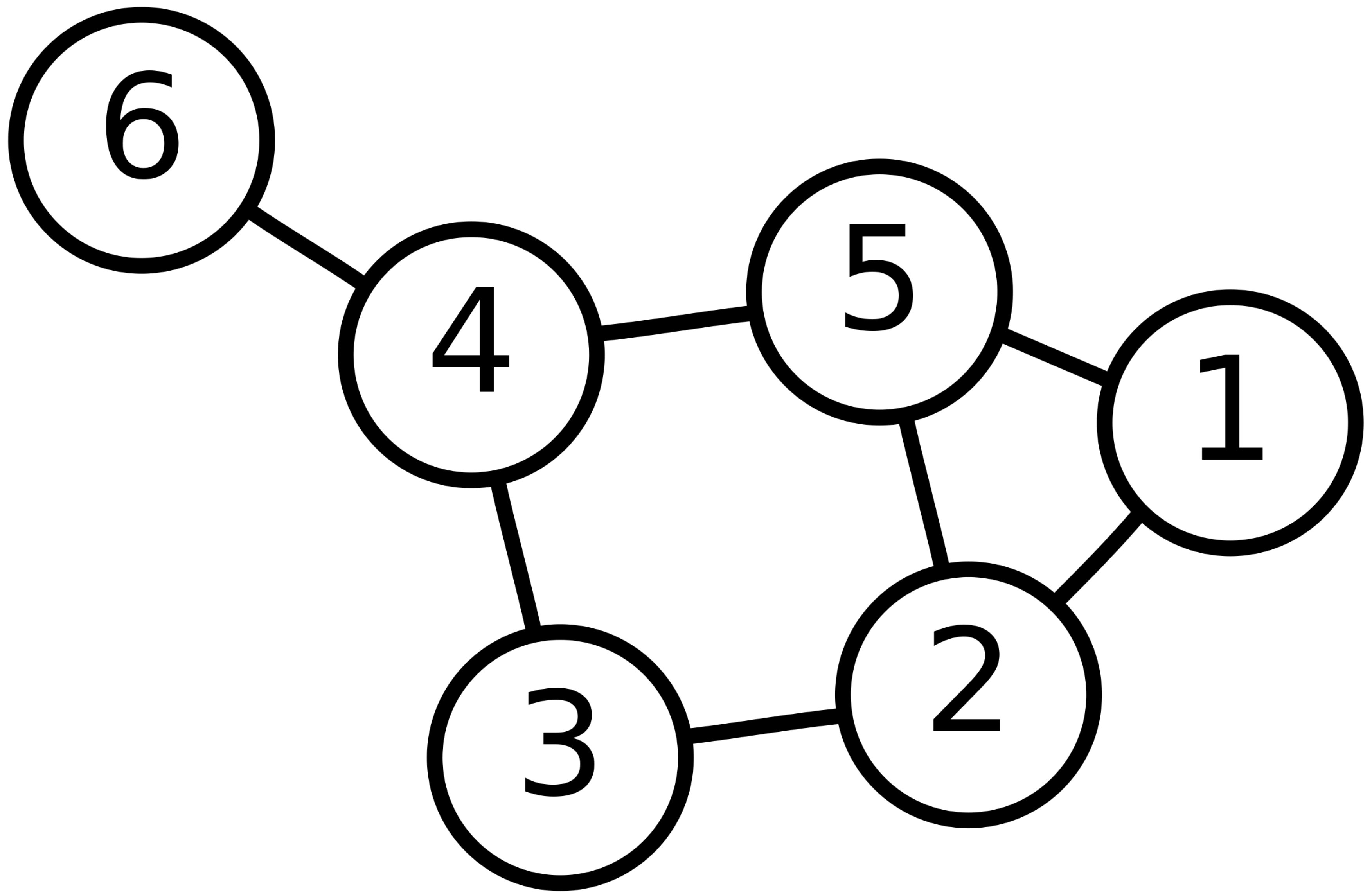
Invertibility is completely determined by how A behaves on $\mathbf{0}$.

Application: Adjacency Matrices

Graphs

Definition (Informal). An **undirected graph** is a collection of nodes with edges between them.



How do we represent these in computers?

Adjacency Matrices

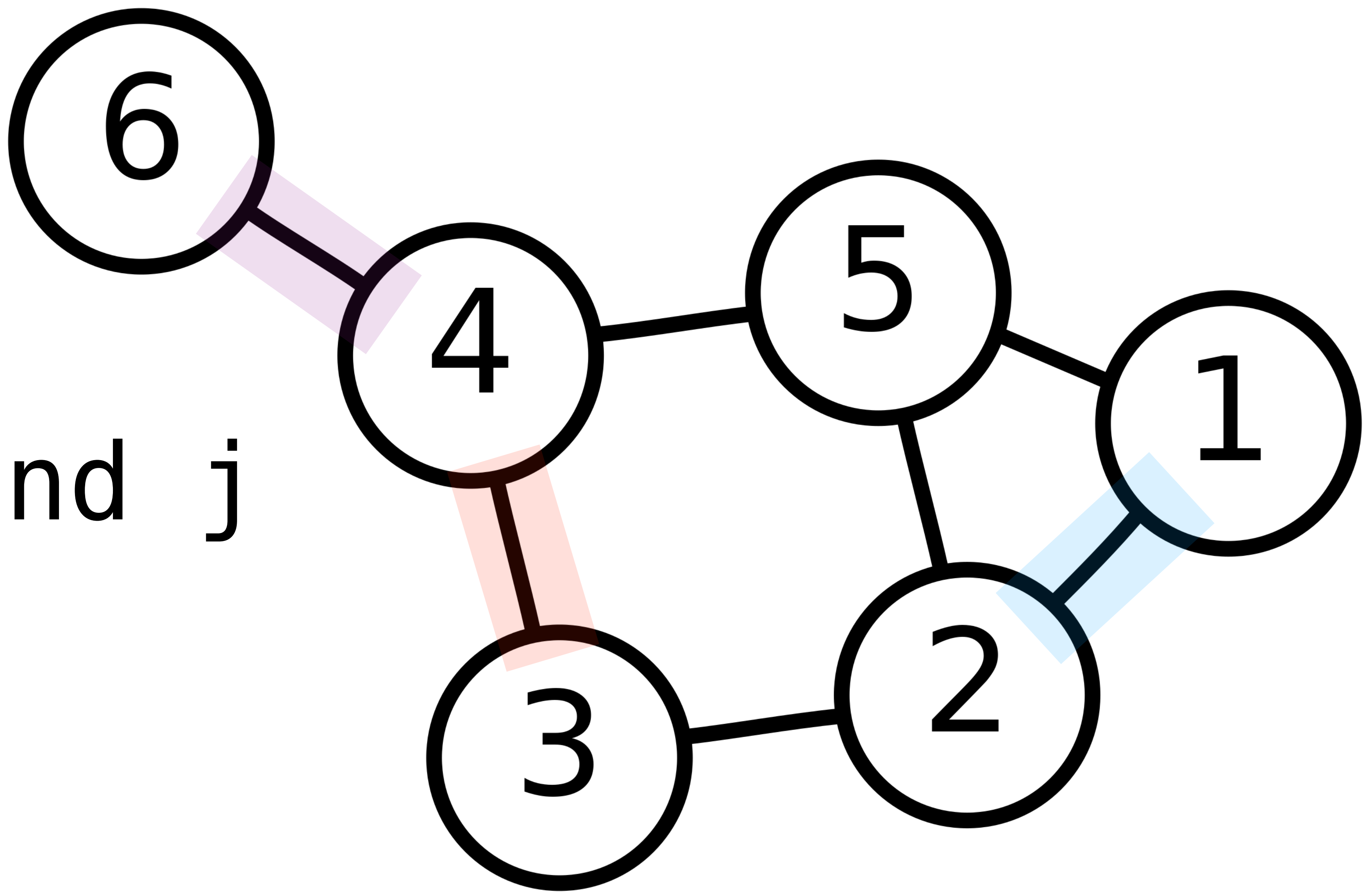
For an undirected graph G we can create the **adjacency matrix** A for G where:

$$A_{ij} = \begin{cases} 1 & \text{there is an edge between i and j} \\ 0 & \text{otherwise} \end{cases}$$

A_{12} A_{34} A_{46}

A_{21} A_{43} A_{64}

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$



Spectral Graph Theory

Once we have an adjacency matrix, we can do linear algebra on graphs.

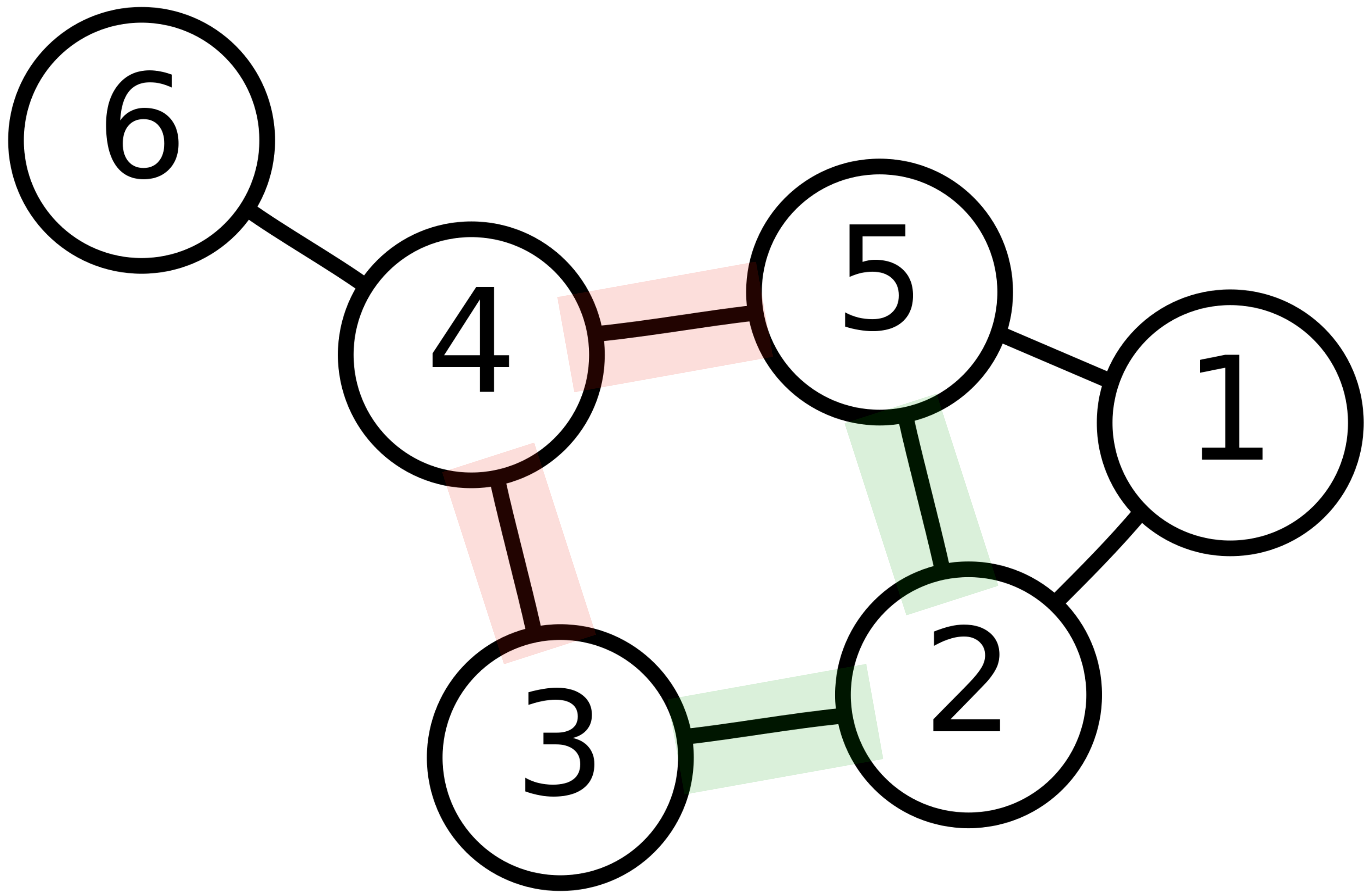
Example: Squared Adjacency Matrices

Given an adjacency matrix A

Can we interpret anything meaningful from A^2?

Example: Squared Adjacency Matrices

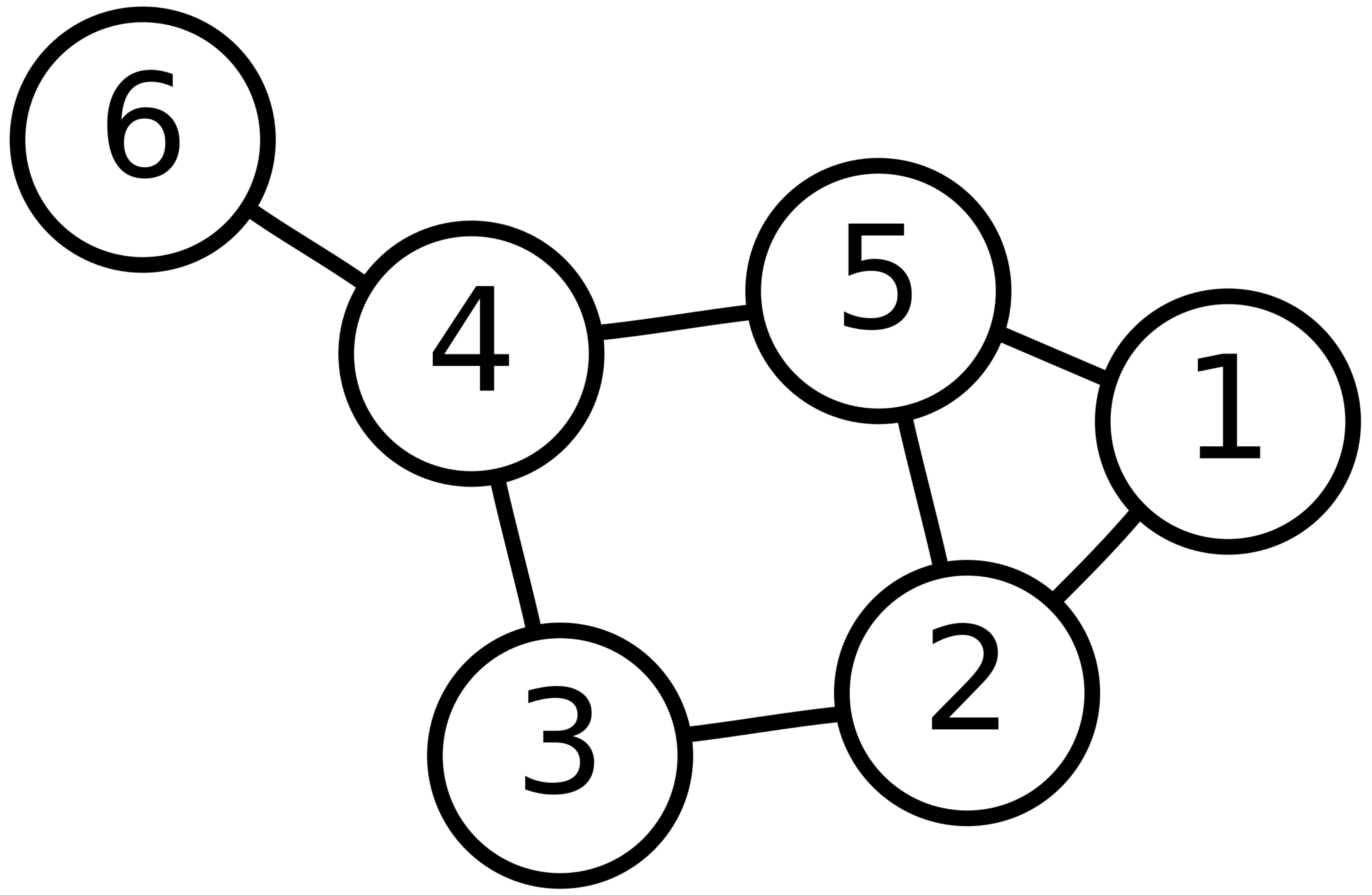
$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$



$$(A^2)_{53} = 1(0) + 1(1) + 0(0) + 1(1) + 0(0) + 0(0) = 2$$

Example: Squared Adjacency Matrices

$$(A^2)_{ij} = A_{i1}A_{1j} + A_{i2}A_{2j} + \ldots + A_{in}A_{nj}$$



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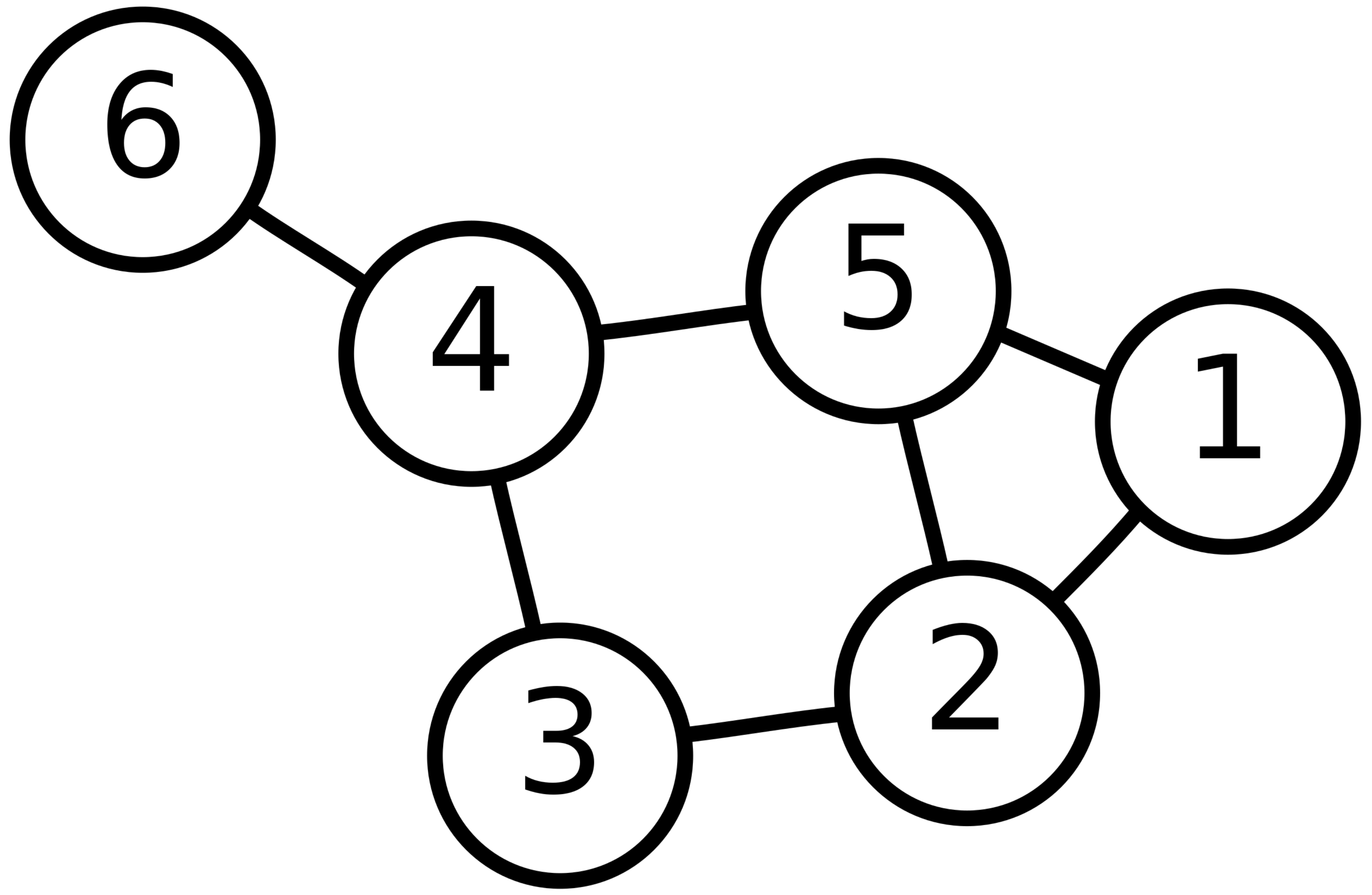

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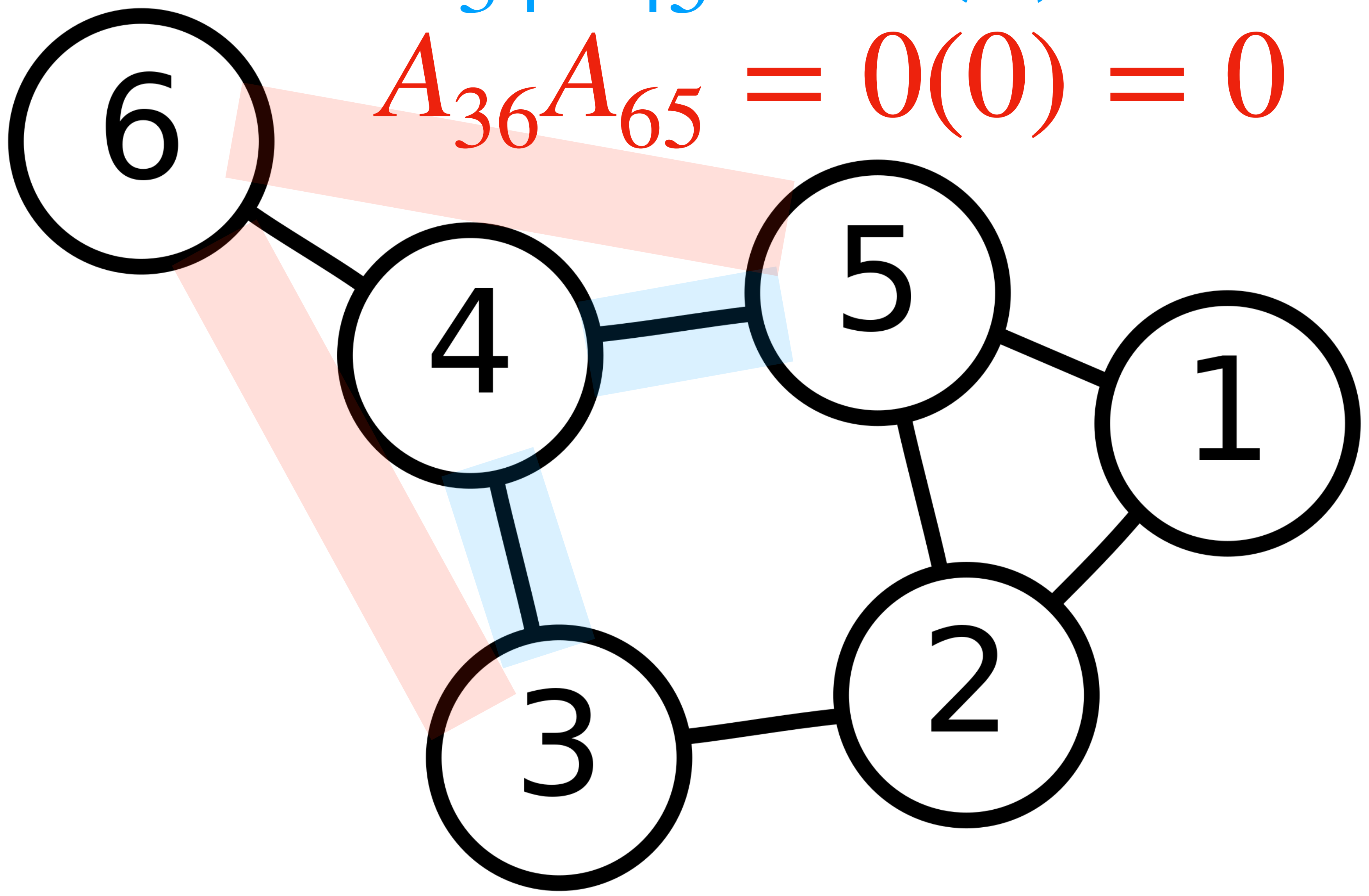
$A_{34}A_{45} = 1(1) = 1$

$A_{36}A_{65} = 0(0) = 0$



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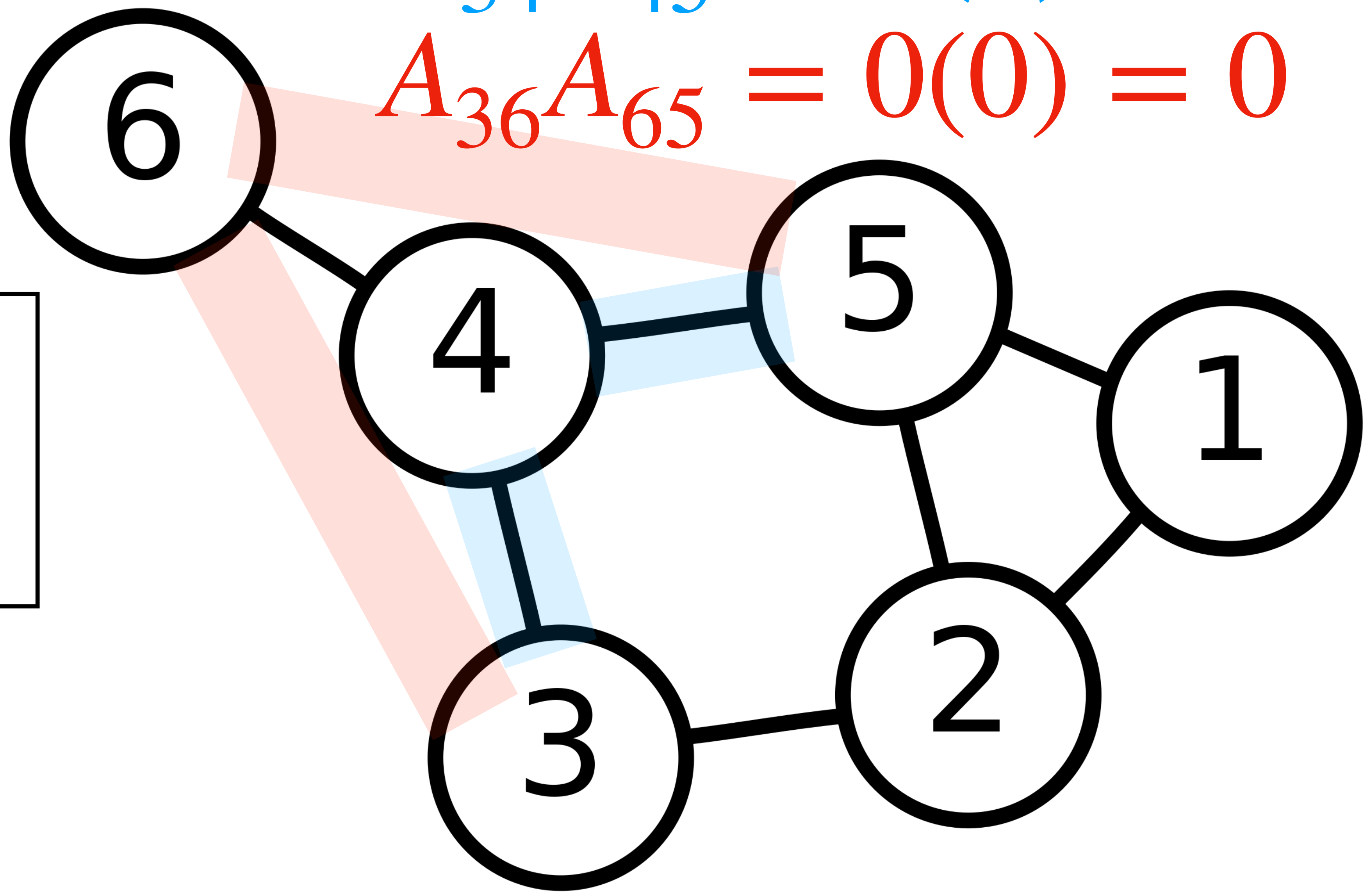
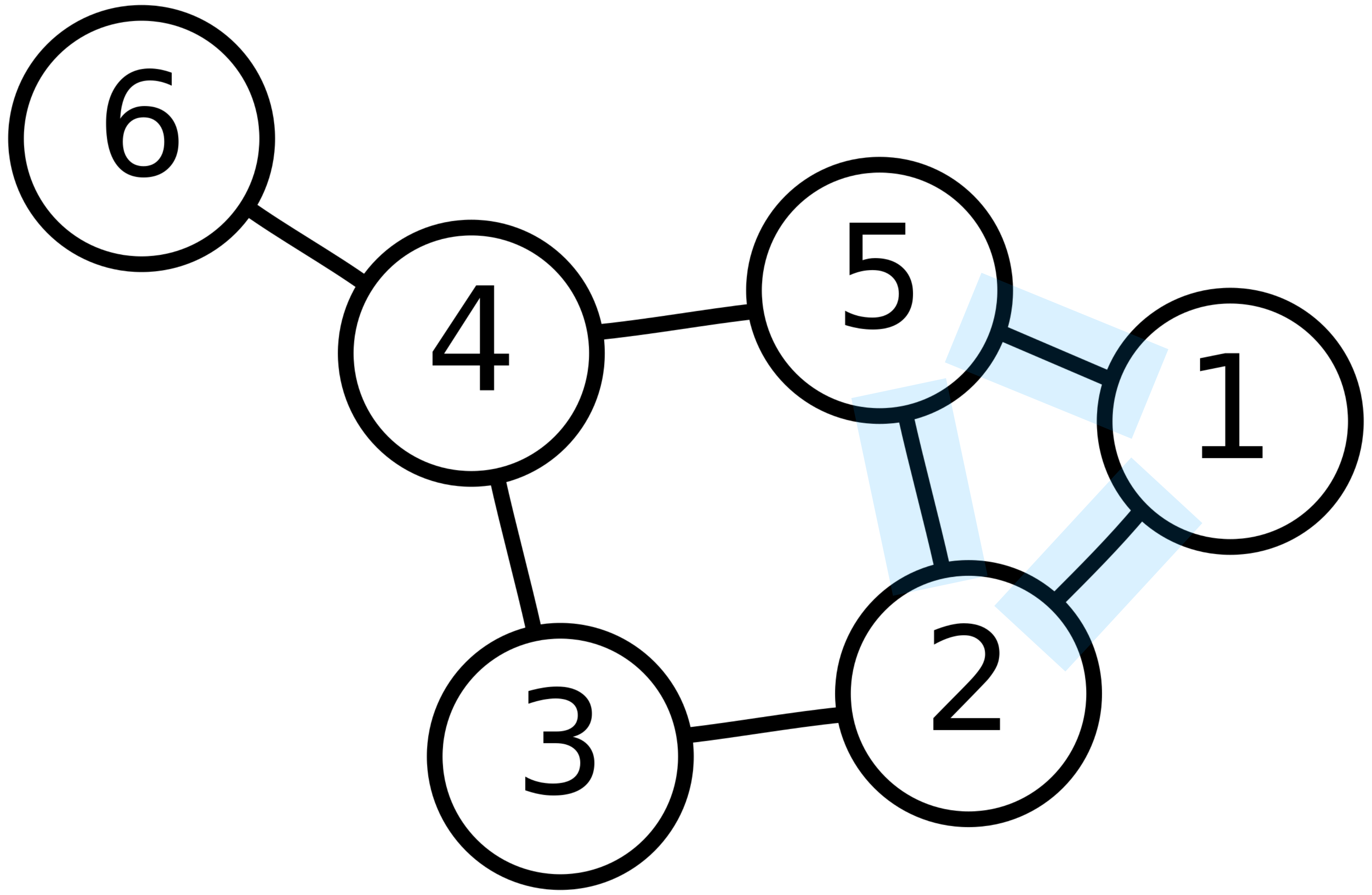


$$(A^2)_{ij} = \boxed{\texttt{number of 2-step paths from i to j}}$$

Application: Triangle Counting

A **triangle** in an undirected graph is a set of three distinct nodes with edges between every pair of nodes.

Triangles in a social network represent mutual friends and tight cohesion (among other things)



Application: Triangle Counting

Theorem. For an adjacency matrix A, the number of triangle containing the edge (i, j) is

$$(A^2)_{ij} A_{ij}$$

Application: Triangle Counting

```
FUNCTION tri_count(A):
    compute A^2
    count ← sum of (A^2)_ij A_ij for all distinct i and j
    RETURN count / 6      # why divided by 6?
```

Summary

We can solve matrix equations by inverting the matrix, though not all matrices have inverses.

We can compute matrix inverses a simultaneous row reduction.

We can connect all the concepts we've defined so far by thinking about them in terms of invertibility (for square matrices).