Invertible Matrix Theorem

Geometric Algorithms
Lecture 11

CAS CS 132

Objectives

1. Recap matrix inverses (it's been a while)
2. Finish up the algebra of matrix inverses
3. Connect everything we've talked about so far via the Invertible Matrix Theorem (IMT)
4. Connect linear algebra to graph theory

Keywords

matrix inverses

invertible matrix theorem

directed/undirected graphs

weighted/unweighted graphs

adjacency matrices

symmetric matrices

triangle counting

Recap

Motivation

$$A\mathbf{x} = \mathbf{b}$$

When can we solve a matrix equation by "*dividing on both sides by A?*"

Motivation

$$A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}$$

When can we solve a matrix equation by "*dividing on both sides by A?*"

Motivation

$$A(A^{-1}b) = \vec{b}$$

$$\mathbf{x} = A^{-1}\mathbf{b}$$

When can we solve a matrix equation by "*dividing on both sides by* A?"

Recall: Matrix Inverses

Recall: Matrix Inverses

Definition. For a $n \times n$ matrix A, an **inverse** of A is a $n \times n$ matrix B such that

$$AB = I_n \text{ (and } BA = I_n\text{)}$$

$(n \times n)$ $(n \times n)$

Recall: Matrix Inverses

Definition. For a $n \times n$ matrix A, an **inverse** of A is a $n \times n$ matrix B such that

$$AB = I_n \text{ (and } BA = I_n\text{)}$$

nonsingular

A is **invertible** if it has an inverse. Otherwise it is **singular**.

Recall: The Identity Matrix

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Definition. The $n \times n$ **identity matrix** is the matrix whose *diagonal* contains all 1s, and all other entries are 0s.

$$I_{ij} = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$

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$$I_{ij} = \begin{cases} 1 & i = j \\ 0 & \texttt{otherwise} \end{cases}$$

Example.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Recall: The Identity Matrix

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The identity matrix implements the "do nothing" transformation. For any $\mathbf{v}$,

$$I\mathbf{v} = \mathbf{v}$$

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$$IA = AI = A$$

Recall: The Identity Matrix

The identity matrix implements the "do nothing" transformation. For any $\mathbf{v}$,

$$I\mathbf{v} = \mathbf{v}$$

It is the "1" of matrices. For any A

$$IA = AI = A$$

These may be different sizes

Recall: The Identity Matrix

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 3 & 4\\5 & 6 & 7 & 8\end{bmatrix} = \begin{bmatrix}1 & 2 & 3 & 4\\5 & 6 & 7 & 8\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 2 & 3 & 4\\5 & 6 & 7 & 8\end{bmatrix}$$

2×2 $\quad 2 \times 4$ $\quad 2 \times 4$ $\quad 4 \times 4$ $\quad 2 \times 4$

Fundamental Questions

How can we determine if a matrix has an inverse?

If a matrix has an inverse how do we compute it?

Fundamental Questions

Answer 1: Try to compute it.

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Fundamental Questions

Answer 1: Try to compute it.

How can we determine if a matrix has an inverse?

If a matrix has an inverse how do we compute it?

Answer 2: the Invertible Matrix Theorem (IMT)

Recall: Computing Inverses in General

$$A \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix} = I$$

If we want a matrix B such that $AB = I$, then the above equation must hold (in the case B has 3 columns).

Can we solve for each $\mathbf{b}_i$?

Recall: In General

$$\begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & A\mathbf{b}_3 \end{bmatrix} = I$$

If we want a matrix B such that $AB = I$, then the above equation must hold (in the case B has 3 columns).

Can we solve for each $\mathbf{b}_i$?

Recall: In General

$$\begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & A\mathbf{b}_3 \end{bmatrix} = [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3]$$

If we want a matrix B such that $AB = I$, then the above equation must hold (in the case B has 3 columns).

Can we solve for each $\mathbf{b}_i$?

Recall: In General

$$A\mathbf{b}_1 = \mathbf{e}_1 \qquad A\mathbf{b}_2 = \mathbf{e}_2 \qquad A\mathbf{b}_3 = \mathbf{e}_3$$

If we want a matrix B such that $AB = I$, then the above equation must hold (in the case B has 3 columns).

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If we want a matrix B such that $AB = I$, then the above equation must hold (in the case B has 3 columns).

Can we solve for each $\mathbf{b}_i$?

We need to solve 3 matrix equations.

Recall: How To: Matrix Inverses

Question. Find the inverse of an invertible $n \times n$ matrix A.

Solution. Solve the equation $A\mathbf{x} = \mathbf{e}_i$ for every standard basis vector. Put those solutions $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_n$ into a single matrix

$$[\mathbf{s}_1 \quad \mathbf{s}_2 \quad \cdots \quad \mathbf{s}_n]$$

Recall: How To: Matrix Inverses

Question. Find the inverse of an invertible $n \times n$ matrix A.

Solution. Row reduce the matrix $[A \;\; I]$ to a matrix $[I \;\; B]$. Then B is the inverse of A.

This is really the same thing. It's a simultaneous reduction.

demo

Algebra of Matrix Inverses

How To: Verifying an Inverse

Question. Given an invertible matrix B and some matrix C, demonstrate that $B^{-1} = C$.

Answer. Show that $BC = I$ (or $CB = I$, but you don't have to do both).

This works because inverses are unique.

Algebraic Properties (Matrix Inverses)

Theorem. For a $n \times n$ invertible matrix A, the matrix A^{-1} is invertible and

$$(A^{-1})^{-1} = A$$

Verify:

$$A^{-1} A = I \Rightarrow$$

$$A = (A^{-1})^{-1}$$

Algebraic Properties (Matrix Inverses)

Theorem. For a $n \times n$ invertible matrix A, the matrix A^T is invertible and

$$(A^T)^{-1} = (A^{-1})^T \qquad (AB)^T = B^T A^T$$

Verify:

$$A^T (A^{-1})^T = (A^{-1}A)^T = (I)^T = I$$

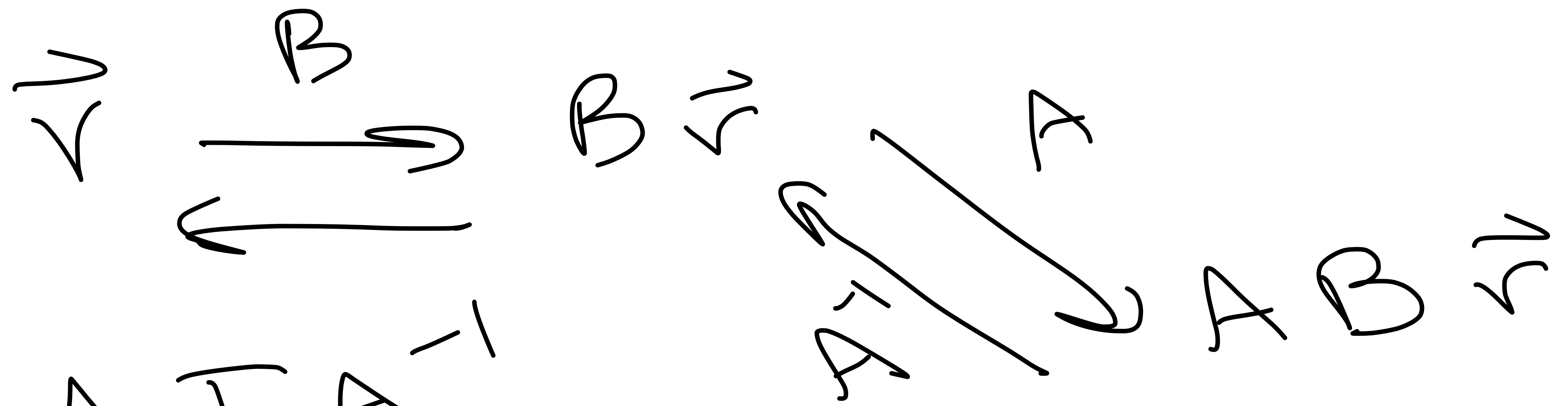
$$(A^{-1})^T = (A^T)^{-1} \quad \checkmark$$

Algebraic Properties (Matrix Inverses)

Theorem. For a $n \times n$ invertible matrices A and B, the matrix AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

Verify:



$$A(BB^{-1})A^{-1} = AIA^{-1}$$

$$= AA^{-1} = I$$

Question

Suppose that A *is a* $n \times n$ *invertible matrix such that* $A = A^T$ *and* B *is a* $m \times n$ *matrix.*

Simplify the expression $A(BA^{-1})^T$ *using the algebraic properties we've seen.*

Answer: B^T

$A(BA^{-1})^T$

$A = A^T$

$$A(BA^{-1})^T = A((A^{-1})^T B^T)$$

$$= A(A^T)^{-1} B^T$$

$$= A A^{-1} B^T$$

$$= B^T$$

Invertible Matrix Theorem

Motivation

Question. How do we know if a square matrix is invertible?

Answer. *Every* perspective we've taken so far can help us answer this question.

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix. Then the following hold.

1. A^T is invertible

Verify: We just did this

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix. Then the following hold.

2. $A\mathbf{x} = \mathbf{b}$ has at <u>least</u> one solution for every $\mathbf{b}$
3. $A\mathbf{x} = \mathbf{b}$ has at <u>most</u> one solution for every $\mathbf{b}$
4. $A\mathbf{x} = \mathbf{b}$ has at <u>exactly</u> one solution for every $\mathbf{b}$

Verify: $A\vec{x} = \vec{b}$

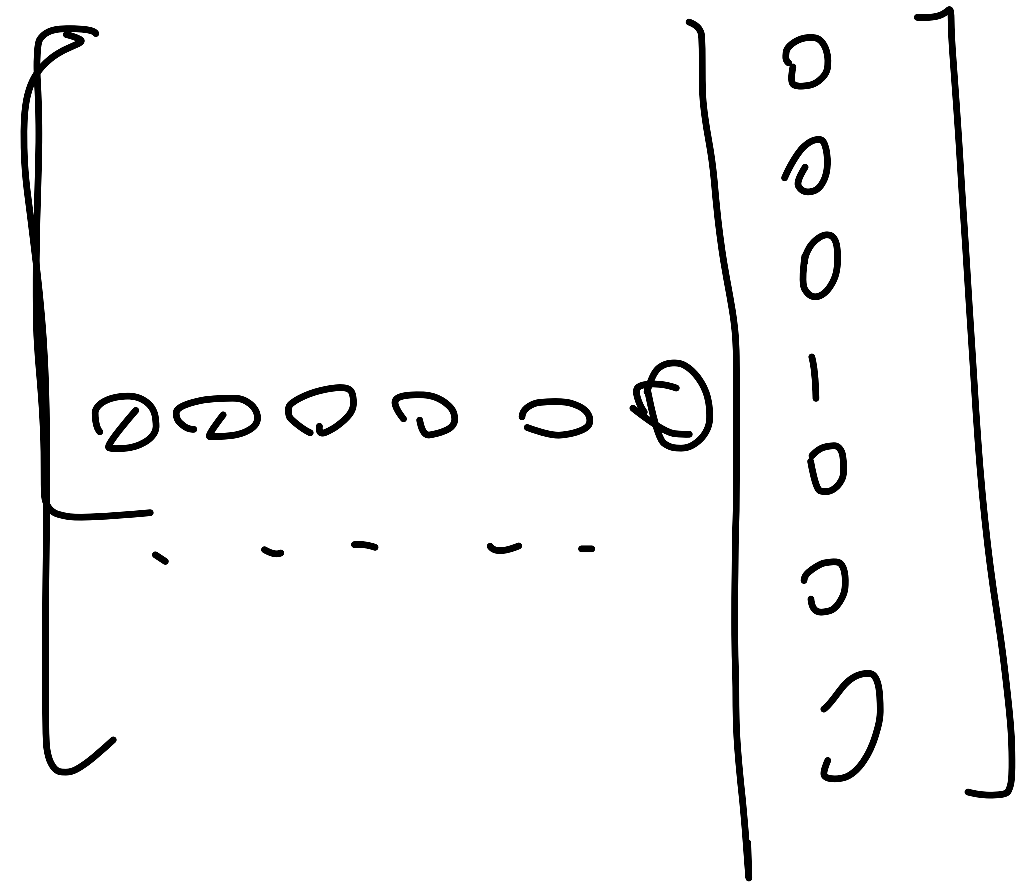
$\vec{x} = A^{-1}\vec{b}$ is the unique solution

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix. Then the following hold.

5. A has a pivot in every <u>column</u>
6. A has a pivot in every <u>row</u>
7. A is row equivalent to I_n

Verify: A has unique solution ⇒
no free variables



Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix. Then the following hold.

8. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
9. The columns of A are linearly independent
10. The columns of A span $\mathbb{R}^n$

Verify: pivot / column $\Rightarrow$ LI

pivot / row $\Rightarrow$ full span

Recall: Onto Transformations

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Definition. A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is ***onto*** if any vector $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at least one vector $\mathbf{v}$ in $\mathbb{R}^n$ (where $T(\mathbf{v}) = \mathbf{b}$).

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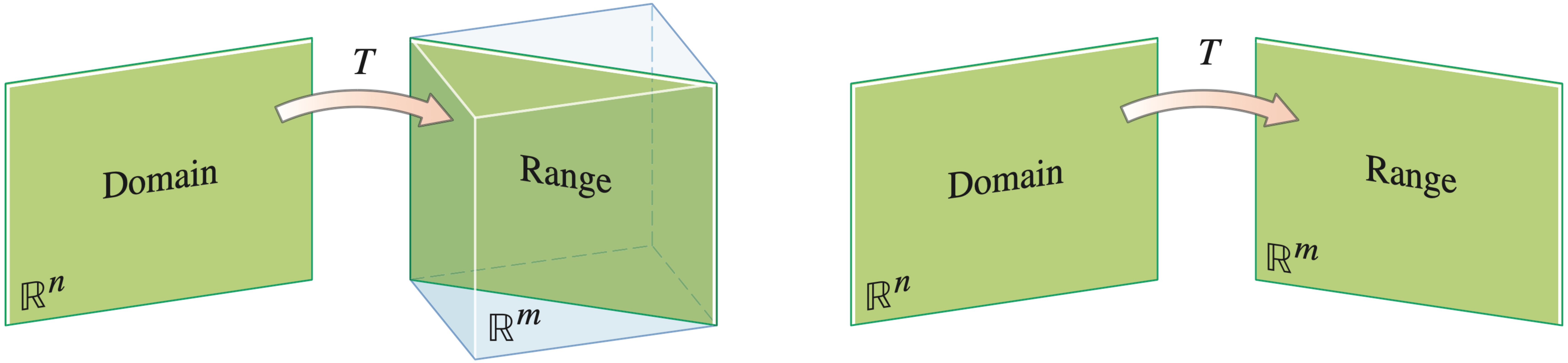

T is *not* onto $\mathbb{R}^m$ — T is onto $\mathbb{R}^m$

image source: Linear Algebra and its Applications. Lay, Lay, and McDonald

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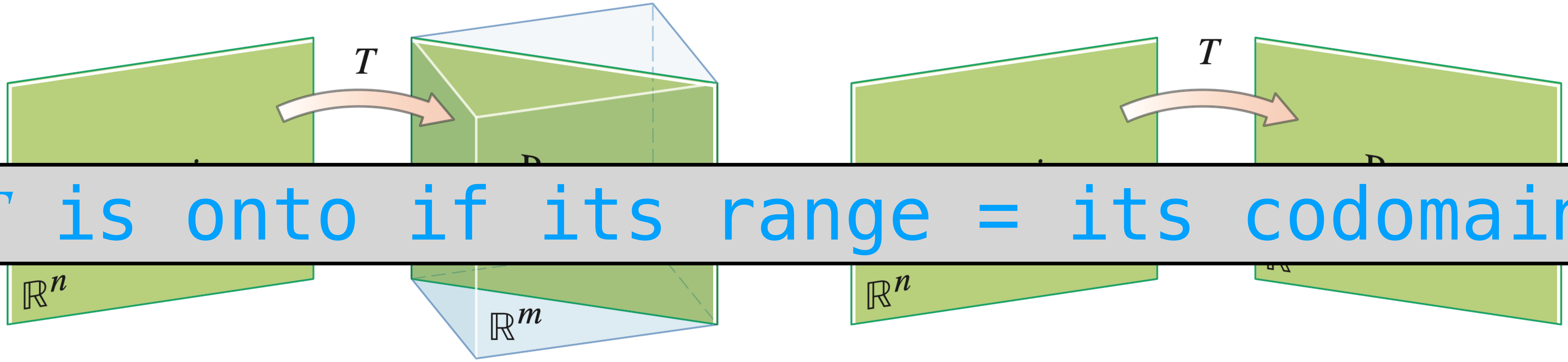

T is onto if its range = its codomain

T is *not* onto $\mathbb{R}^m$ T is onto $\mathbb{R}^m$

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Recall: One-to-one Transformations

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Definition. A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is ***one-to-one*** if any vector $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at most one vector $\mathbf{v}$ in $\mathbb{R}^n$ (where $T(\mathbf{v}) = \mathbf{b}$).

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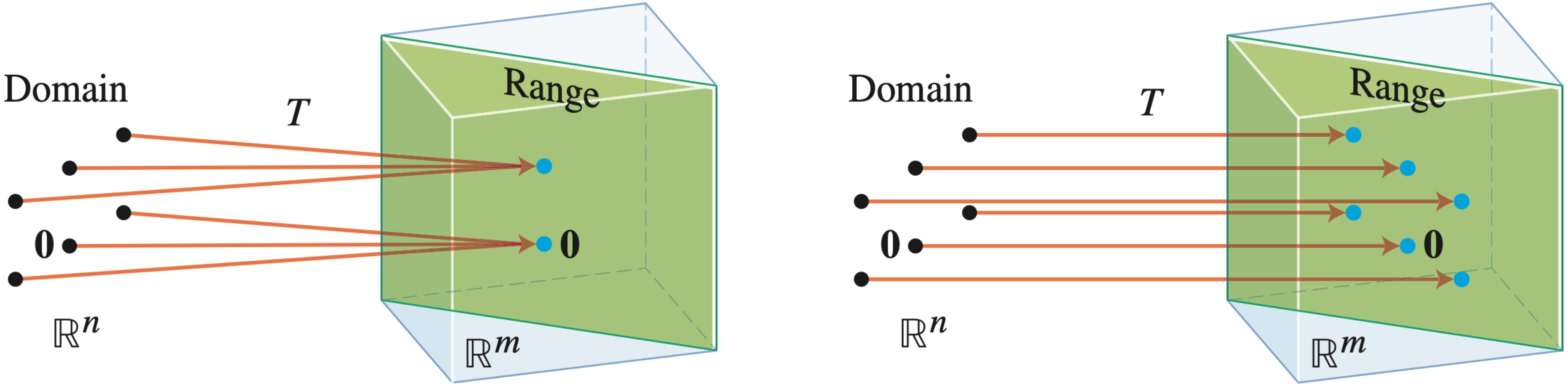


T is *not* one-to-one   *T* is one-to-one

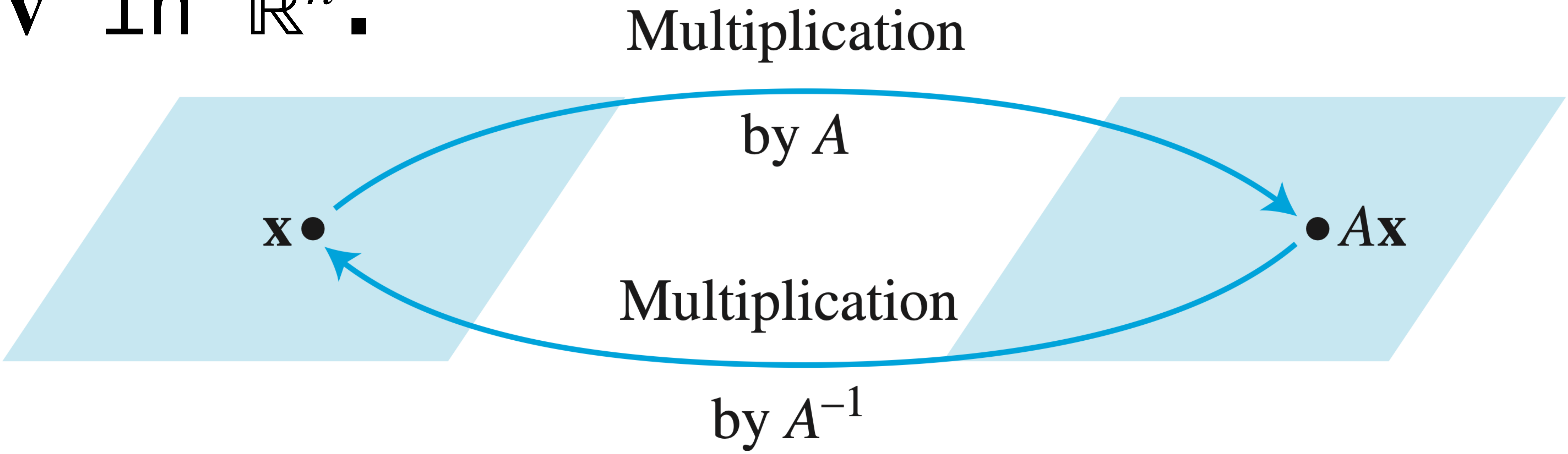
image source: Linear Algebra and its Applications. Lay, Lay, and McDonald

Recall: Invertible Transformations

Definition. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is **invertible** if there is a linear transformation S such that

$$S(T(\mathbf{v})) = \mathbf{v} \quad \text{and} \quad T(S(\mathbf{v})) = \mathbf{v}$$

for any $\mathbf{v}$ in $\mathbb{R}^n$.



Recall: One-to-One Correspondence

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Definition. A transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is a **one-to-one correspondence** (bijection) if any vector $\mathbf{b}$ in $\mathbb{R}^n$ is the image of **exactly** one vector $\mathbf{v}$ in $\mathbb{R}^n$ (where $T(\mathbf{v}) = \mathbf{b}$).

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A transformation is a 1-1 correspondence if it is 1-1 and onto.

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A transformation is a 1-1 correspondence if it is 1-1 and onto.

Invertible transformations are 1-1 correspondences.

Invertible Matrix Theorem

Theorem. Suppose A is a $n \times n$ invertible matrix. Then the following hold.

11. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto
12. $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one
13. $\mathbf{x} \mapsto A\mathbf{x}$ is a one-to-one correspondence
14. $\mathbf{x} \mapsto A\mathbf{x}$ is invertible

Verify:

Taking Stock: IMT

The following are logically equivalent:

1. A is invertible
2. A^T is invertible
3. $A\mathbf{x} = \mathbf{b}$ has at least one solution for any $\mathbf{b}$
4. $A\mathbf{x} = \mathbf{b}$ has at most one solution for any $\mathbf{b}$
5. $A\mathbf{x} = \mathbf{b}$ has a unique solution for any $\mathbf{b}$
6. A has n pivots (per row and per column)
7. A is row equivalent to I
8. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
9. The columns of A are linearly independent
10. The columns of A span $\mathbb{R}^n$
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These all express the **same thing**

(this is a stronger statement than we just verified)

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!! only for square matrices !!

We get a lot of information for free

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Theorem. If A is square, then

A **is 1–1** if and only if A **is onto**

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We only need to check one of these.

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Theorem. If A is square, then

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We only need to check one of these.

Warning. Remember this only applies square matrices.

We get a lot of information for free

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Theorem. If A is square, then

A **is invertible** $\equiv$ $A\mathbf{x} = \mathbf{0}$ **implies** $\mathbf{x} = \mathbf{0}$

We get a lot of information for free

Theorem. If A is square, then

$$A \textbf{ is invertible} \quad \equiv \quad A\mathbf{x} = \mathbf{0} \textbf{ implies } \mathbf{x} = \mathbf{0}$$

Invertibility is completely determined by how A behaves on $\mathbf{0}$.

Question (Conceptual)

True *or* ***False:*** *If A is invertible, and B is row equivalent to A (we can transform B into A by a sequence of row operations), then B is also invertible.*

Answer: True

Row reductions don't change the number of pivots.

Question

3×3

If $[\mathbf{a}_1 \;\; \mathbf{a}_2 \;\; \mathbf{a}_3]$ *is invertible, then is* $\left[(\mathbf{a}_1 + \mathbf{a}_2 - 2\mathbf{a}_3) \;\; (\mathbf{a}_2 + 5\mathbf{a}_3) \;\; \mathbf{a}_3\right]$ *also invertible? Justify your answer.*

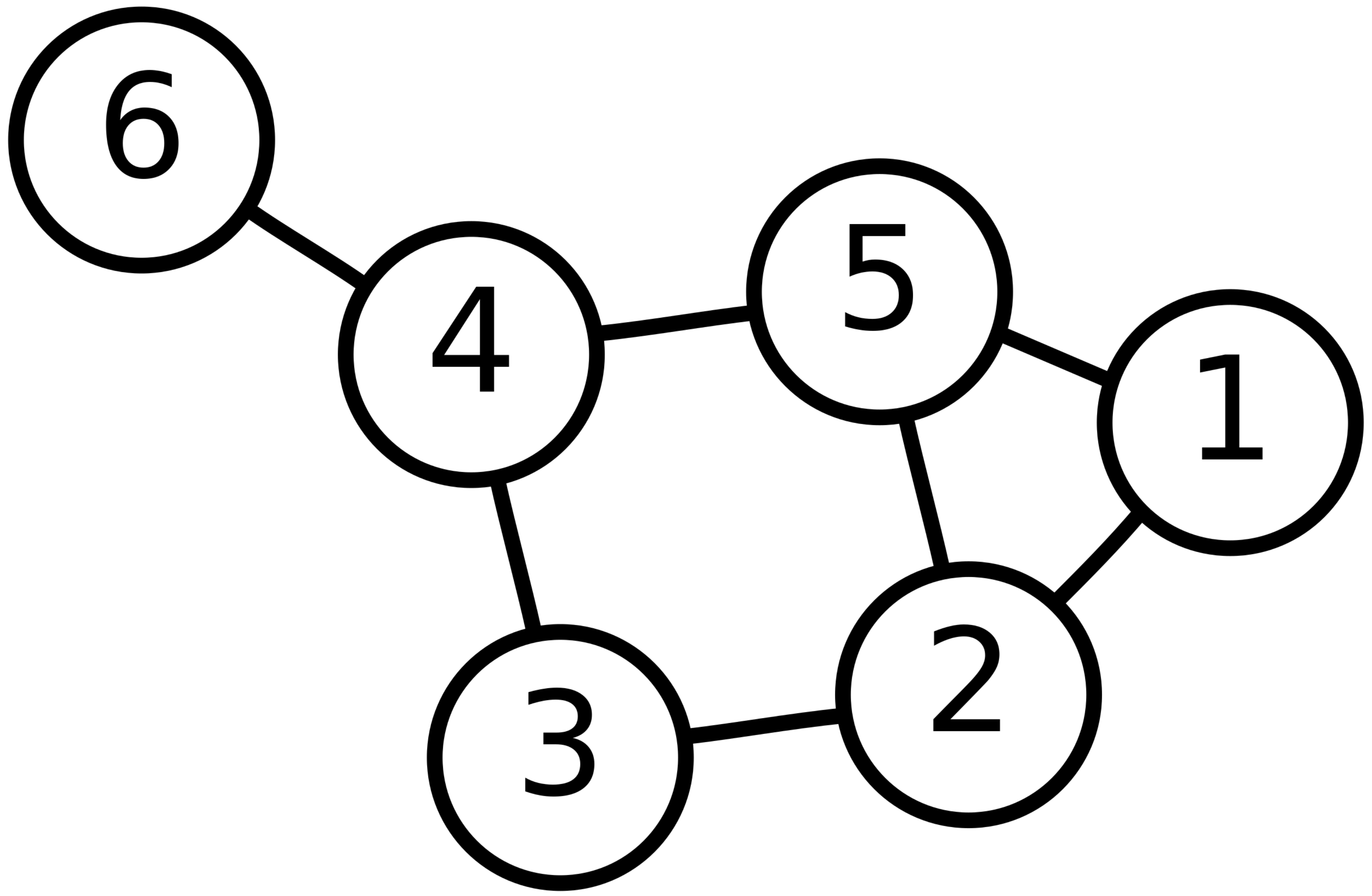
Answer

Consider $[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3]^T$. We can get to $\left[(\mathbf{a}_1 + \mathbf{a}_2 - 2\mathbf{a}_3) \quad (\mathbf{a}_2 + 5\mathbf{a}_3) \quad \mathbf{a}_3\right]^T$ by <u>row operations</u>

Adjacency Matrices

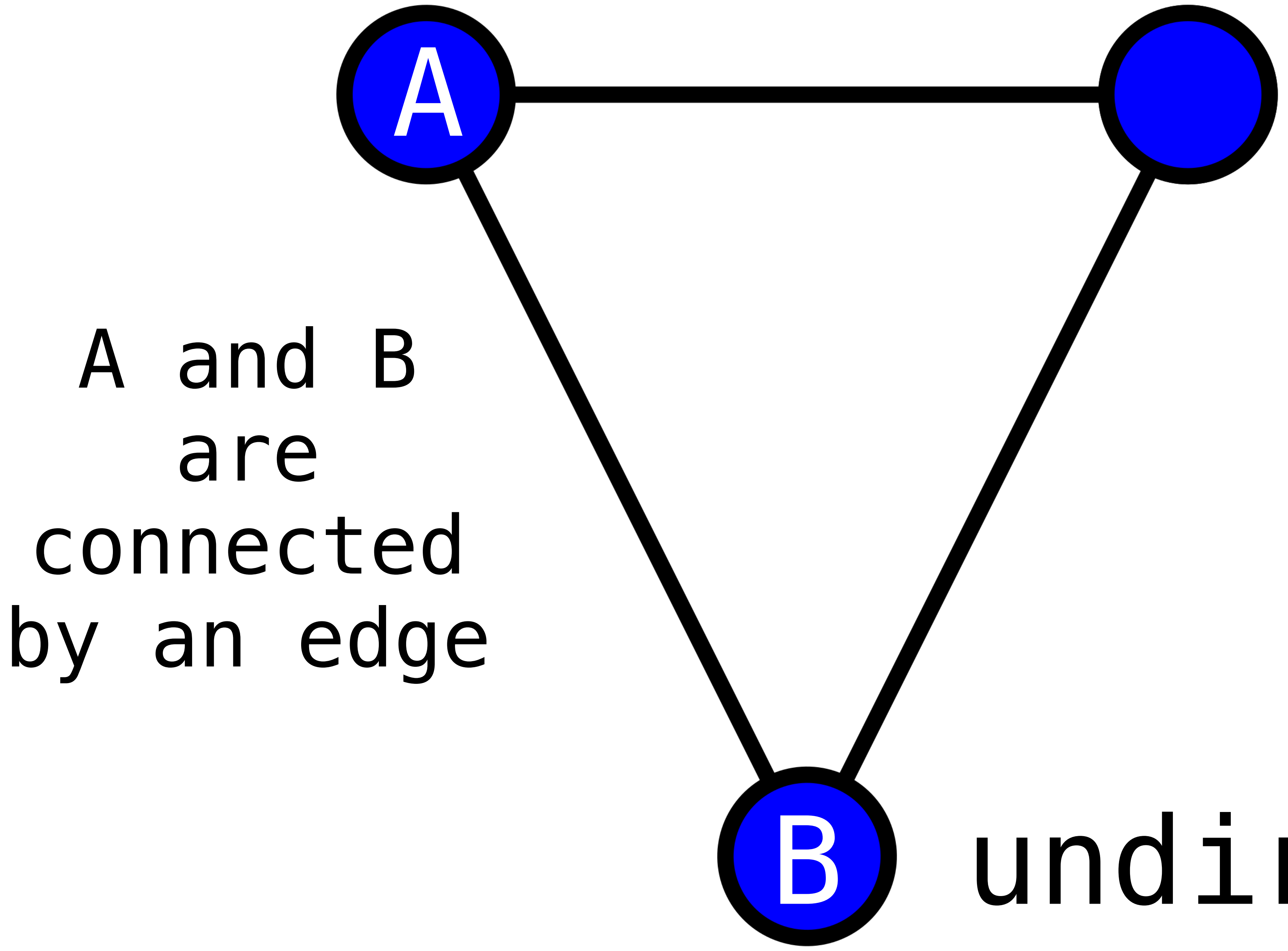
Graphs

Definition (Informal). A **graph** is a collection of nodes with edges between them.

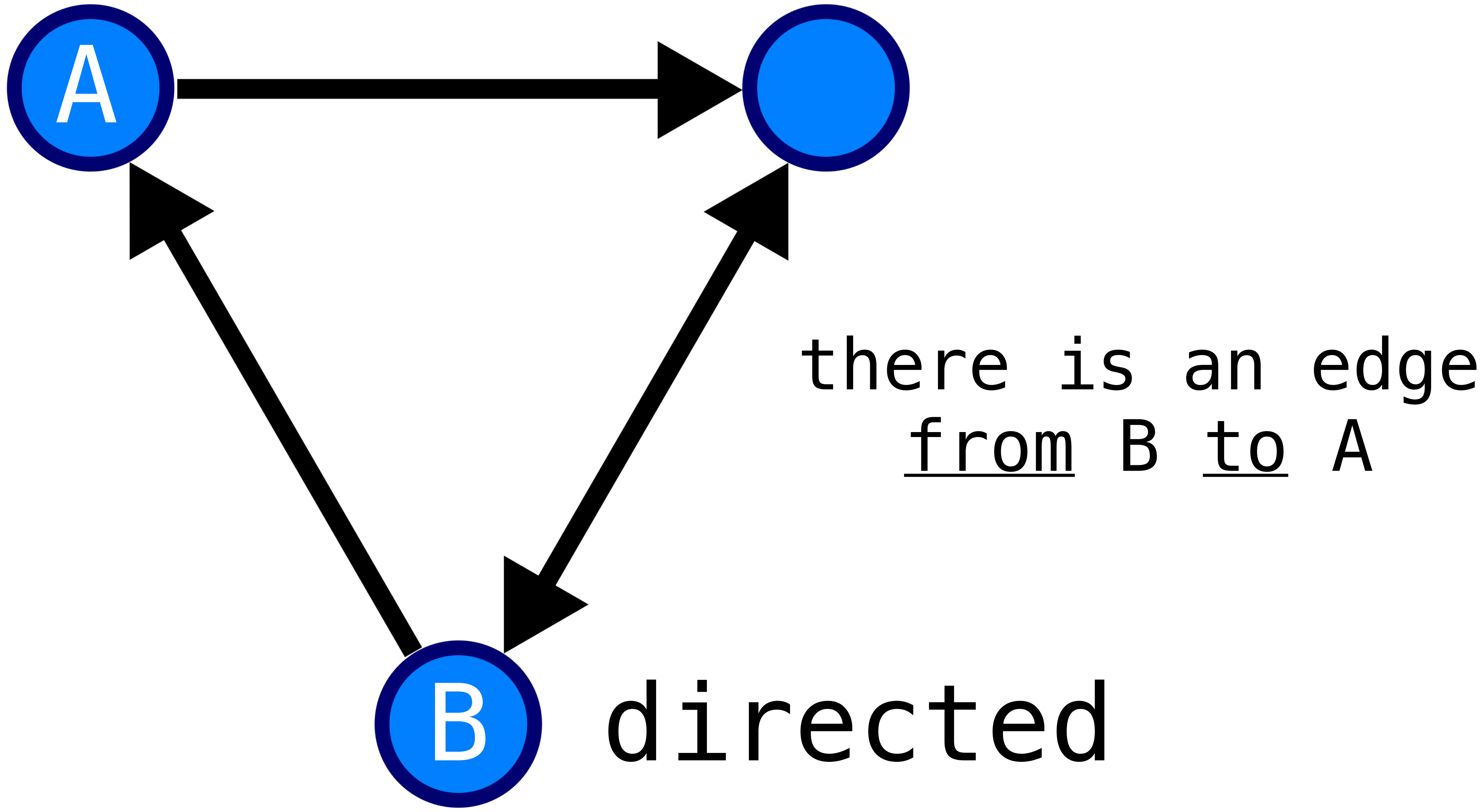


Directed vs. Undirected Graphs

A graph is **directed** if its edges have a direction.



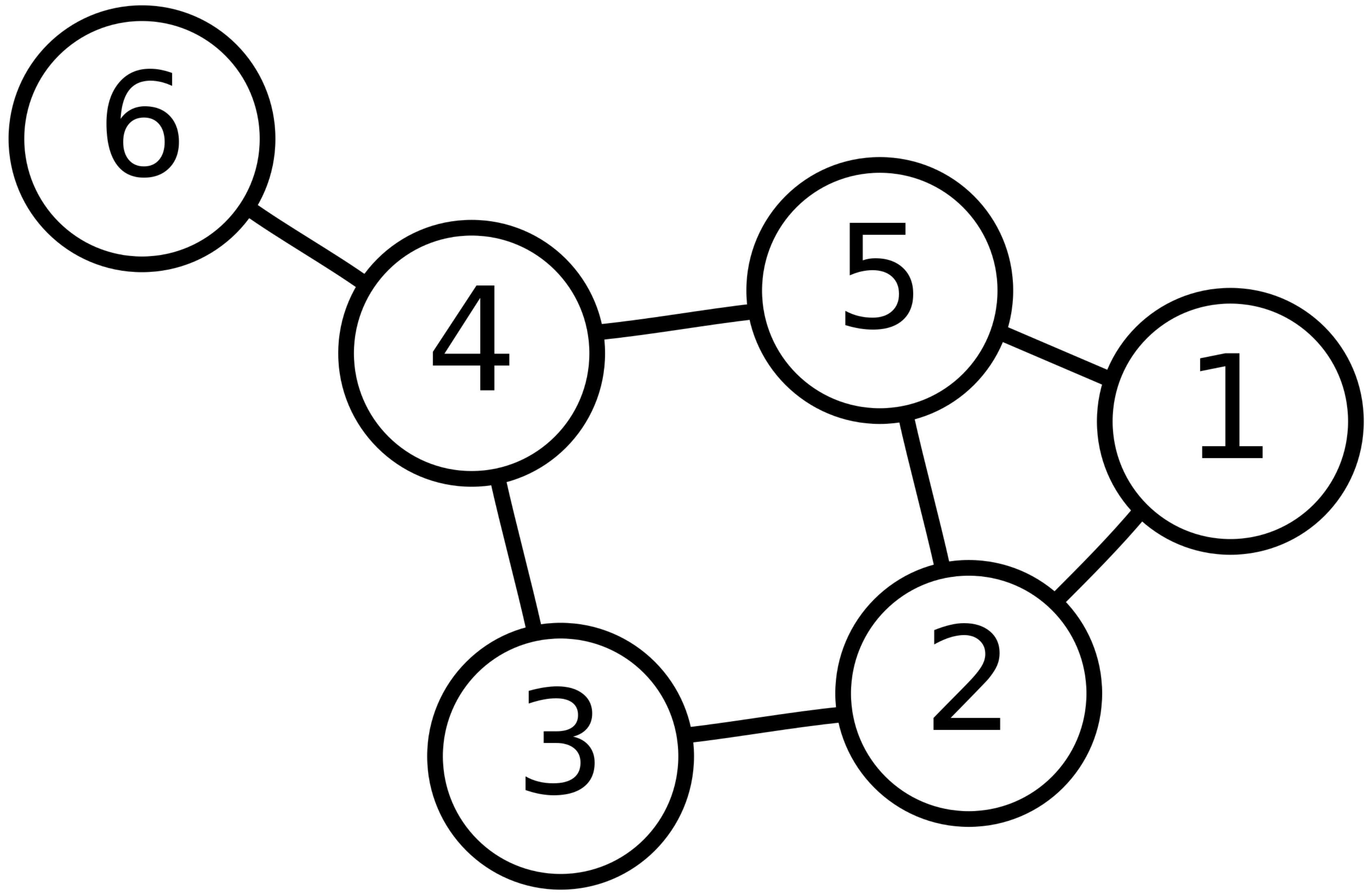
undirected



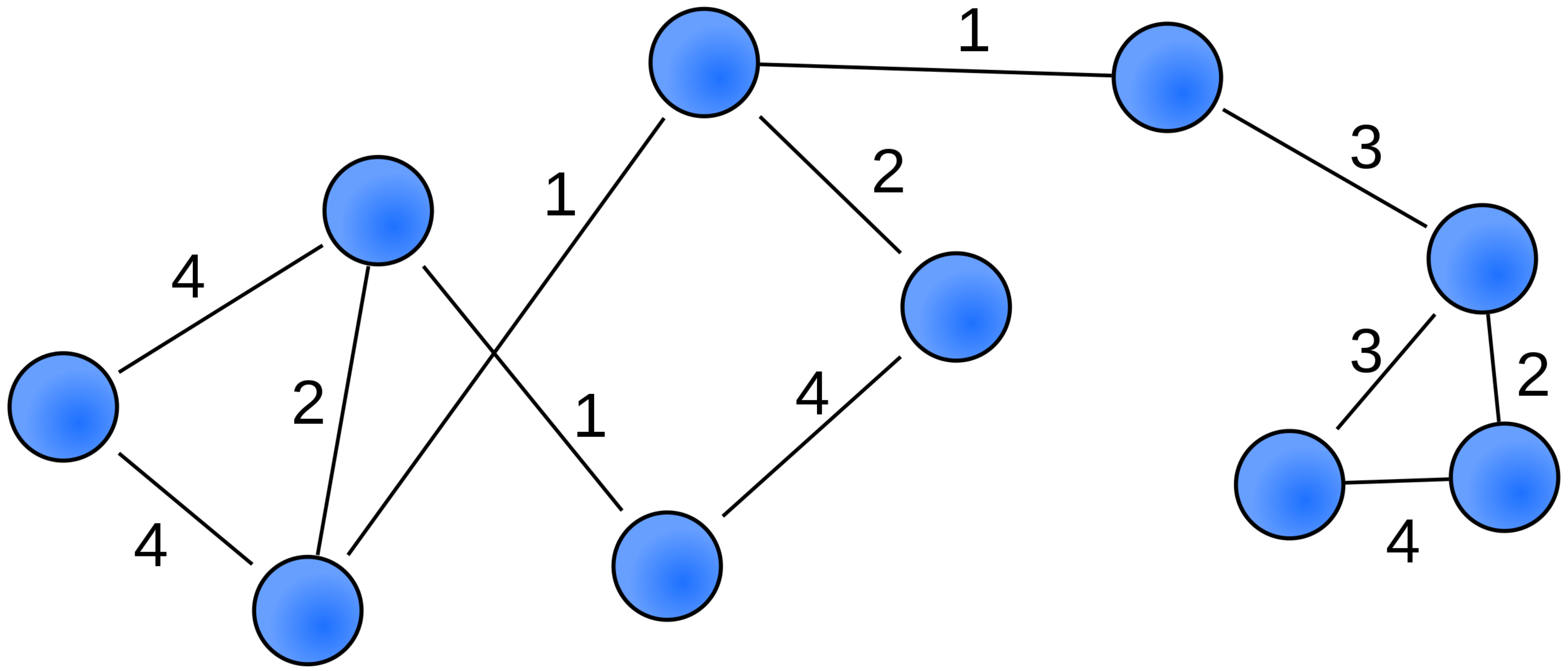
directed

Weighted vs Unweighted graphs

A graph is **weighted** if its edges have associated values.



unweighted



weighted

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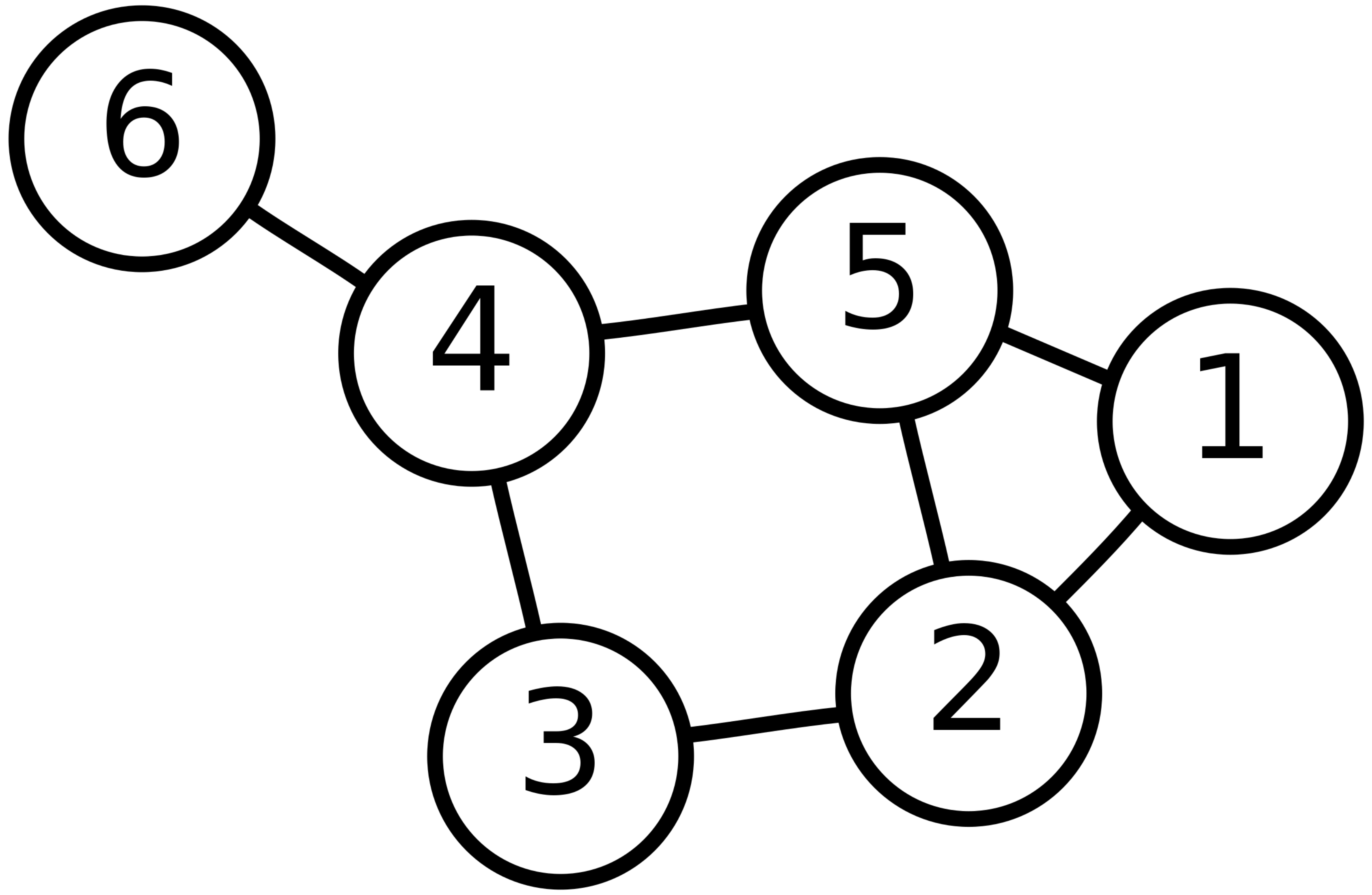
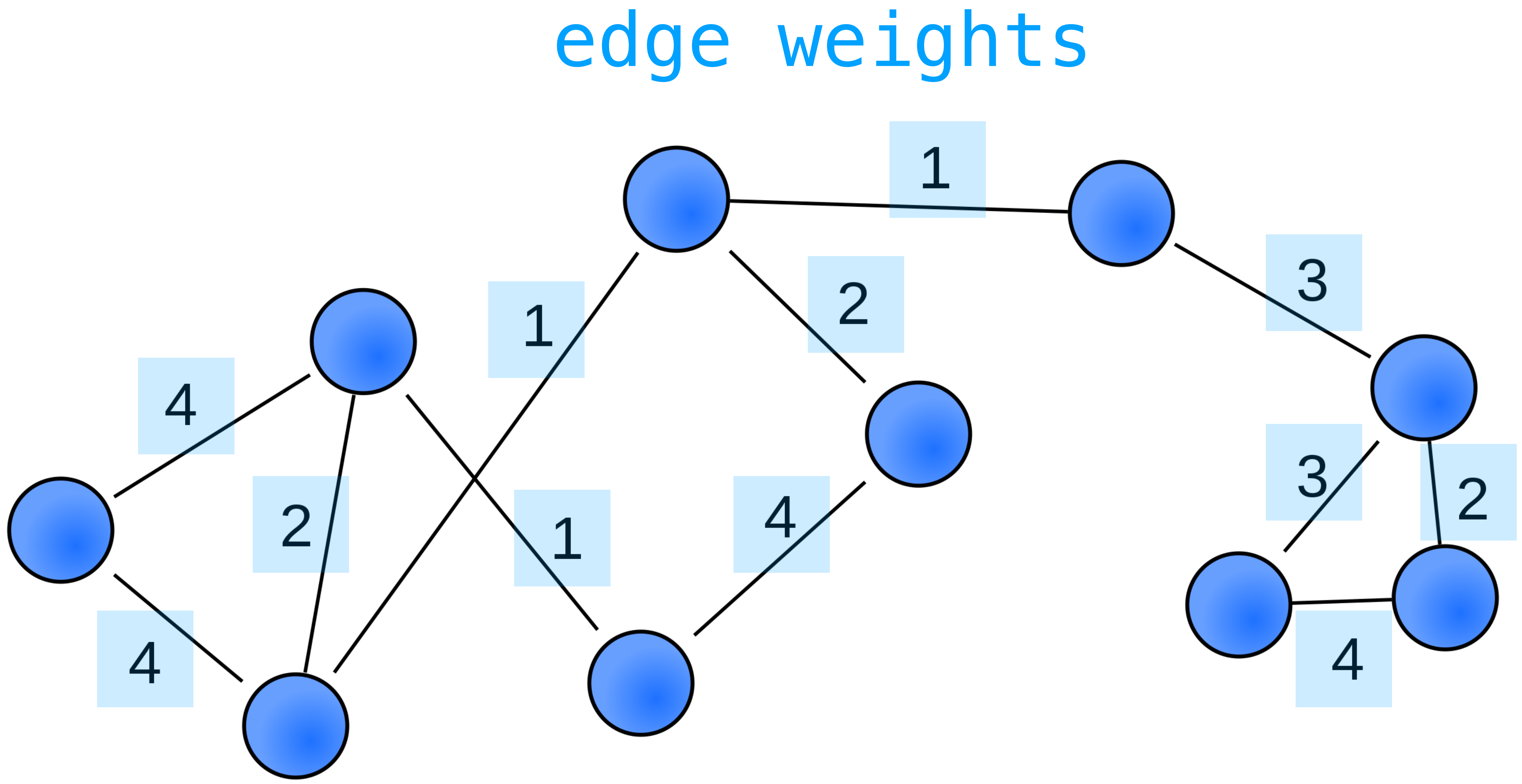


unweighted



weighted

Four Kinds of Graphs

	directed	undirected
weighted	nodes are traffic lights edges are streets weights are number of lanes	nodes are musicians edges are collaborations weights are number of collaborations
unweighted	nodes are instagram users edges are follows	nodes are bodies of land edges are pedestrian bridges

Four Kinds of Graphs

	directed	undirected
weighted	nodes are traffic lights edges are streets weights are number of lanes	nodes are musicians edges are collaborations weights are number of collaborations
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Four Kinds of Graphs

	directed	undirected
weighted	nodes are traffic lights edges are streets weights are number of lanes Markov Chains	nodes are musicians edges are collaborations weights are number of collaborations
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Fundamental Question

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How do we represent a graph formally in a computer?

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How do we represent a graph formally in a computer?

There are a couple ways, but one way is to use matrices.

Adjacency Matrices

Let G be an undirected unweighted graph with its nodes labeled by numbers 1 through n.

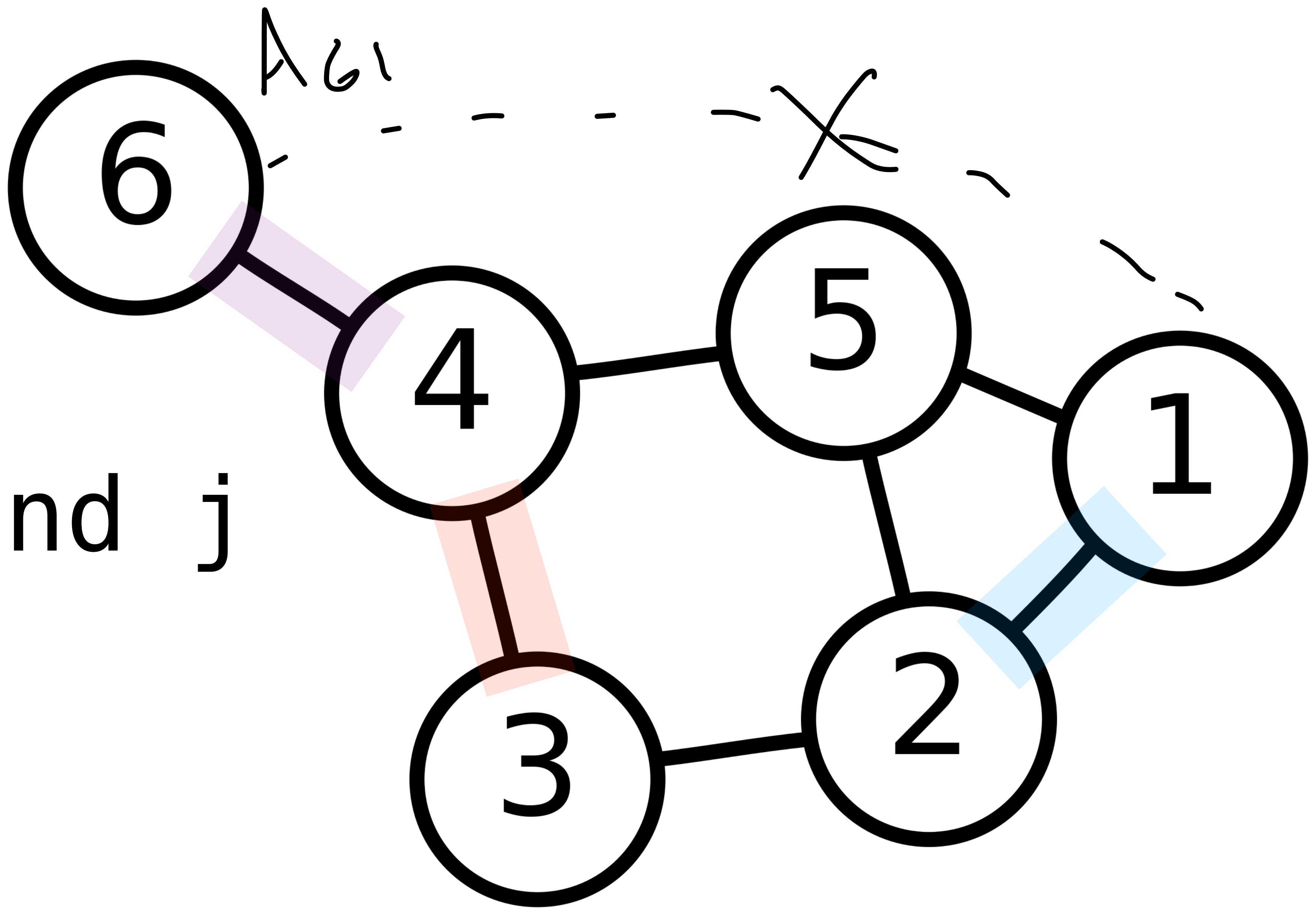
We can create the **adjacency matrix** A for G as follows.

$$A_{ij} = \begin{cases} 1 & \text{there is an edge between i and j} \\ 0 & \text{otherwise} \end{cases}$$

A_{12} A_{34} A_{46}

A_{21} A_{43} A_{64}

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$



Symmetric Matrices

Definition. A $n \times n$ matrix is **symmetric** if

$$A^T = A$$

Example.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Spectral Graph Theory

Once we have an adjacency matrix, we can do linear algebra on graphs.

Example: Squared Adjacency Matrices

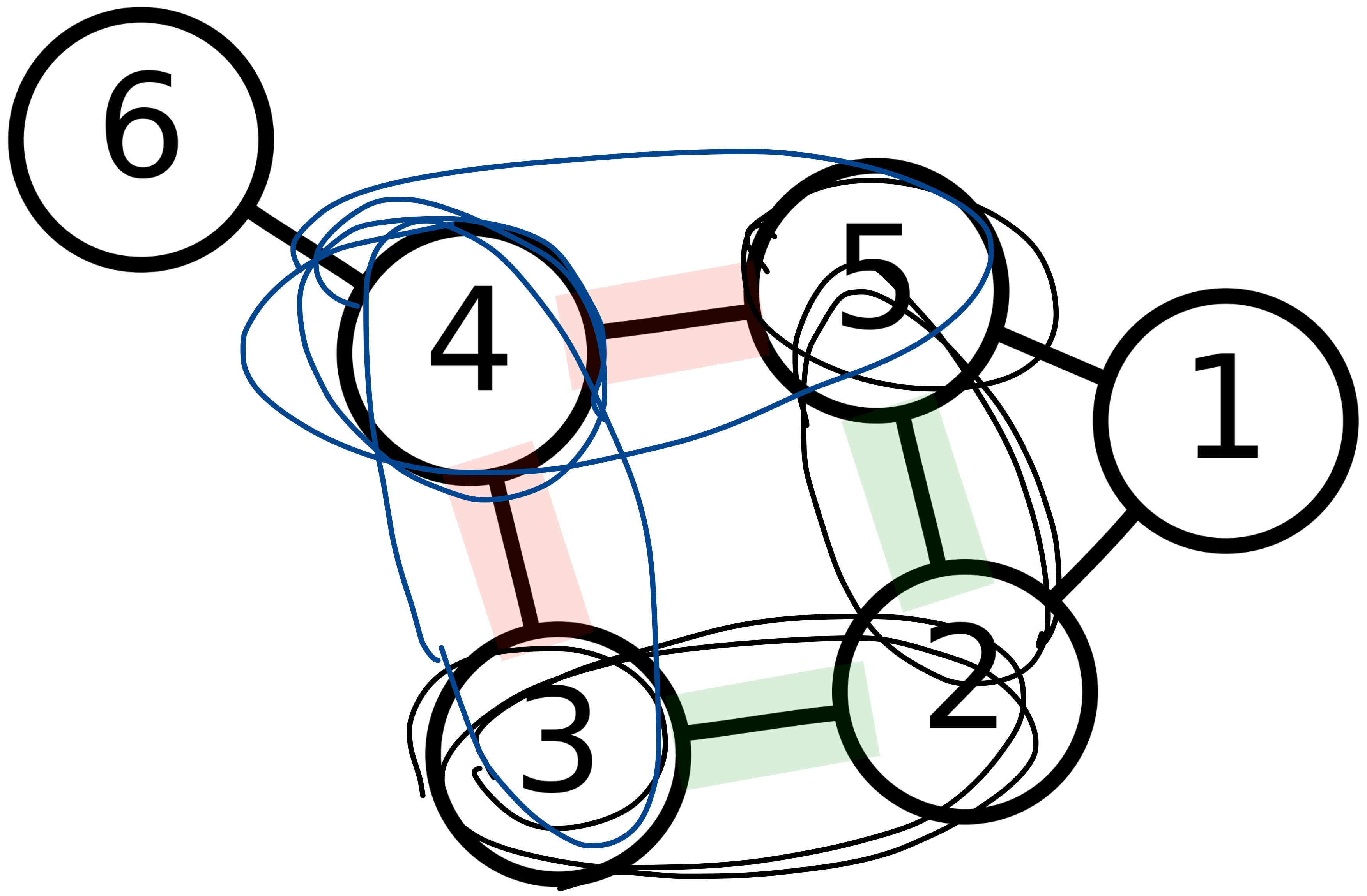
Given an adjacency matrix A, can we interpret anything meaningful from A^2?

Example: Squared Adjacency Matrices

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

A23

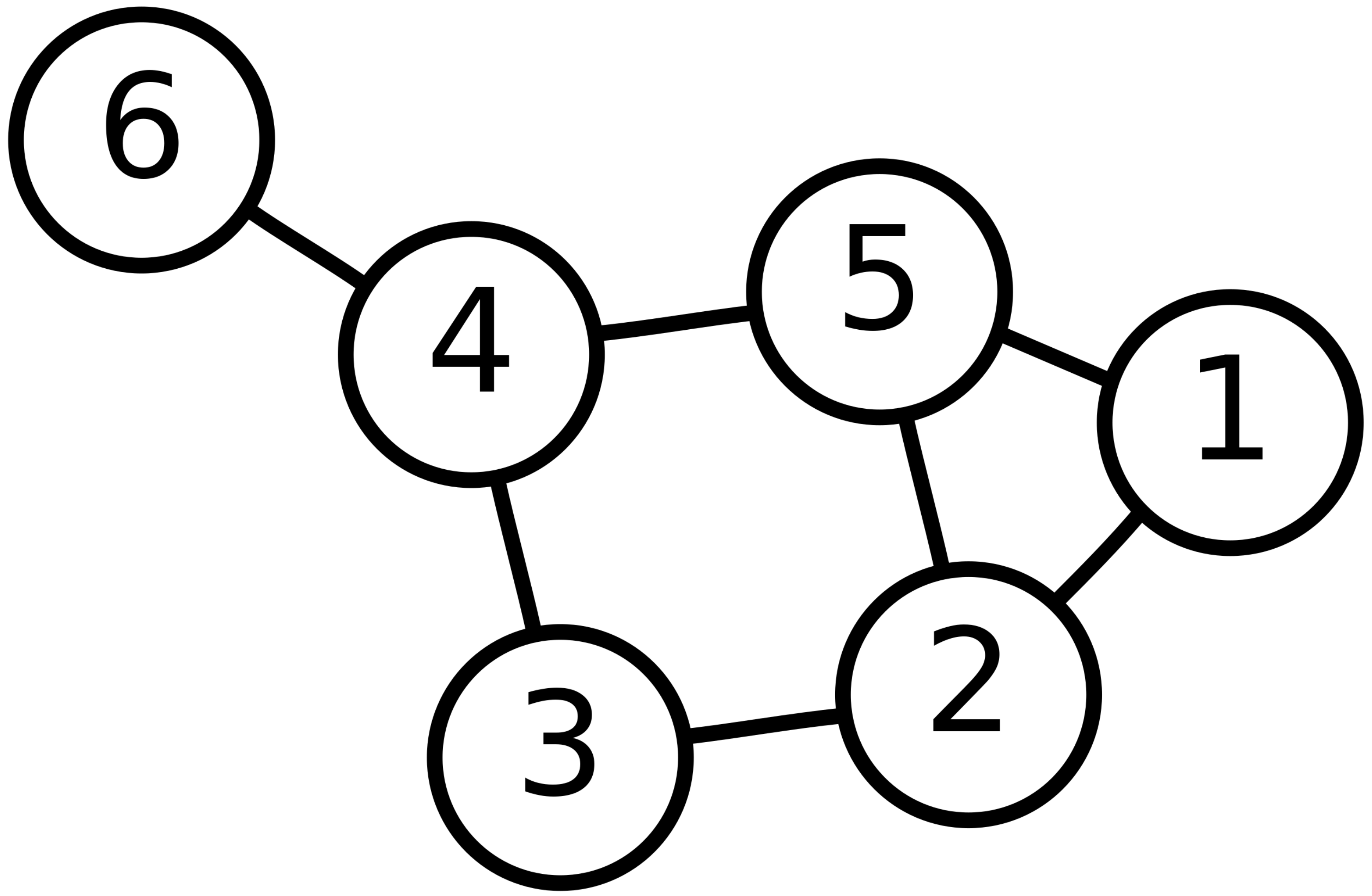
A52 A54 A43



$$(A^2)_{53} = 1(0) + 1(1) + 0(0) + 1(1) + 0(0) + 0(0) = 2$$

Example: Squared Adjacency Matrices

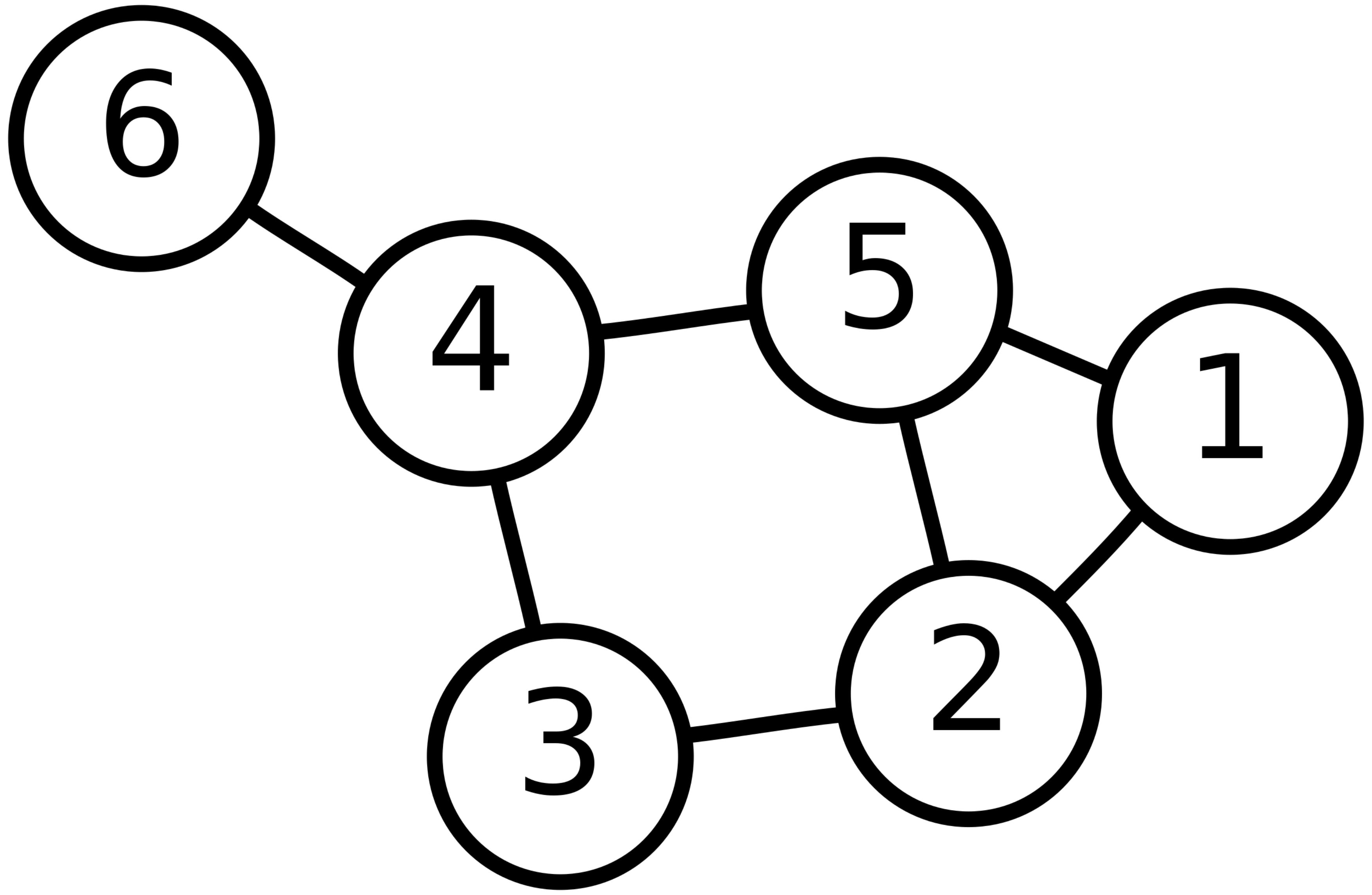
$$(A^2)_{ij} = A_{i1}A_{1j} + A_{i2}A_{2j} + \ldots + A_{in}A_{nj}$$



Example: Squared Adjacency Matrices

$$(A^2)_{ij} = A_{i1}A_{1j} + A_{i2}A_{2j} + \ldots + A_{in}A_{nj}$$

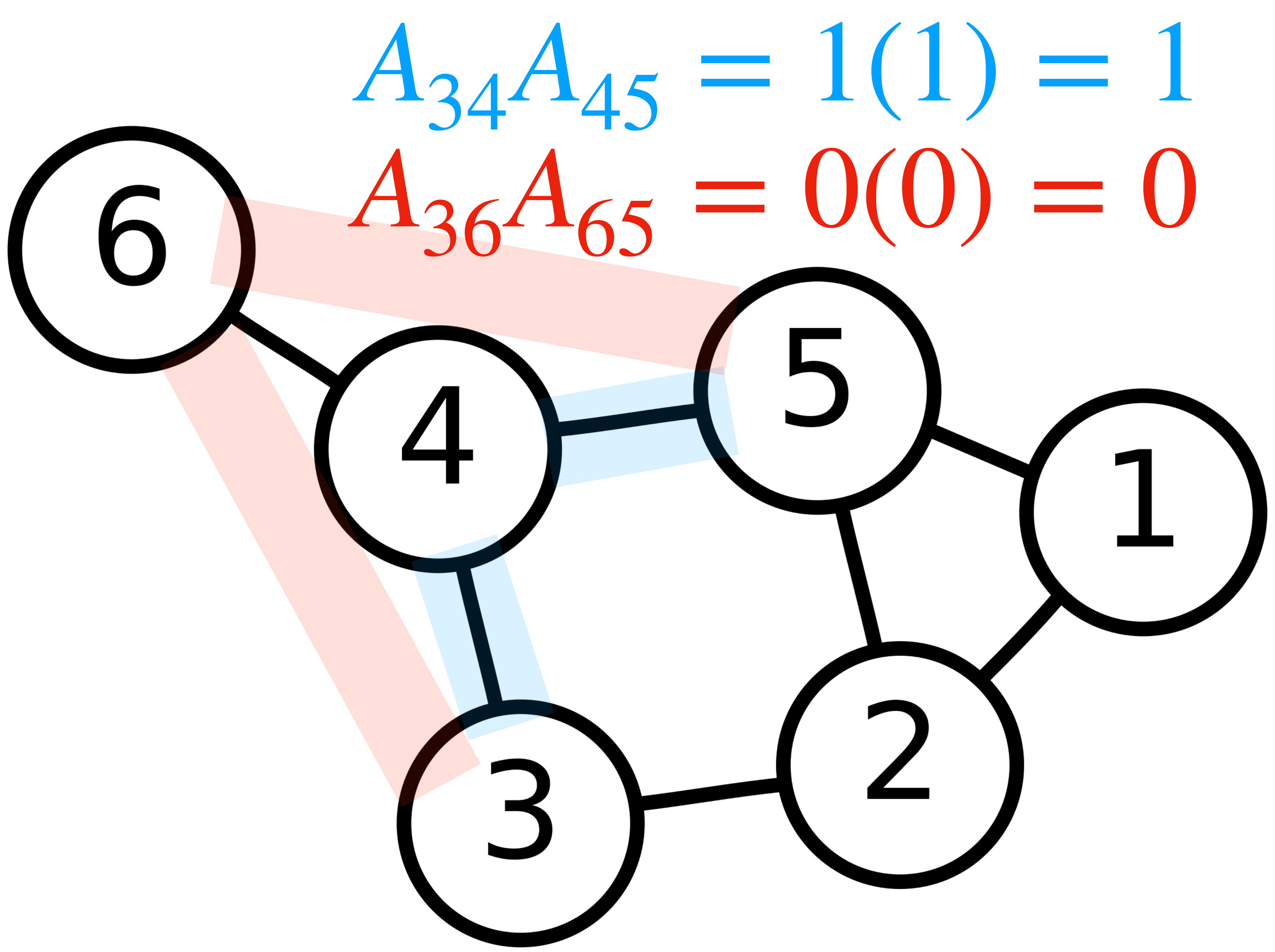
$$A_{ik}A_{kj} = \begin{cases} 1 & \texttt{there are edges i to k and k to j} \\ 0 & \texttt{otherwise} \end{cases}$$



Example: Squared Adjacency Matrices

$$(A^2)_{ij} = A_{i1}A_{1j} + A_{i2}A_{2j} + \ldots + A_{in}A_{nj}$$

$$A_{ik}A_{kj} = \begin{cases} 1 & \texttt{there are edges i to k and k to j} \\ 0 & \texttt{otherwise} \end{cases}$$



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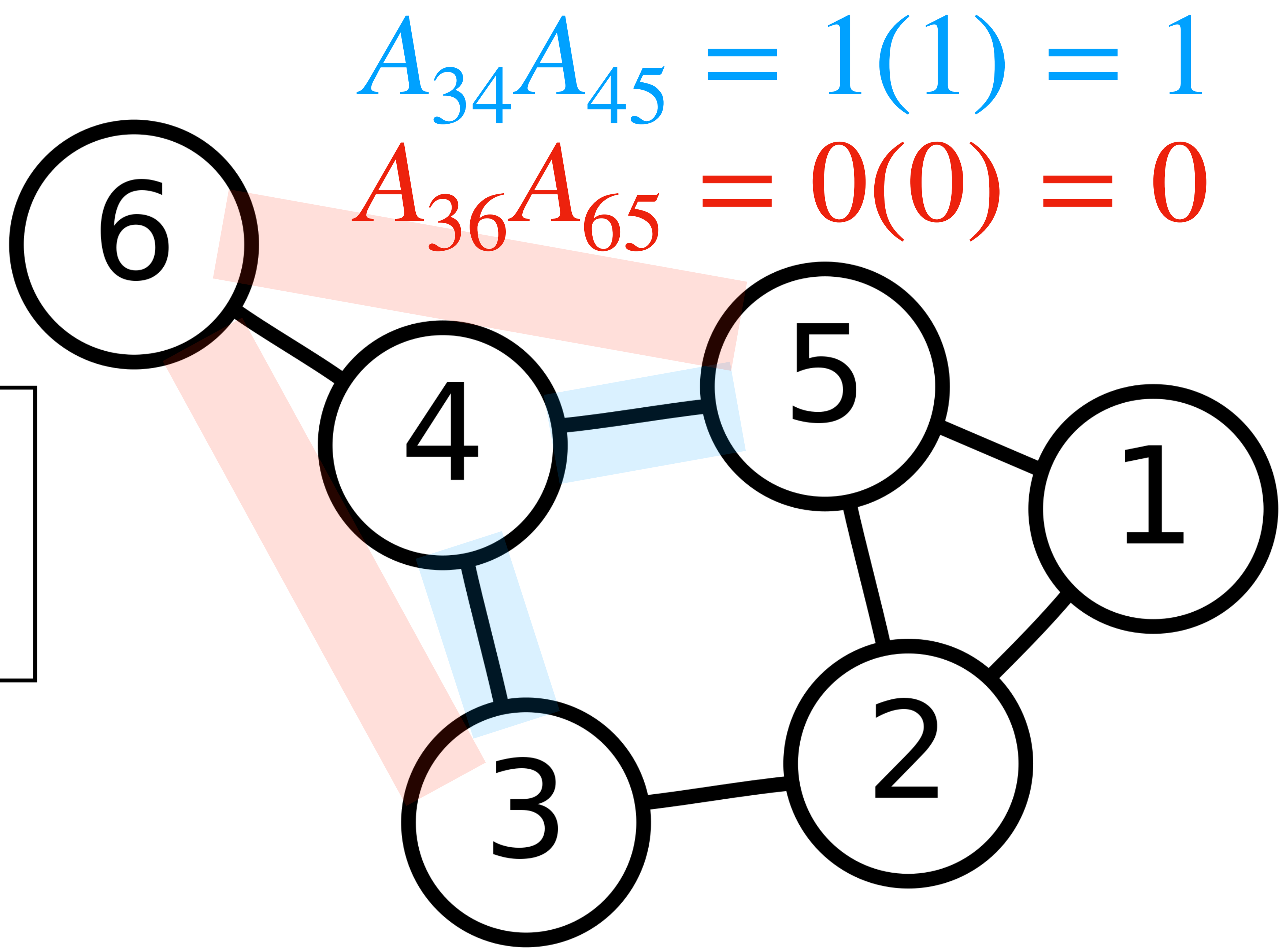
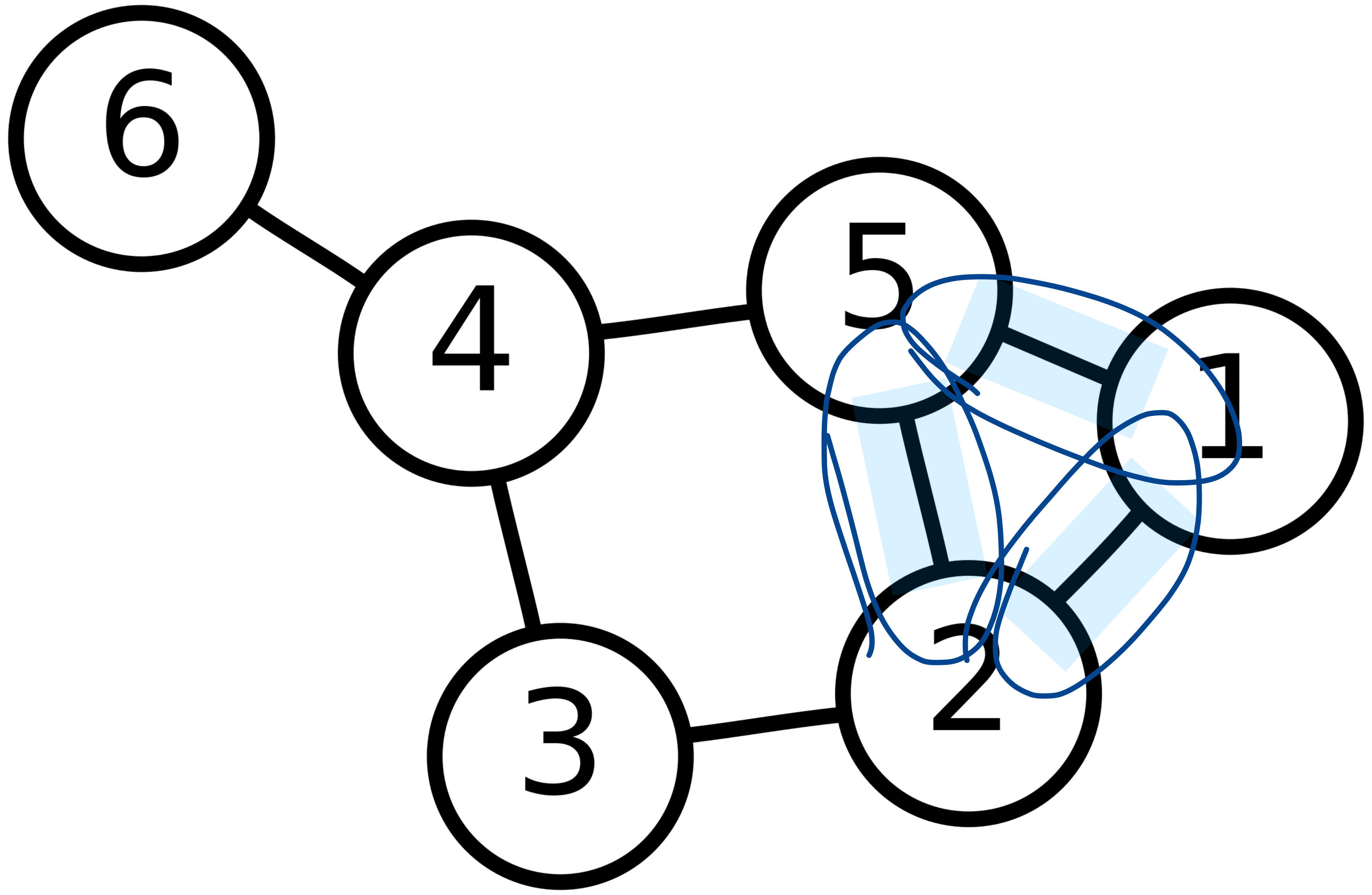
$(A^2)_{ij} =$ **number of 2-step paths from i to j**



Application: Triangle Counting

A **triangle** in an undirected graph is a set of three distinct nodes with edges between every pair of nodes.

Triangles in a social network represent mutual friends and tight cohesion (among other things)



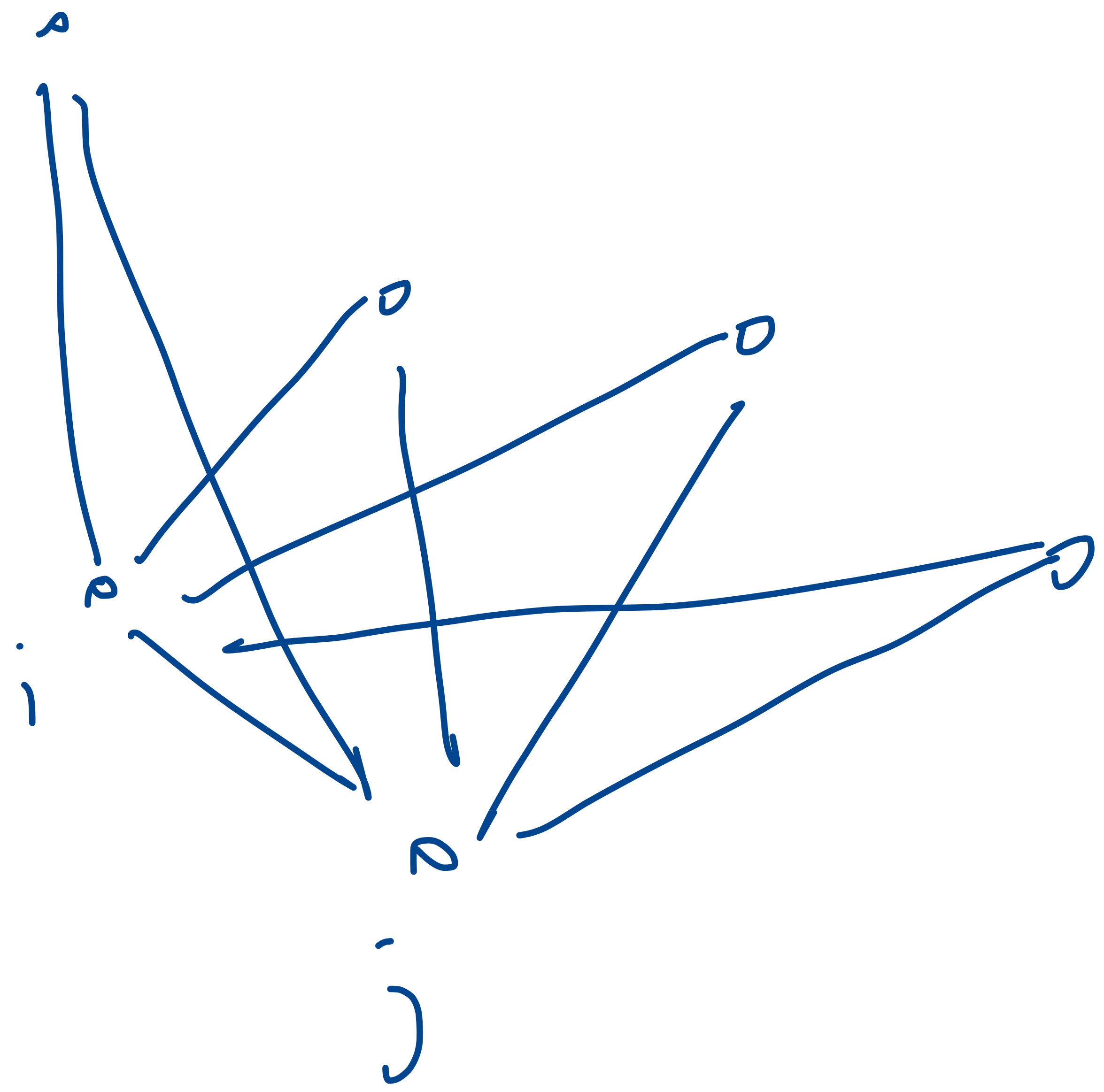
Application: Triangle Counting (Naive)

```
FUNCTION tri_count_naive(A):
  count = 0
  for i from 1 to n:
    for j from i to n:
      for k from j to n:
        if A_ij = 1 and A_jk = 1 and A_ki = 1: # an edge between each pair
          count += 1:
  RETURN count
```

Application: Triangle Counting

Theorem. For an adjacency matrix A, the number of triangle containing the edge (i,j) is

$$(A^2)_{ij} * A_{ij}$$



Verify:

Application: Triangle Counting

```
FUNCTION tri_count(A):
    compute A^2
    count ← sum of (A^2)_ij * A_ij for all distinct i and j
    RETURN count / 6      # why divided by 6?
```

Application: Triangle Counting

```
FUNCTION tri_count(A):
   # in NumPy '*' is entry-wise multiplication
   count ← sum of the entries of A^2 * A
   RETURN count / 6
```

Application: Triangle Counting

```
FUNCTION tri_count(A):
  # in NumPy '*' is entry-wise multiplication
  # and 'np.sum' sums the entry of a matrix
  RETURN np.sum((A @ A) * A) / 6
```

demo

Summary

The algebra of matrices can help us simplify matrix expressions.

The invertible matrix theorem connects all the perspectives we've taken so far.

Adjacency matrices are linear algebraic representations of graphs.