Dimension and Rank

Geometric Algorithms
Lecture 16

CAS CS 132

Introduction

Recap Problem

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \qquad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$$

Consider the subspace H generated by $\mathbf{v}_1$ and $\mathbf{v}_2$. Show that $\mathbf{v}_3$ and $\mathbf{v}_4$ form a basis for H.

Answer

$$\mathbf{v}_1 = \begin{bmatrix}1\\2\\0\end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix}-1\\0\\1\end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix}1\\0\\0\end{bmatrix} \qquad \mathbf{v}_4 = \begin{bmatrix}2\\1\\2\end{bmatrix}$$

Hint. Show that $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *are in the span of* $\mathbf{v}_3$ *and* $\mathbf{v}_4$

Objectives

1. Learn techniques to find bases for the column space and the null space of a matrix

2. Briefly discuss the coordinate systems.

3. Introduce the fundamental notion of dimension, which quantifies how "large" a space is

4. Relate the dimension of the column space and the null space of a matrix

Keywords

basis

column space

null space

coordinate system

change of basis

dimension

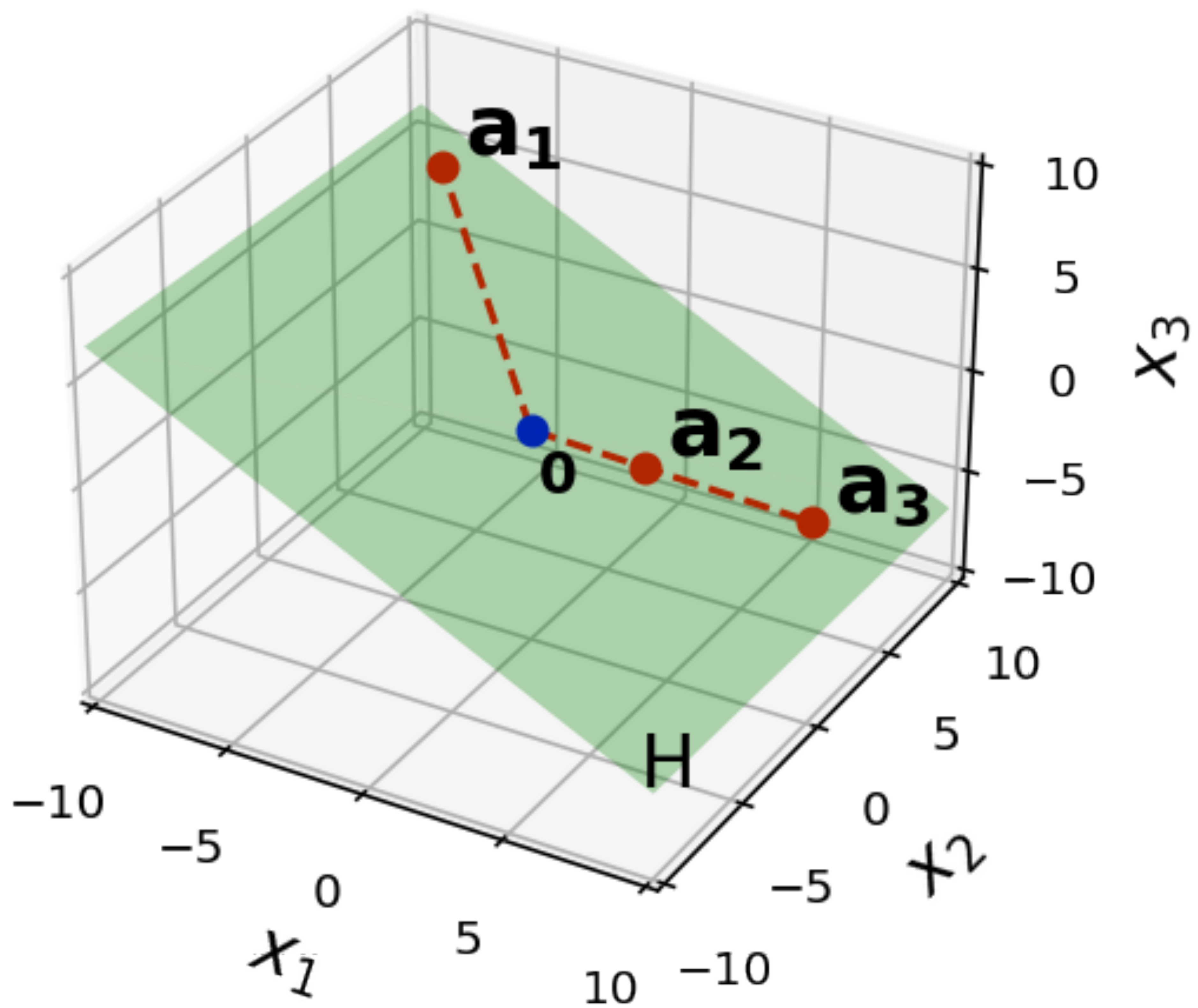
rank

rank theorem

invertible matrix theorem (extended)

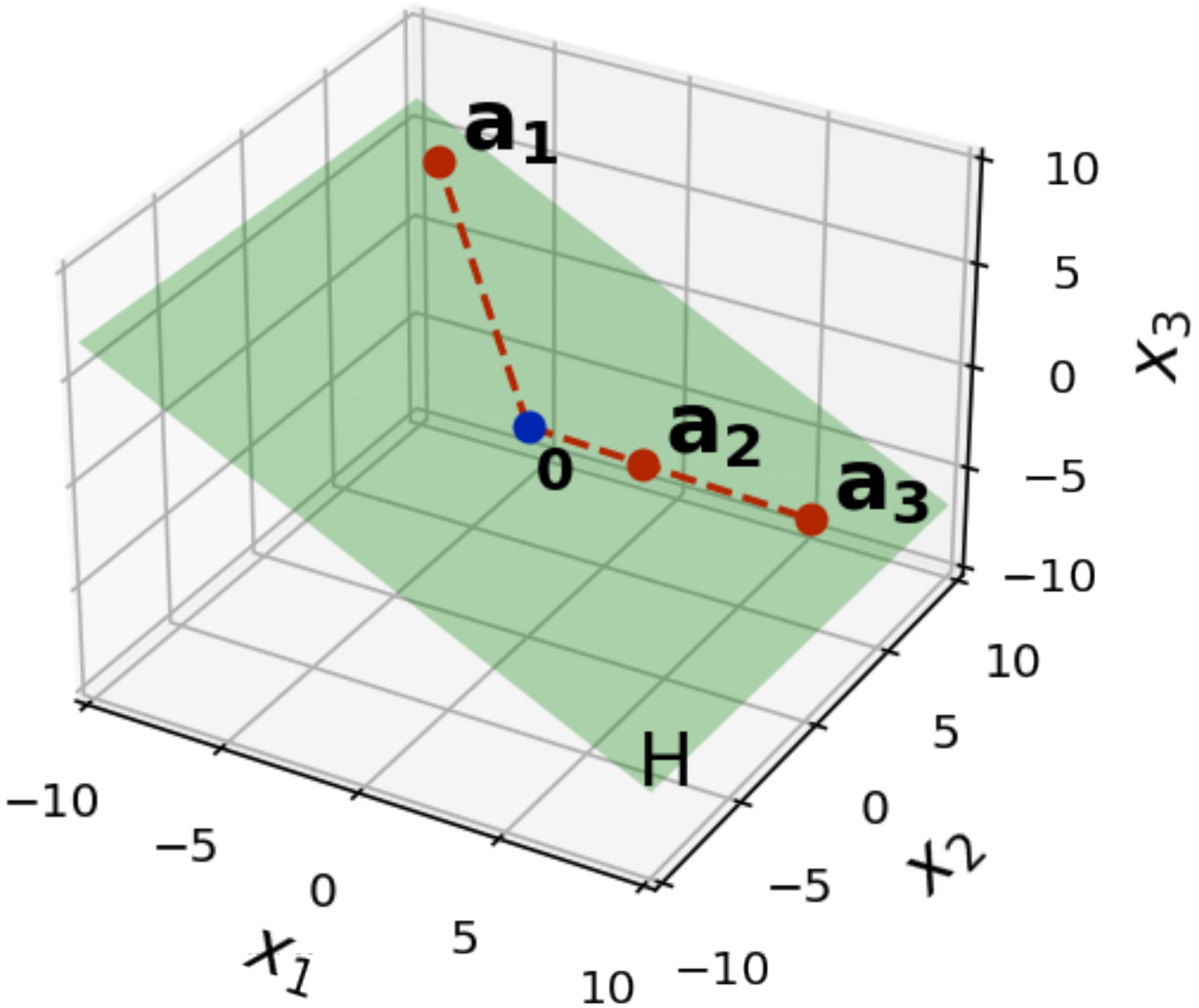
Recap

Recall: The Idea Behind Subspaces



Recall: The Idea Behind Subspaces

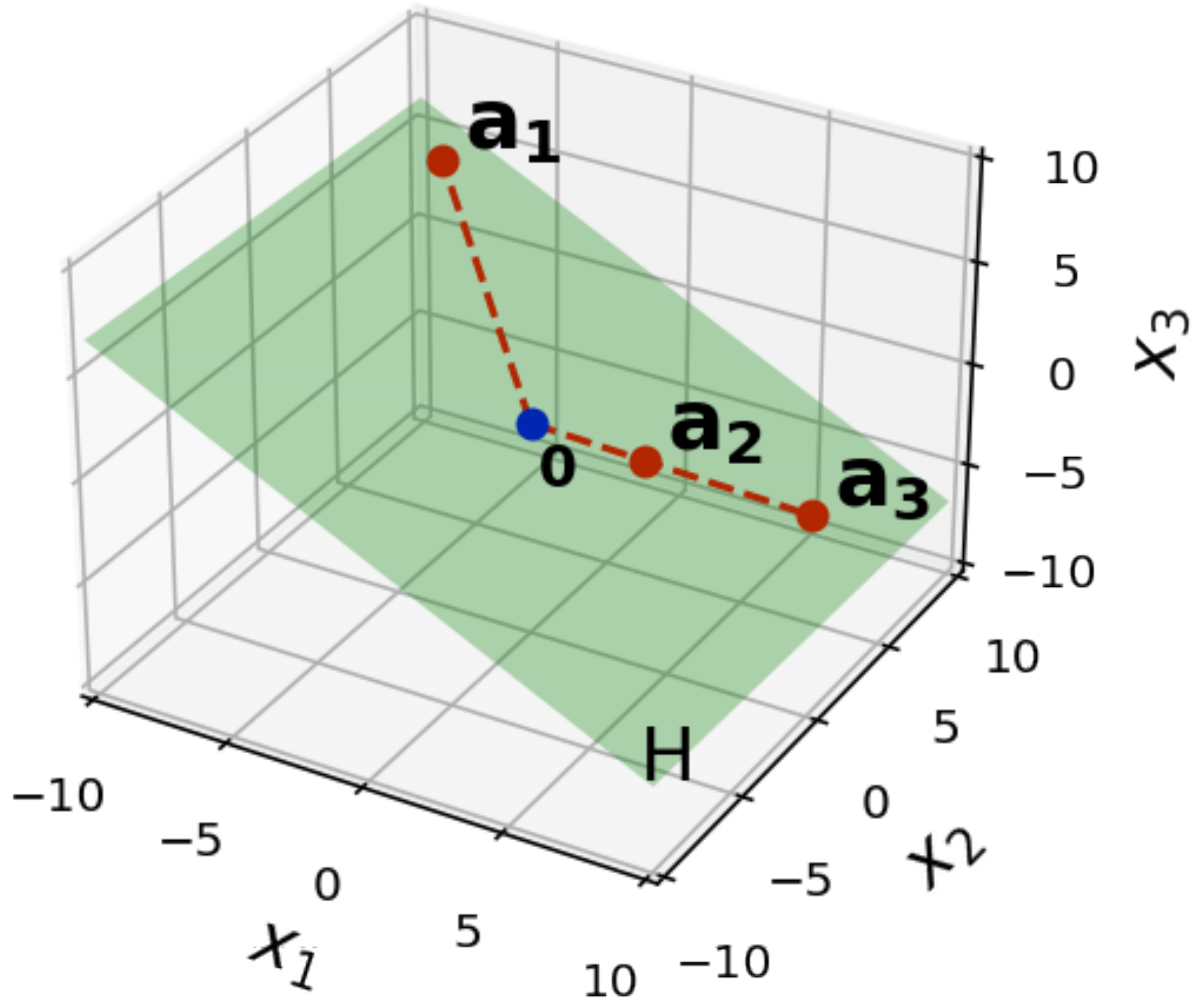
"sub" means "part of" or "below"



Recall: The Idea Behind Subspaces

"sub" means "part of" or "below"

A plane in $\mathbb{R}^3$ looks like a (possibly tilted) copy of $\mathbb{R}^2$



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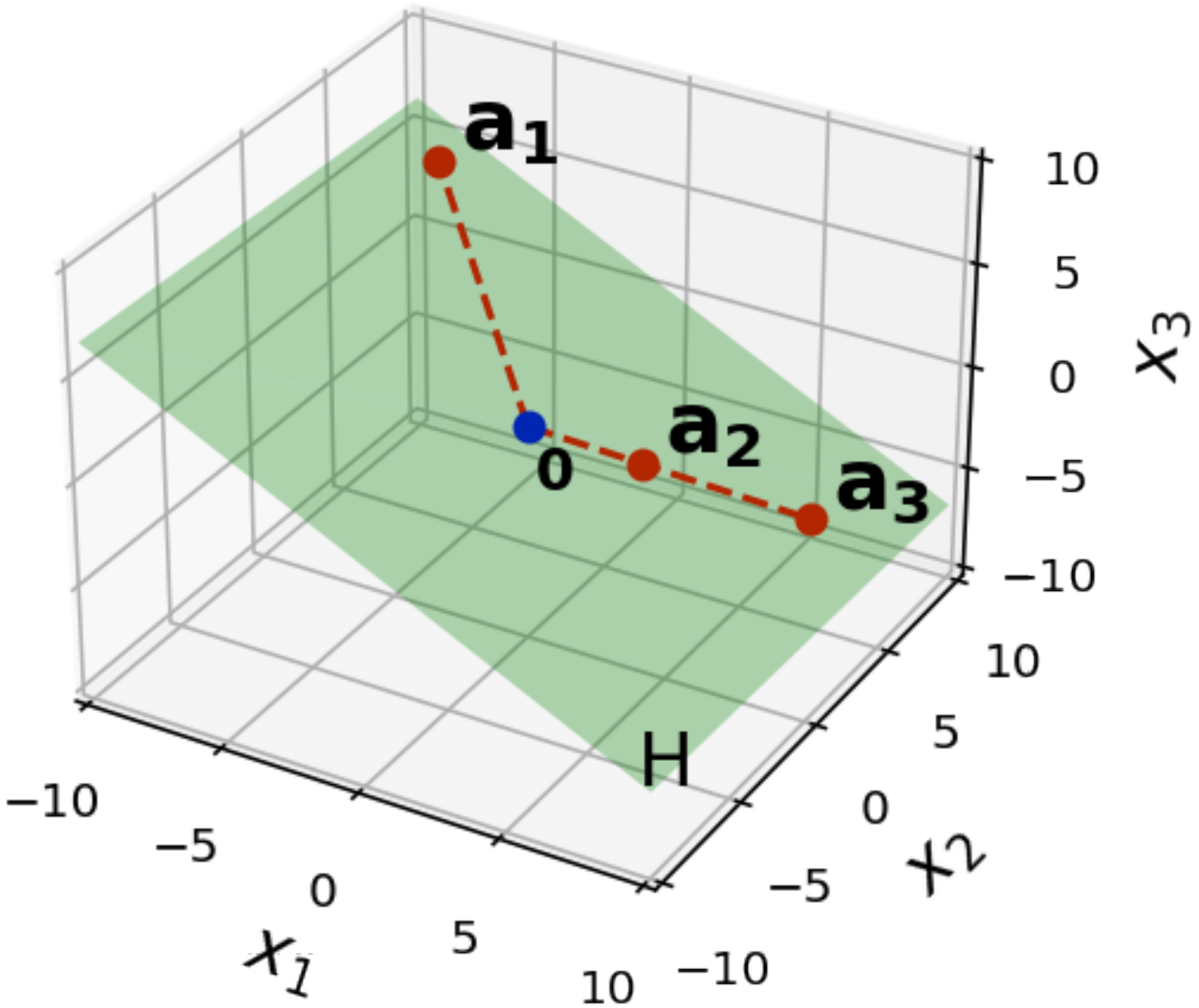
A plane in $\mathbb{R}^3$ looks like a (possibly tilted) copy of $\mathbb{R}^2$

Subspaces *generalize* of this idea.



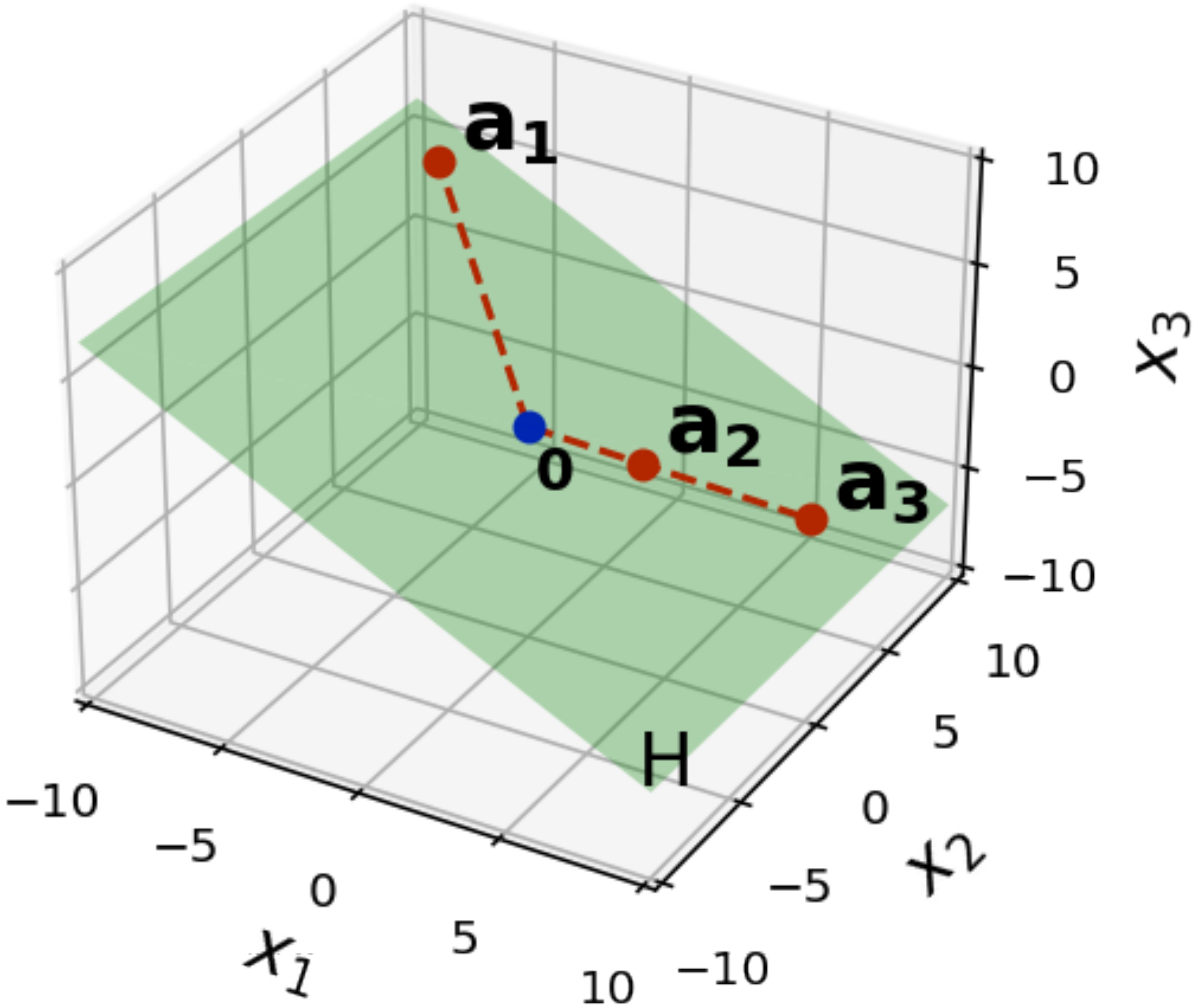
Recall: The Idea Behind Subspaces

"sub" means "part of" or "below"

A plane in $\mathbb{R}^3$ looks like a (possibly tilted) copy of $\mathbb{R}^2$

Subspaces *generalize* of this idea.

For example, there can be a "possibly tilted copy" of $\mathbb{R}^3$ sitting in $\mathbb{R}^5$



Recall: Subspace (Algebraic Definition)

Definition. A **subspace** of $\mathbb{R}^n$ is a set H of vectors in $\mathbb{R}^n$ such that

1. for every $\mathbf{u}$ and $\mathbf{v}$ in H, the vector $\mathbf{u} + \mathbf{v}$ is in H

2. for every $\mathbf{u}$ in H and scalar c, the vector $c\mathbf{u}$ is in H

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2. for every $\mathbf{u}$ in H and scalar c, the vector $c\mathbf{u}$ is in H **H is closed under scaling**

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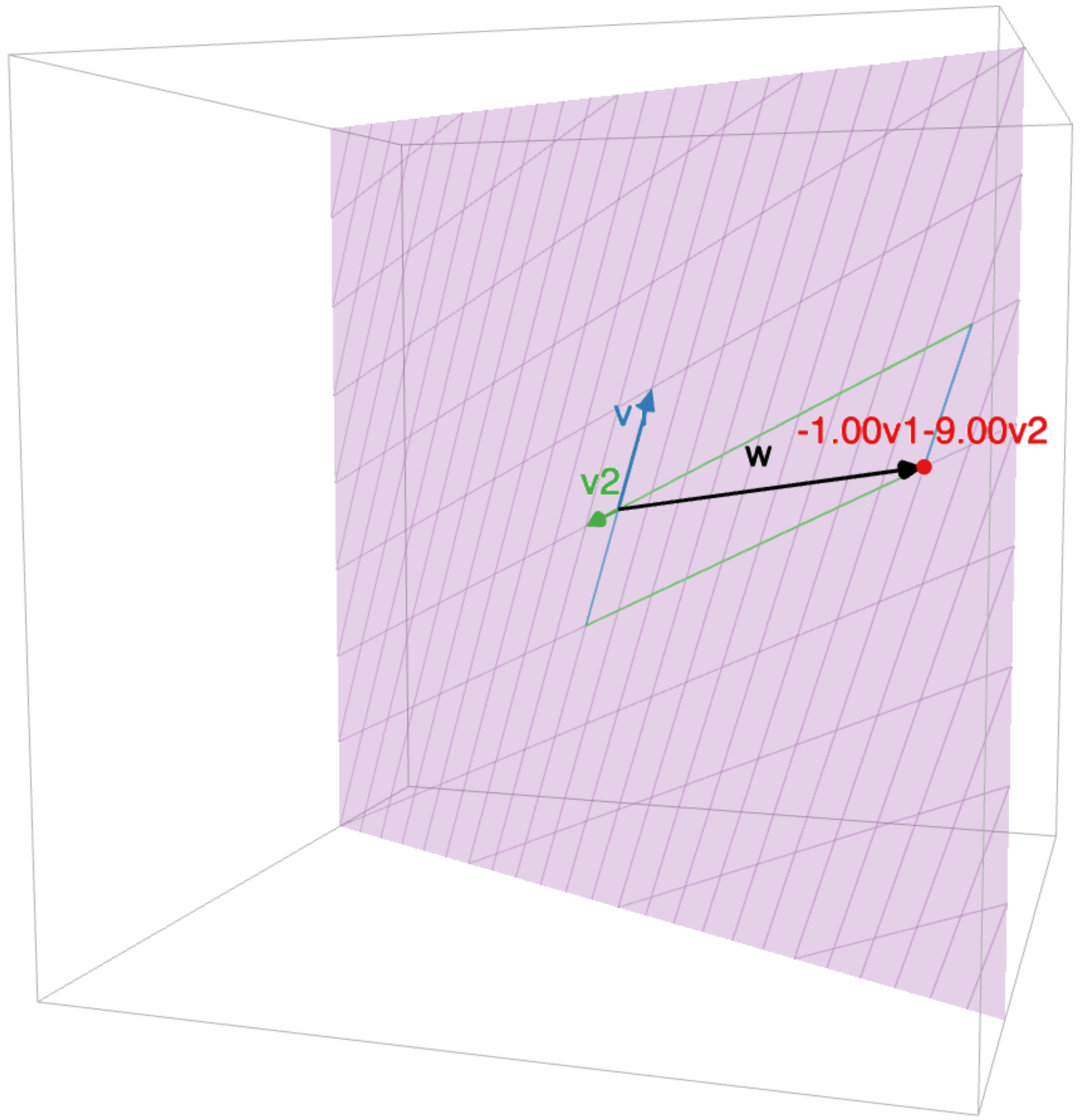
!! Subspaces must "live" somewhere !!

Recall: How to Think About this Definition

It's not possible to "leave" H by addition or scaling.

(recall this is also how we discussed spans)

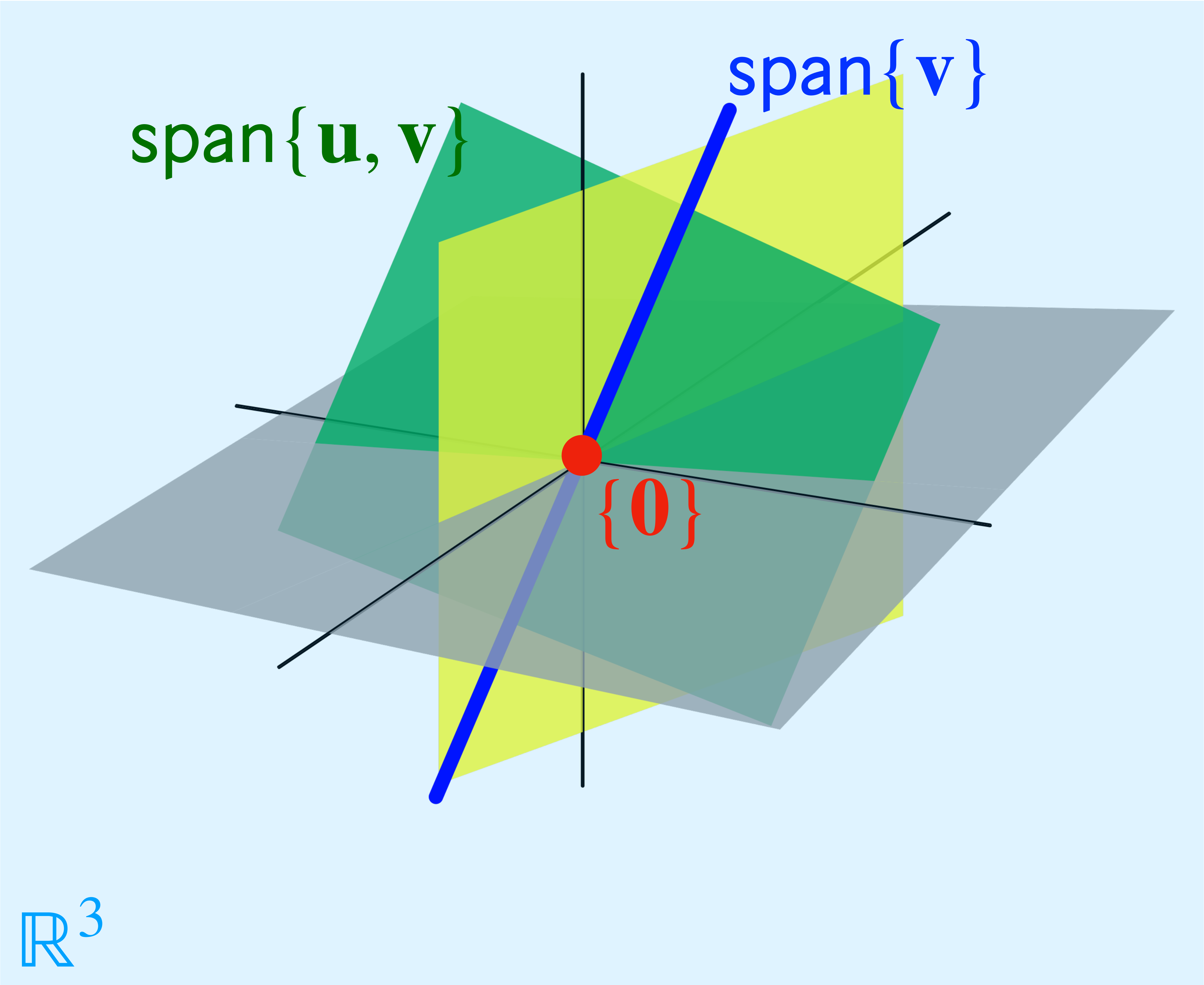


https://textbooks.math.gatech.edu/ila/spans.html

Recall: Subspace in $\mathbb{R}^3$ (Geometrically)

There are only 4 kinds of subspaces of $\mathbb{R}^3$:

1. $\{\mathbf{0}\}$ just the origin
2. lines (through the origin)
3. planes (through the origin)
4. All of $\mathbb{R}^3$



https://commons.wikimedia.org/wiki/File:Linear_subspaces_with_shading.svg

Column Space

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Definition. The **column space** of a matrix A, written $\text{Col}(A)$ or $\text{Col}\,A$, is the set of all linear combinations of the columns of A.

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The column space of a matrix is the span of its columns.

The column space of a matrix is the range of the linear transformation it implements.

Subspace of What?

$$m \left| \quad \overset{n}{\overline{\begin{bmatrix} | & | & \cdots & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_{n-1} & \mathbf{a}_n \\ | & | & \cdots & | & | \end{bmatrix}}} \right.$$

$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \ldots c_n\mathbf{a}_n$ **is a vector in** $\mathbb{R}^m$

$\text{Col}(A)$

is a subspace of

$\mathbb{R}^m$

Null Space

Null Space

Definition. The **null space** of a matrix A, written $\text{Nul}(A)$ or $\text{Nul}\,A$, is the set of all solutions to the homogenous equation

$$A\mathbf{x} = \mathbf{0}$$

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$$A\mathbf{x} = \mathbf{0}$$

The null space of a matrix A is the set of all vectors that are mapped to the zero vector by A.

Subspace of What?

n columns

m rows

$$A\mathbf{v} = \mathbf{0}$$

$m \times n$ $\quad n \times 1 \quad$ $m \times 1$

$\mathbf{v}$ **is a vector in** $\mathbb{R}^n$

$\text{Nul}(A)$

is a subspace of

$\mathbb{R}^n$

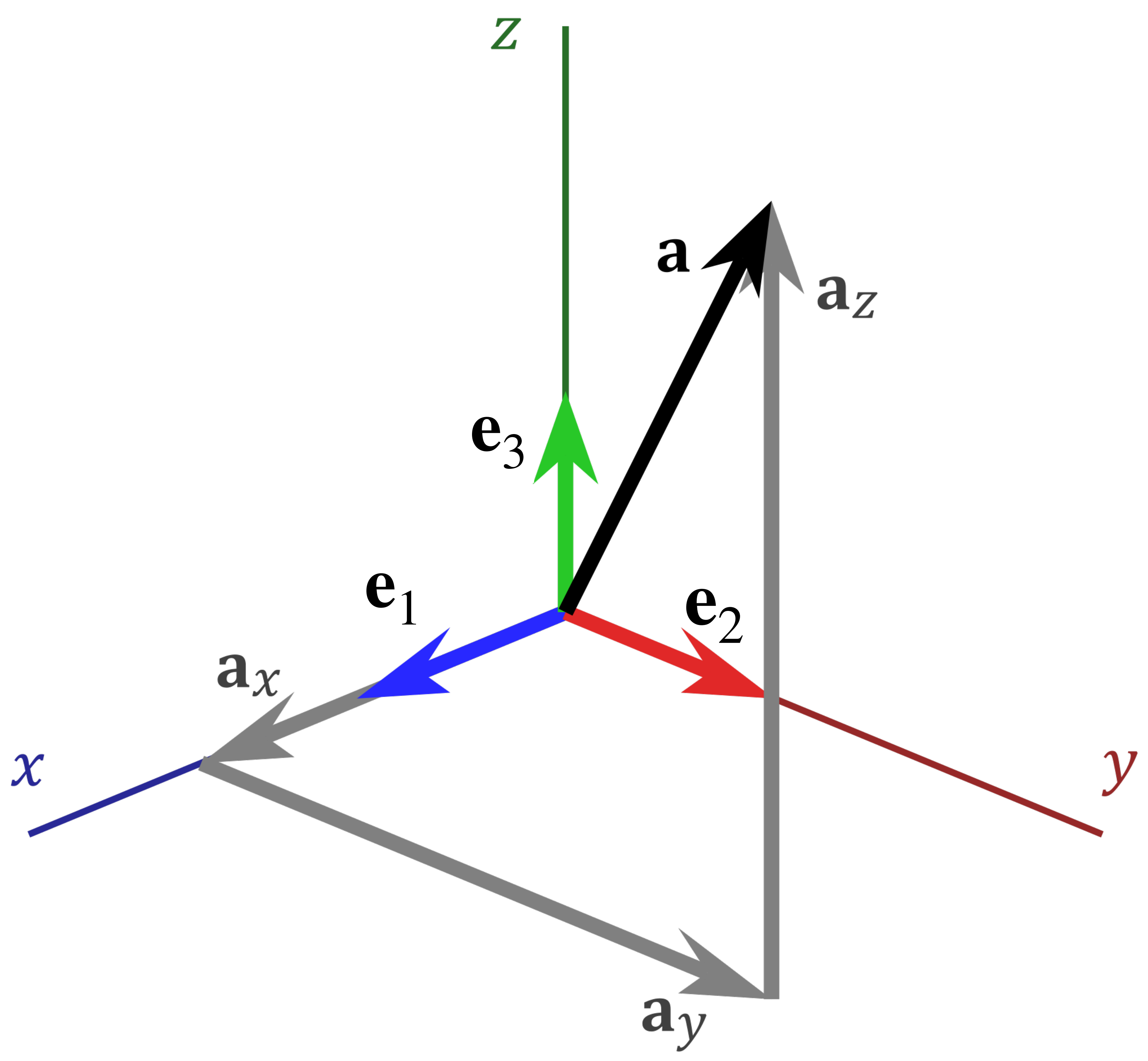
Recall: Basis

Recall: Basis

Definition. A **basis** for a subspace H of $\mathbb{R}^n$ is a linearly independent set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of vectors that spans H (in symbols: $H = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$).

A basis is a *minimal* set of vectors which spans all of H.

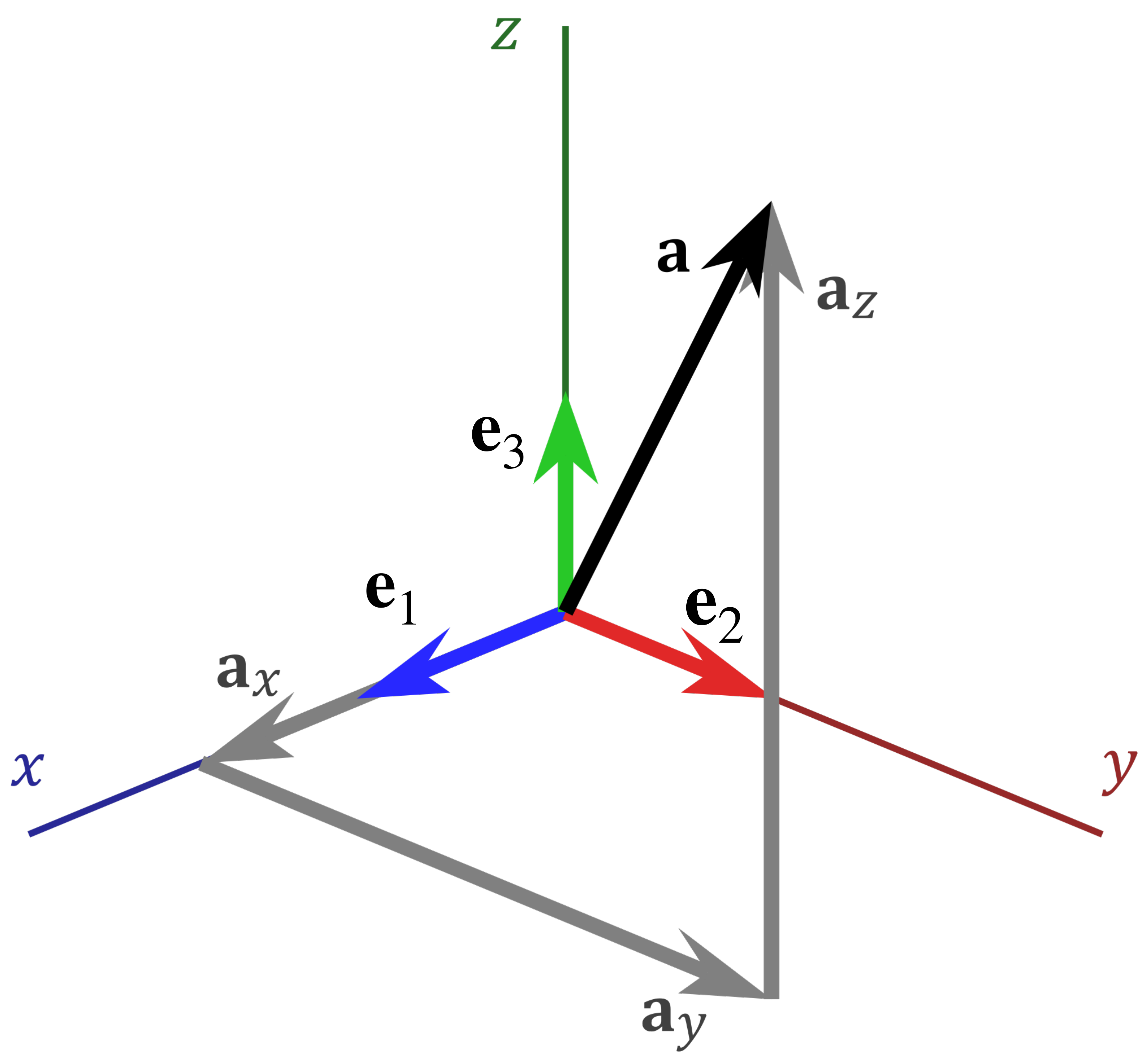
Recall: What's interesting about the standard basis?



Recall: What's interesting about the standard basis?

The n standard basis vectors in $\mathbb{R}^n$:

» are linearly independent
» span all of $\mathbb{R}^n$

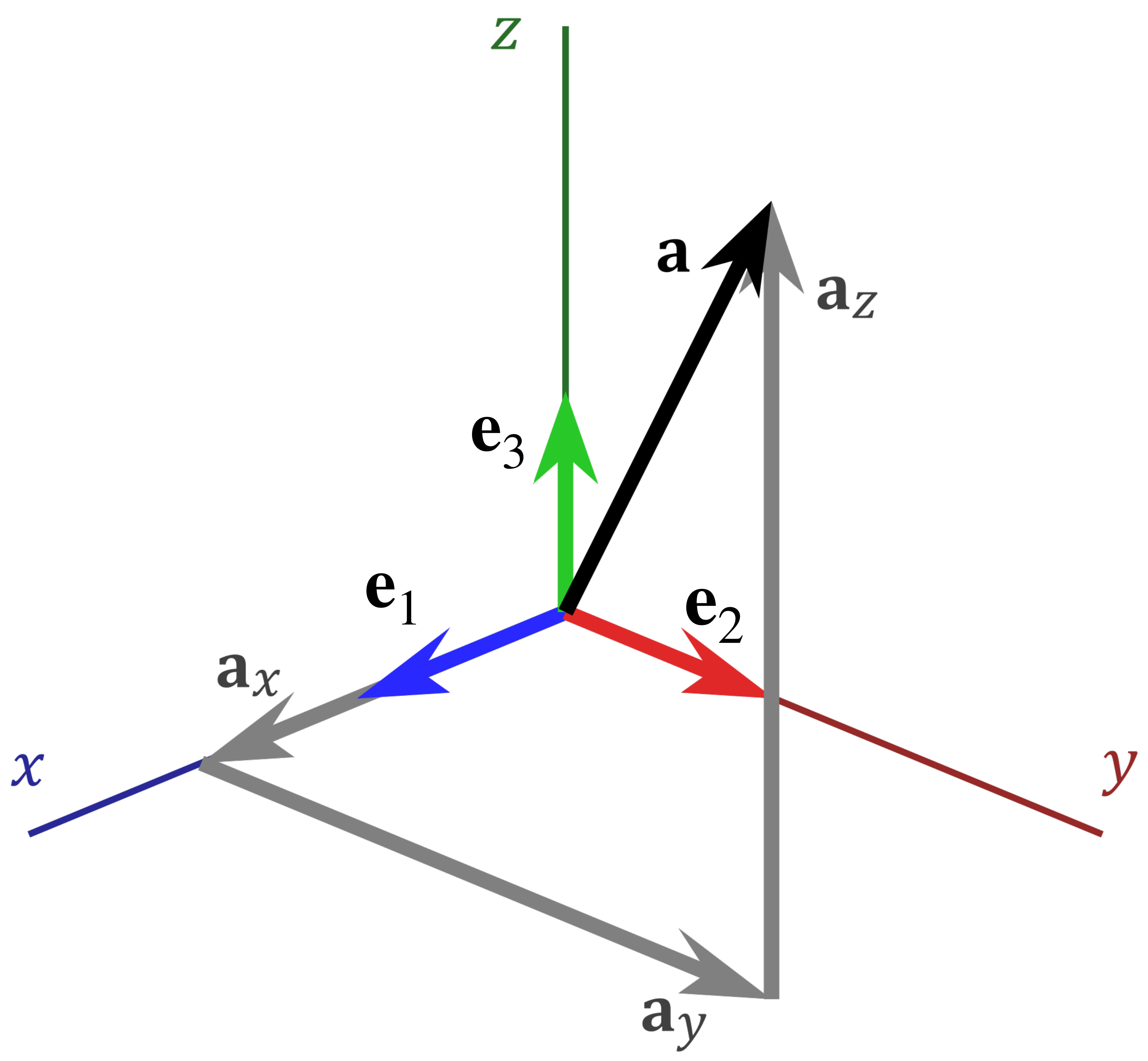


Recall: What's interesting about the standard basis?

The n standard basis vectors in $\mathbb{R}^n$:

» are linearly independent
» span all of $\mathbb{R}^n$

Their span is as "large" as possible while the set of vectors generating the span is as "small" as possible.



Recall: Example: Standard basis

The standard basis is a basis of $\mathbb{R}^n$.

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \ldots + v_n\mathbf{e}_n$$

Every column vector can be written in <u>exactly one way</u> as a linear combination of standard basis vectors

Recall: Example: Column Space of Invertible Matrices

Fact. The columns of an invertible $n \times n$ matrix form a basis of $\mathbb{R}^n$.

Verify: IMT tells us

columns of A are L.I. ✓

columns of A span $\mathbb{R}^n$ ✓

Bases of Column Space and Null Space

The Goal of this Section

Determine how to find bases for the **column space** and the **null space** of a given matrix.

How to: Finding a basis for the null space

Question. Given a $m \times n$ matrix A find a basis for $\mathrm{Nul}(A)$.

How to: Finding a basis for the null space

Question. Given a $m \times n$ matrix A find a basis for $\text{Nul}(A)$.

The idea. Describe the solutions of $A\mathbf{x} = 0$ as linear combination of vectors

Example

$$A \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Suppose A has the above reduced echelon form.

Let's write down a general form solution for A:

x_1, x_2, x_3, x_4, x_5

$$\left[\begin{array}{ccccc|c} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$A\vec{x} = \vec{0}$

$x_1 = 2x_2 + x_4 - 3x_5$

x_2 is free

$x_3 = -2x_4 + 2x_5$

x_4 is free

x_5 is free

Parametric Solutions

We can think of our general form solution as a <u>(linear) transformation</u>.

$$x_1 = 2x_2 + x_4 - 3x_5$$

x_2 is free

$$x_3 = (-2)x_4 + 2x_5$$

x_4 is free

x_5 is free

$\equiv$

"given values for x_2, x_3, and x_4, I can give you a solution"

Parametric Solutions

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$$\equiv$$

$$\begin{bmatrix} s \\ t \\ u \end{bmatrix} \mapsto \begin{bmatrix} 2s + t - 3u \\ s \\ (-2)t + 2u \\ t \\ u \end{bmatrix}$$

Parametric Solutions

We can think of our general form solution as a <u>(linear) transformation</u>.

!! this transformation is only linear !!
!! in the case of homogeneous equations !!

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Example

$$\begin{bmatrix} s \\ t \\ u \end{bmatrix} \mapsto \begin{bmatrix} 2s + t - 3u \\ s \\ (-2)t + 2u \\ t \\ u \end{bmatrix}$$

Let's find the matrix implementing this linear transformation:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & -3 \\ 1 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

implements

Example

$$\begin{bmatrix} 2 & 1 & -3 \\ 1 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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Every solution to $A\mathbf{x} = \mathbf{0}$ can be written as an *image* of this transformation.

Example

$$\begin{bmatrix} 2 & 1 & -3 \\ 1 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Every solution to $A\mathbf{x} = \mathbf{0}$ can be written as an *image* of this transformation.

So every solution can be written as a linear combination of its columns.

Example

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Every solution to $A\mathbf{x} = \mathbf{0}$ can be written as an *image* of this transformation.

So every solution can be written as a linear combination of its <u>columns</u>.

The columns of this matrix <u>span</u> $\text{Nul}(A)$.

Example

$$\begin{bmatrix} 2 & 1 & -3 \\ 1 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

x_2 is free

x_4 is free

x_5 is free

The columns of this matrix are linearly independent.

Verify: suppose

$$a_1 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$a_1(1) + a_2(0) + a_3(0) = 0$

$a_1 = 0$

$a_1(0) + a_2(1) + a_3(0) = 0$

$a_2 = 0$

$a_3 = 0$

Example

$$\begin{bmatrix} 2 & 1 & -3 \\ 1 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The columns of this matrix span $\text{Nul}(A)$.

The columns of this matrix are linearly independent.

The columns of this matrix form a basis for $\text{Nul}(A)$.

Example

Alternatively, we can think of writing a general form solution so that it is a linear combination of vectors with free variables as weights:

$x_1 = 2x_2 + x_4 - 3x_5$
x_2 is free
$x_3 = (-2)x_4 + 2x_5$
x_4 is free
x_5 is free

$$x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

How to: Finding a basis for the null space

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Question. Given a $m \times n$ matrix A find a basis for $\text{Nul}(A)$.

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Solution.

1. Find a general form solution for $A\mathbf{x} = \mathbf{0}$.

How to: Finding a basis for the null space

Question. Given a $m \times n$ matrix A find a basis for $\mathrm{Nul}(A)$.

Solution.

1. Find a general form solution for $A\mathbf{x} = \mathbf{0}$.
2. Write this solution as a linear combination of vectors <u>where the free variables are the weights</u>.

How to: Finding a basis for the null space

Question. Given a $m \times n$ matrix A find a basis for Nul(A).

Solution.

1. Find a general form solution for $A\mathbf{x} = \mathbf{0}$.
2. Write this solution as a linear combination of vectors where the free variables are the weights.
3. The resulting vectors form a basis for Nul(A).

An Observation

The *number* of vectors in the basis we found is the same as the number of free variables in our general form solution.

$x_1 = 2x_2 + x_4 - 3x_5$

x_2 is free

$x_3 = (-2)x_4 + 2x_5$

x_4 is free

x_5 is free

$$\equiv$$

$$\begin{bmatrix} s \\ t \\ u \end{bmatrix} \mapsto \begin{bmatrix} 2s + t - 3u \\ s \\ (-2)t + 2u \\ t \\ u \end{bmatrix}$$

moving on to column space...

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We already know the columns of A span $\mathrm{Col}(A)$.

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We already know the columns of A span $\mathsf{Col}(A)$.

So we also already know *some* subset of columns of A form a basis for $\mathsf{Col}(A)$.

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Question. Given a $m \times n$ matrix A, find a basis for $\text{Col}(A)$.

We already know the columns of A span $\text{Col}(A)$.

So we also already know *some* subset of columns of A form a basis for $\text{Col}(A)$.

Which columns of A should we choose?

$\text{Col}(A) = \mathbb{R}^4$

$A = 4\Big[\quad\overset{10}{}\quad\Big]$

Column Space and Reduced Echelon form

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4 \quad \mathbf{a}_5] \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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The idea. What if we cover up the non-pivot columns?

Column Space and Reduced Echelon form

$$\begin{bmatrix} \mathbf{a}_1 & \text{[redacted]} & \mathbf{a}_3 & \text{[redacted]} & \text{[redacted]} \end{bmatrix} \sim \begin{bmatrix} 1 & \text{[redacted]} & 0 & \text{[redacted]} & \text{[redacted]} \\ 0 & & 1 & & \\ 0 & & 0 & & \end{bmatrix}$$

The idea. What if we cover up the non-pivot columns?

Then we see $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_3 \end{bmatrix}$ has 2 pivots.

Column Space and Reduced Echelon form

$$[\mathbf{a}_1 \ \text{[redacted]} \ \mathbf{a}_3 \ \text{[redacted]} \ \text{[redacted]}] \sim \begin{bmatrix} 1 & \text{[redacted]} & 0 & \text{[redacted]} & \text{[redacted]} \\ 0 & & 1 & & \\ 0 & & 0 & & \end{bmatrix}$$

The idea. What if we cover up the non-pivot columns?

Then we see $[\mathbf{a}_1 \ \ \mathbf{a}_3]$ has 2 pivots.

So the pivot columns are linearly independent.

Column Space and Reduced Echelon form

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4 \quad \mathbf{a}_5] \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Column Space and Reduced Echelon form

$$\begin{bmatrix} \overset{2}{\mathbf{a}_1} & \overset{1}{\mathbf{a}_2} & \text{[redacted]} \end{bmatrix} \sim \begin{bmatrix} \overset{2}{1} & \overset{1}{-2} & \text{[redacted]} \\ 0 & 0 & \\ 0 & 0 & \end{bmatrix}$$

Observation. $[2\ 1\ 0\ 0\ 0]^T$ is a solution to the system $A\mathbf{x} = \mathbf{0}$.

Column Space and Reduced Echelon form

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So $2\mathbf{a}_1 + \mathbf{a}_2 = \mathbf{0}$ and $\mathbf{a}_2 = (-2)\mathbf{a}_1$.

Column Space and Reduced Echelon form

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In general, every non-pivot column of A can be written as a linear combination pivots in front of it.

Column Space and Reduced Echelon form

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So $2\mathbf{a}_1 + \mathbf{a}_2 = \mathbf{0}$ and $\mathbf{a}_2 = (-2)\mathbf{a}_1$.

In general, every non-pivot column of A can be written as a linear combination pivots in front of it.

This tells us that $\mathbf{a}_1$ and $\mathbf{a}_3$ <u>span</u> $\text{Col}(A)$.

Column Space and Reduced Echelon form

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4 \quad \mathbf{a}_5] \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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The takeaway. The pivot columns of A form a basis for $\mathrm{Col}(A)$.

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The takeaway. The pivot columns of A form a basis for $\text{Col}(A)$.

!! IMPORTANT !!

Choose the columns of A.

($\mathbf{e}_1$ and $\mathbf{e}_2$ do not necessarily form a basis for $\text{Col}(A)$)

How To: Finding a basis for the column space

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Solution.

1. Find the pivot columns in an echelon form of A.

How To: Finding a basis for the column space

Question. Given a $m \times n$ matrix A, find a basis for $\text{Col}(A)$.

Solution.

1. Find the pivot columns in an echelon form of A.

2. The associated columns <u>in A</u> form a basis for $\text{Col}(A)$.

An Observation

The *number* of vectors in the basis we found is the same as the number of basic variable or equivalently the number of pivot columns.

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4 \quad \mathbf{a}_5] \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4 \quad \mathbf{a}_5] \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Question

$$A = \begin{bmatrix} 1 & -2 & 19 & 0 & -4 \\ 1 & 0 & 9 & 1 & 1 \\ 1 & -1 & 14 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 9 & 0 & 0 \\ 0 & 1 & -5 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Find a bases for the column space and null space of A.

$$A = \begin{bmatrix} 1 & -2 & 19 & 0 & -4 \\ 1 & 0 & 9 & 1 & 1 \\ 1 & -1 & 14 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 9 & 0 & 0 \\ 0 & 1 & -5 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Answer

basis for Col(A): $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 9 \\ 1 \\ 1 \end{bmatrix} \right\}$

basis for Nul(A): $\left\{ \begin{bmatrix} -9 \\ 5 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

$x_1 = -9x_3$

$x_2 = 5x_3 - 2x_5$

x_3 is free

$x_4 = -x_5$

x_5 is free

$$x_3 \begin{bmatrix} -9 \\ 5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

moving on...

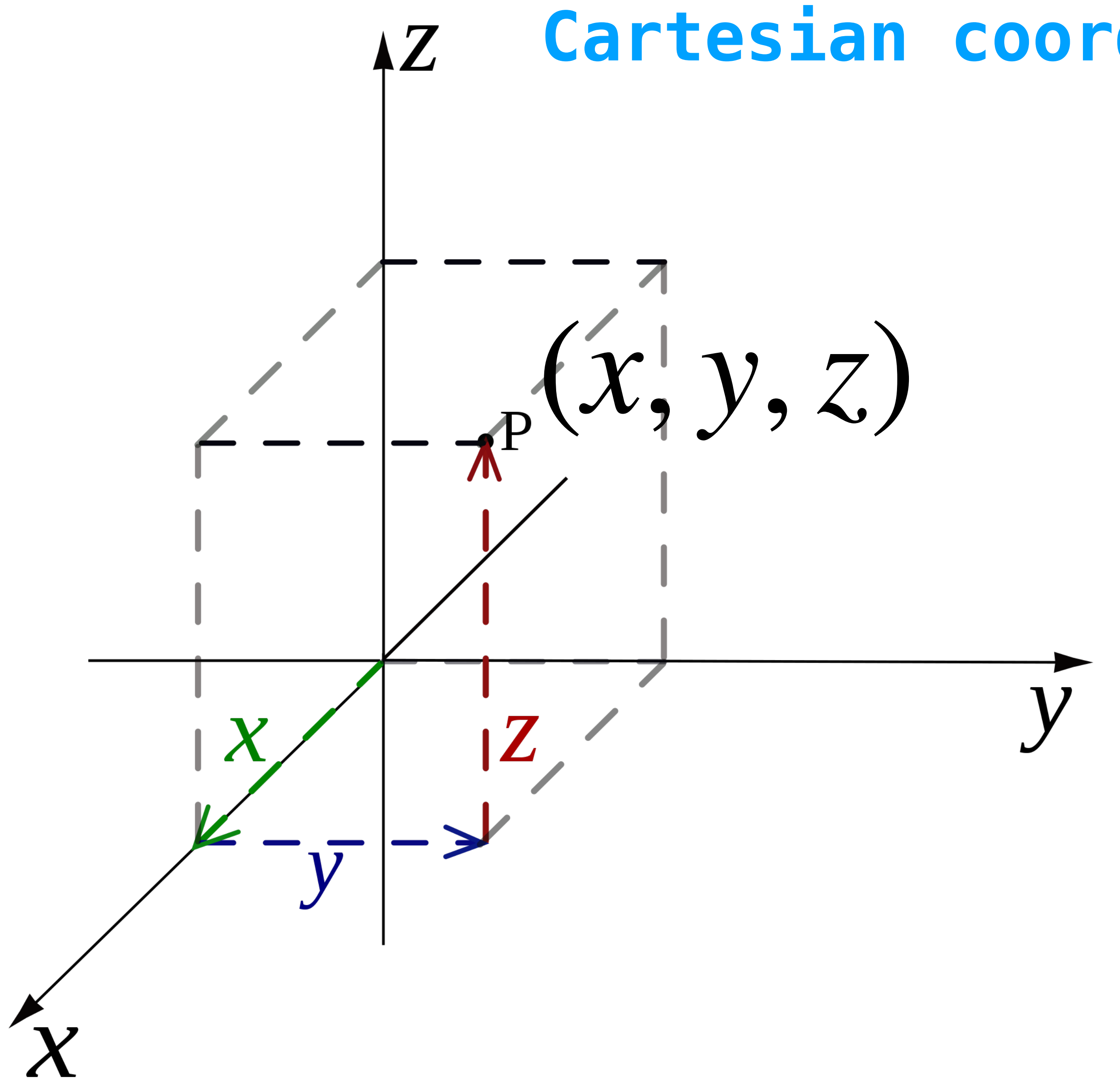
Coordinate Systems

At a High Level

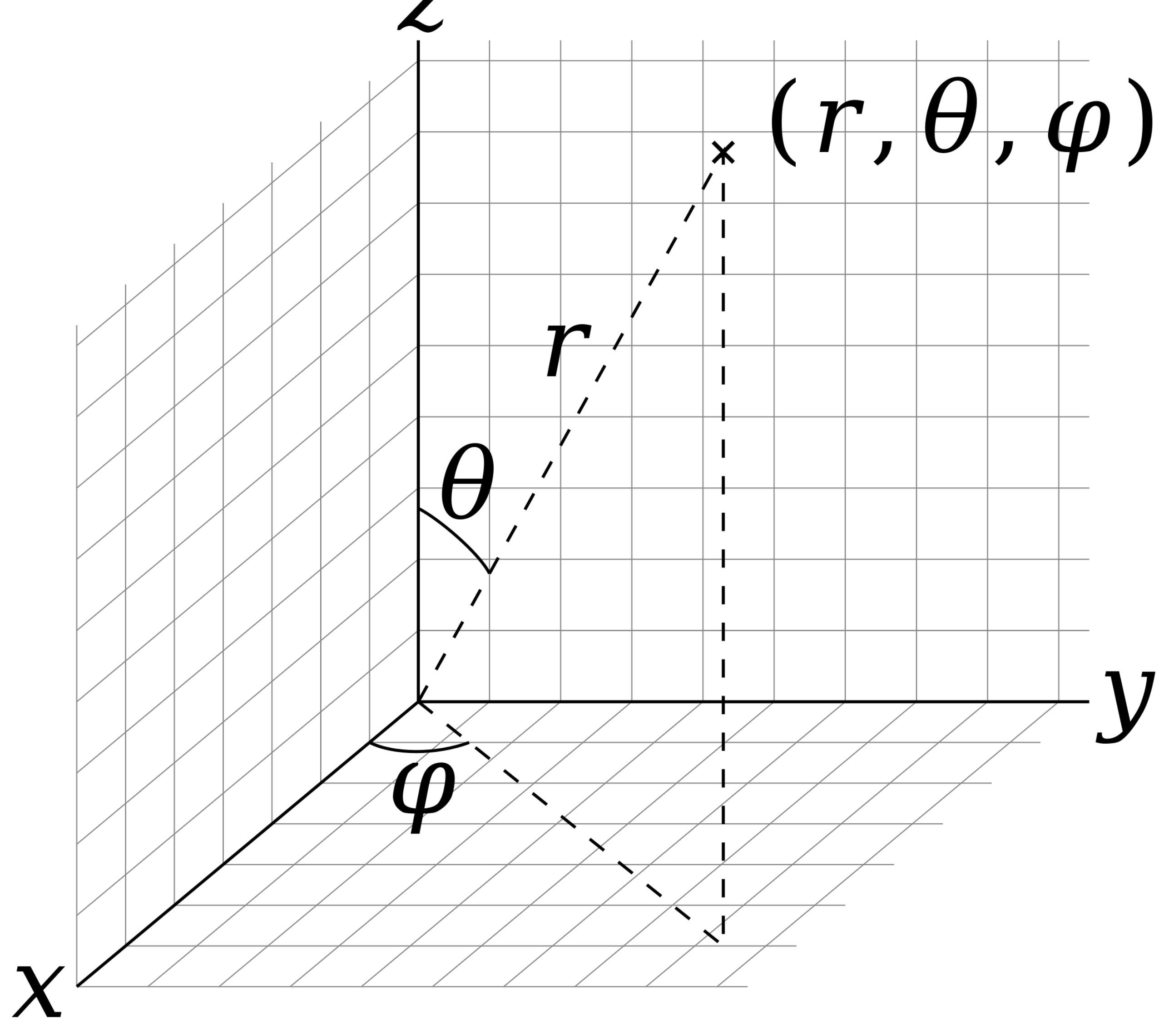
A coordinate system is a way of representing positions in terms of a sequence of numbers.

Examples.

Cartesian coordinates



Polar coordinates



Question (Conceptual)*

*And a bit of a trick question

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Is $(2.3, 0.01, 5)$ *a polar coordinate or a cartesian coordinate?*

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This question is non-sensical.

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Question (Conceptual)*

Is $(2.3, 0.01, 5)$ *a polar coordinate or a cartesian coordinate?*

This question is non-sensical.

It's just a sequence of numbers. We need to be *told* if it should be interpreted in the **polar** coordinate system or the **Cartesian** coordinate system.

*And a bit of a trick question

Bases define Coordinate Systems

Given a basis $\mathscr{B}$ of a subspace H, there is **exactly one way** to write every vector in H as a linear combination of vectors in $\mathscr{B}$.

Verify: $\mathcal{B} = \{ \vec{b}_1, \vec{b}_2, \vec{b}_3 \}$

$$\vec{r} = a_1\vec{b}_1 + a_2\vec{b}_2 + a_3\vec{b}_3 \qquad \vec{r} = c_1\vec{b}_1 + c_2\vec{b}_2 + c_3\vec{b}_3$$

$$(a_1 - c_1)\vec{b}_1 + (a_2 - c_2)\vec{b}_2 + (a_3 - c_3)\vec{b}_3 = a_1\vec{b}_1 + a_2\vec{b}_2 + a_3\vec{b}_3 - c_1 b_1 + c_2 b_2 + c_3 b_3 = \vec{r} - \vec{r} = \vec{0}$$

$$(a_1 - c_1) = 0 \quad (a_2 - c_2) = 0 \quad (a_3 - c_3) = 0$$

Bases define Coordinate Systems

Given a basis $\mathscr{B}$ of a subspace H, there is **exactly one way** to write every vector in H as a linear combination of vectors in $\mathscr{B}$.

Every basis provides a way to write down *coordinates* of a vector.

And every time we write down a vector, we are **assuming a coordinate system.**

what do we mean by this?

A Thought Experiment

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Imagine doing this whole class from the beginning, but never saying *what vectors are.*

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(This is actually how we would do linear algebra if this were a math class)

A Thought Experiment

Imagine doing this whole class from the beginning, but never saying *what vectors are.*

(This is actually how we would do linear algebra if this were a math class)

Then one day, you get tired of talking about "abstract" vectors, you want to work with *numbers.*

A Thought Experiment

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Because we've learned everything up to now, we know that there is a basis $\mathbf{b}_1$, $\mathbf{b}_2$,..., $\mathbf{b}_n$ for the space $\mathbb{R}^n$.

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So given $\mathbf{v}$, if we know how to write it in terms of the basis, we can write...

$$\mathbf{v} = 2\mathbf{b}_1 + 3\mathbf{b}_2 + \ldots + (-0.1)\mathbf{b}_n$$

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$$\mathbf{v} = 2\mathbf{b}_1 + 3\mathbf{b}_2 + \ldots + (-0.1)\mathbf{b}_n$$

$$\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ \vdots \\ -0.1 \end{bmatrix}$$

and then choose those weights as a representation of v as a sequence of numbers

But wait...

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`This` *`depends`* `on the choice of basis.`

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If we started with $\mathbf{c}_1$, $\mathbf{c}_2$,..., $\mathbf{c}_n$ then we would get some other representation.

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If we started with $\mathbf{c}_1$, $\mathbf{c}_2$,..., $\mathbf{c}_n$ then we would get some other representation.

$$\mathbf{v} = (-10)\mathbf{c}_1 + (4.3)\mathbf{c}_2 + \ldots + 0\mathbf{c}_n = \begin{bmatrix} -10 \\ 4.3 \\ \vdots \\ 0 \end{bmatrix}$$

But wait...

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Every basis defined a different coordinate system

Standard Basis

The standard basis defines the Cartesian coordinate system for $\mathbb{R}^n$.

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \ldots + v_n\mathbf{e}_n$$

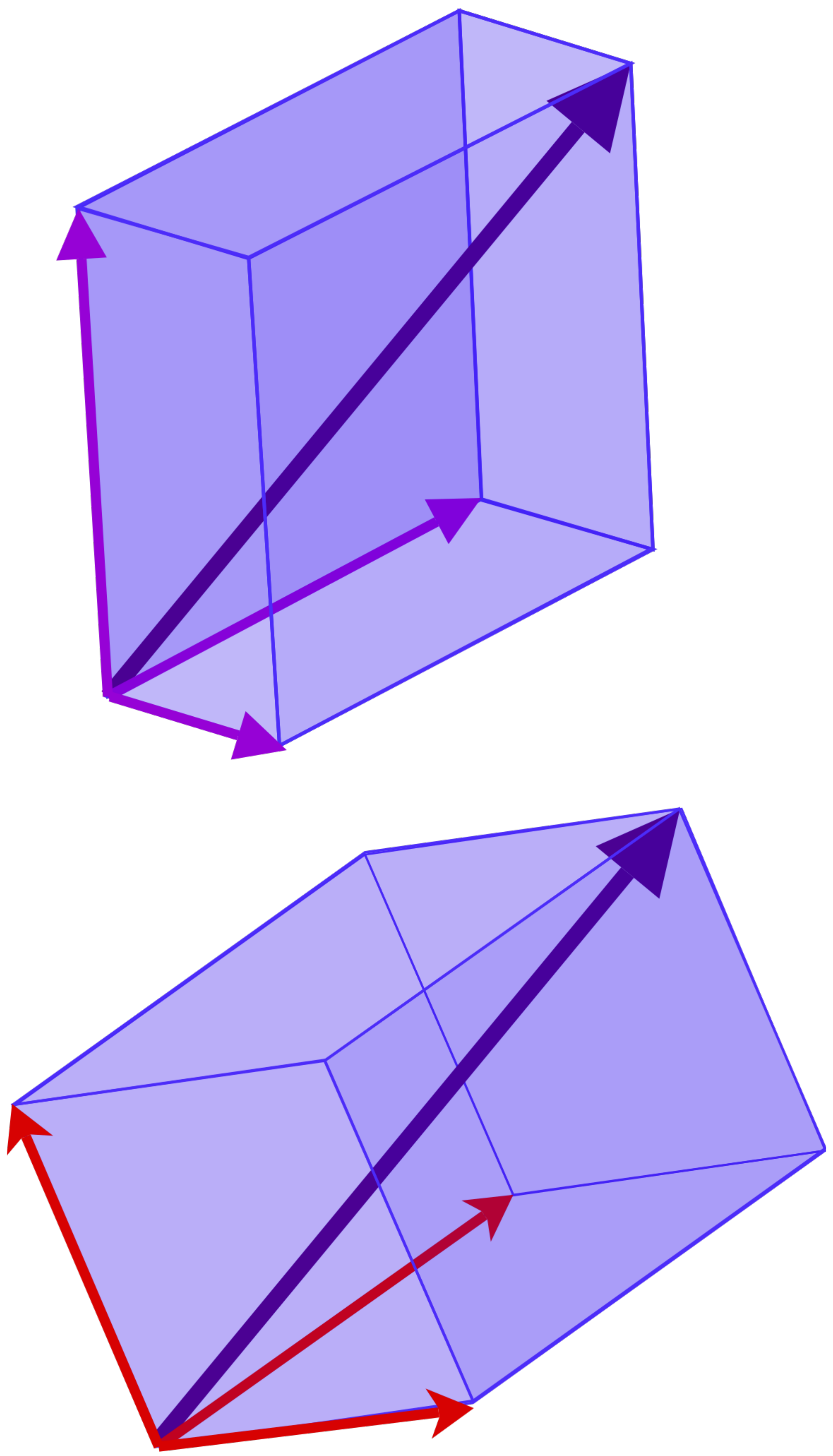
Column vectors are just weights for a linear combination of the standard basis

but we can also use
different coordinate systems

How to think about this

Changing the coordinate system "warps space".

The question is: how to we represent a vector $\mathbf{v}$ in the warped space if we wanted it to "be in the same place"?



https://commons.wikimedia.org/wiki/File:3d_two_bases_same_vector.svg

Coordinate Vectors

Coordinate Vectors

Let $\mathbf{v}$ be a vector in a subspace H of $\mathbb{R}^n$ and let $\mathscr{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k\}$ be a basis of H where

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Definition. The **coordinate vector of $\mathbf{v}$ relative to $\mathscr{B}$** is

$$[\mathbf{v}]_{\mathscr{B}} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix}$$

Coordinate Vectors and the Standard Basis

When we write down a vector $\mathbf{v}$ in $\mathbb{R}^n$, we're really writing down a coordinate vector **relative to the standard basis** $\mathscr{E}$**.**

$$[\mathbf{v}]_{\mathscr{E}} = \mathbf{v}$$

How do we find coordinate vectors?

For an arbitrary basis $\mathscr{B}$, to determine $[\mathbf{v}]_{\mathscr{B}}$, we need to find weights $a_1, \ldots, a_k$ such that

$$a_1\mathbf{b}_1 + \ldots + a_k\mathbf{b}_k = \mathbf{v}$$

This is just <u>solving a vector equation</u>.

Example: 2D Case

Write the coordinate vector for $\begin{bmatrix}1\\6\end{bmatrix}$ relative to the basis $\mathcal{B} = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}\right\}$ for $\mathbb{R}^2$

$$x_1\begin{bmatrix}1\\0\end{bmatrix} + x_2\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1\\6\end{bmatrix}$$

$$\left[\begin{array}{cc|c}1 & 1 & 1\\0 & 2 & 6\end{array}\right] \sim \left[\begin{array}{ccc}1 & 1 & 1\\0 & 1 & 3\end{array}\right] \sim \left[\begin{array}{cc|c}1 & 0 & -2\\0 & 1 & 3\end{array}\right]$$

$$\left[\begin{bmatrix}1\\6\end{bmatrix}\right]_{\mathcal{B}} = \begin{bmatrix}-2\\3\end{bmatrix}$$

Example: 2D Case (Geometrically)

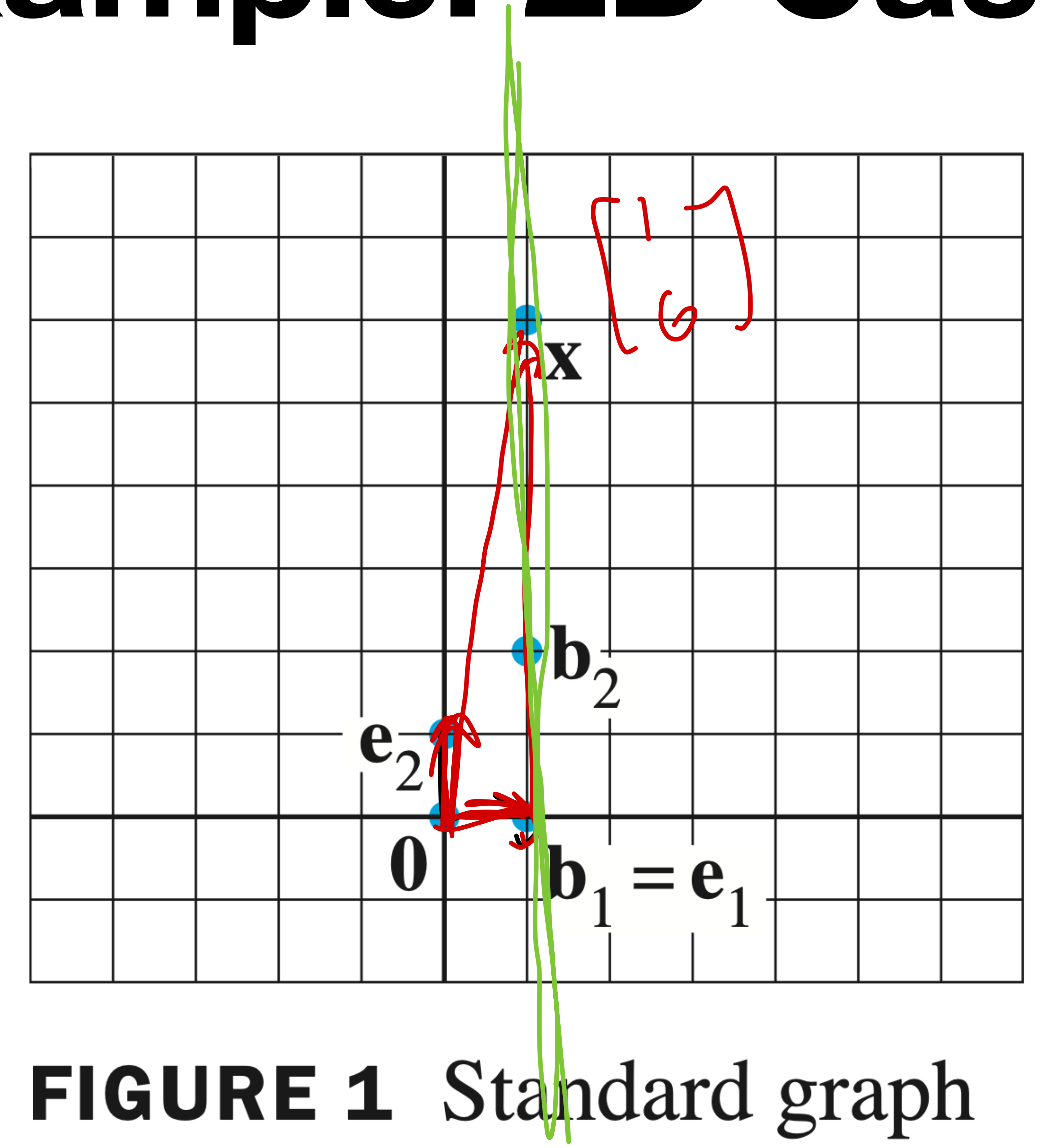


FIGURE 1 Standard graph paper.

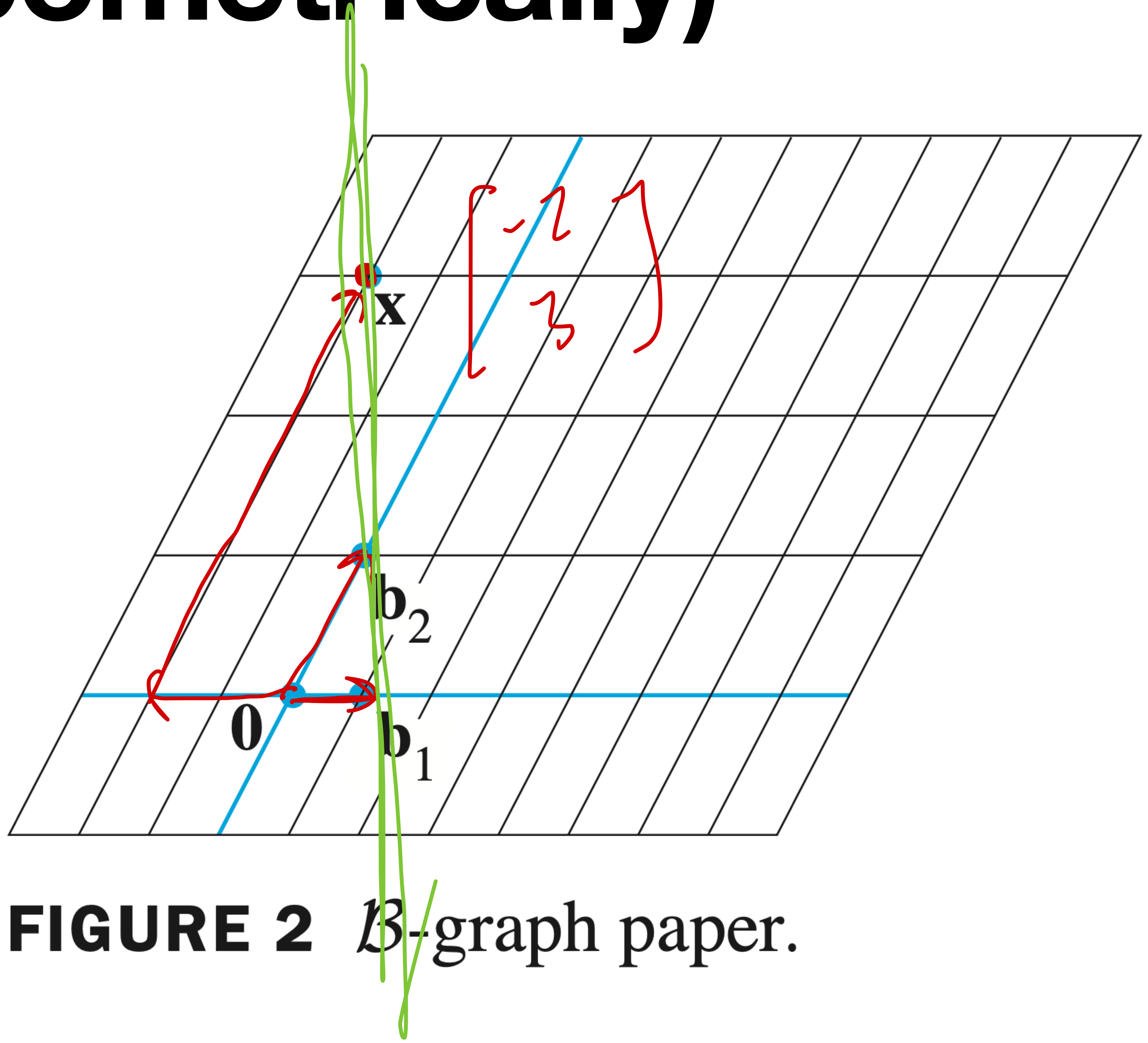


FIGURE 2 $\mathcal{B}$-graph paper.

$\mathcal{B}$ **defines a "different grid for our graph paper"**

Linear Algebra and its Applications (Lay, Lay, McDonald)

How To: Coordinate Vectors

Question. Find the coordinate vector for $\mathbf{v}$ in the subspace H relative to the basis $\mathbf{b}_1, \ldots, \mathbf{b}_k$.

Solution. Solve the vector equation

$$x_1\mathbf{b}_1 + \ldots + x_k\mathbf{b}_k = \mathbf{v}$$

A solution $(a_1, \ldots, a_k)$ means

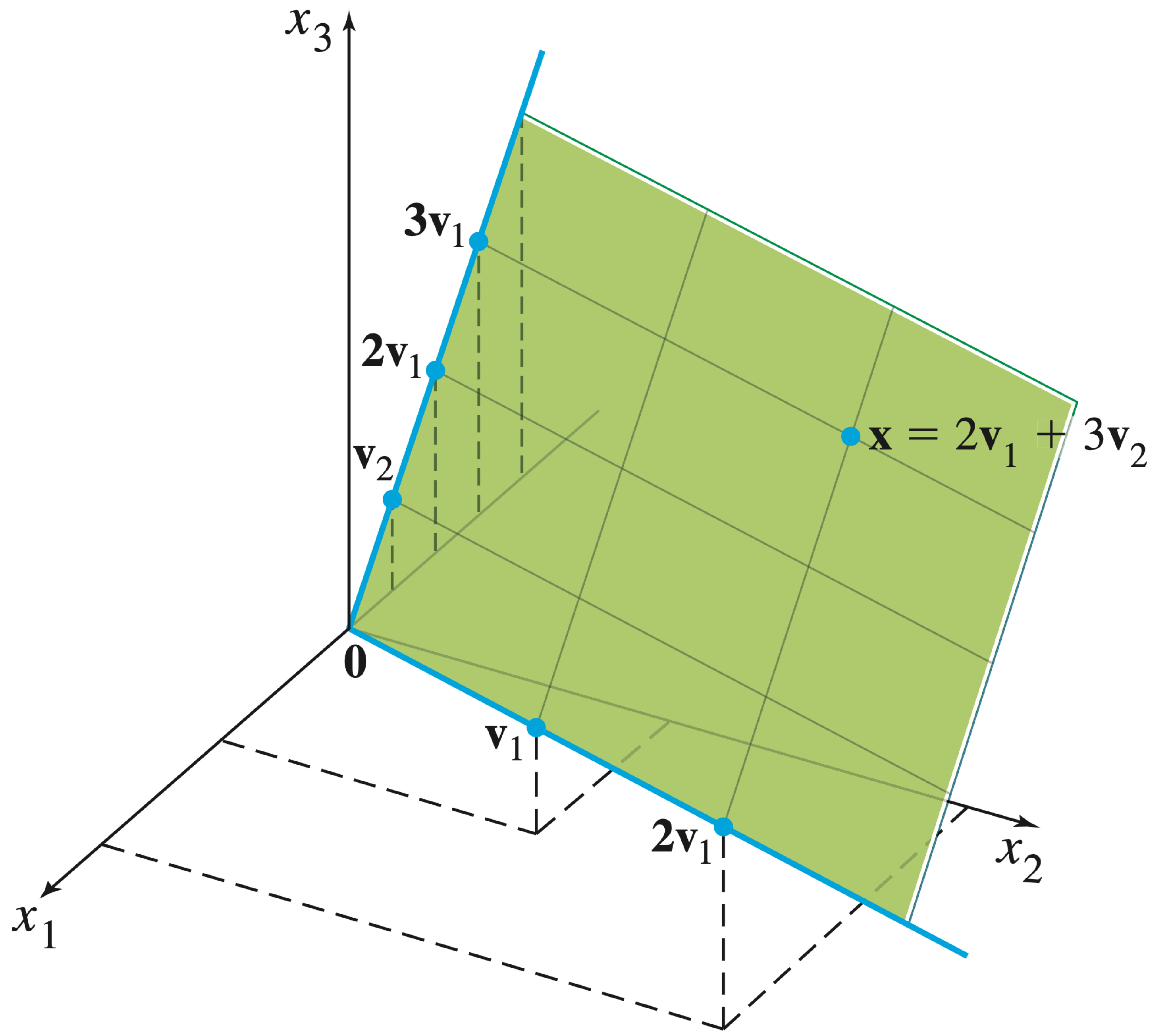
$$[\mathbf{v}]_{\mathscr{B}} = \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix}$$

Example: 3D Case

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$$

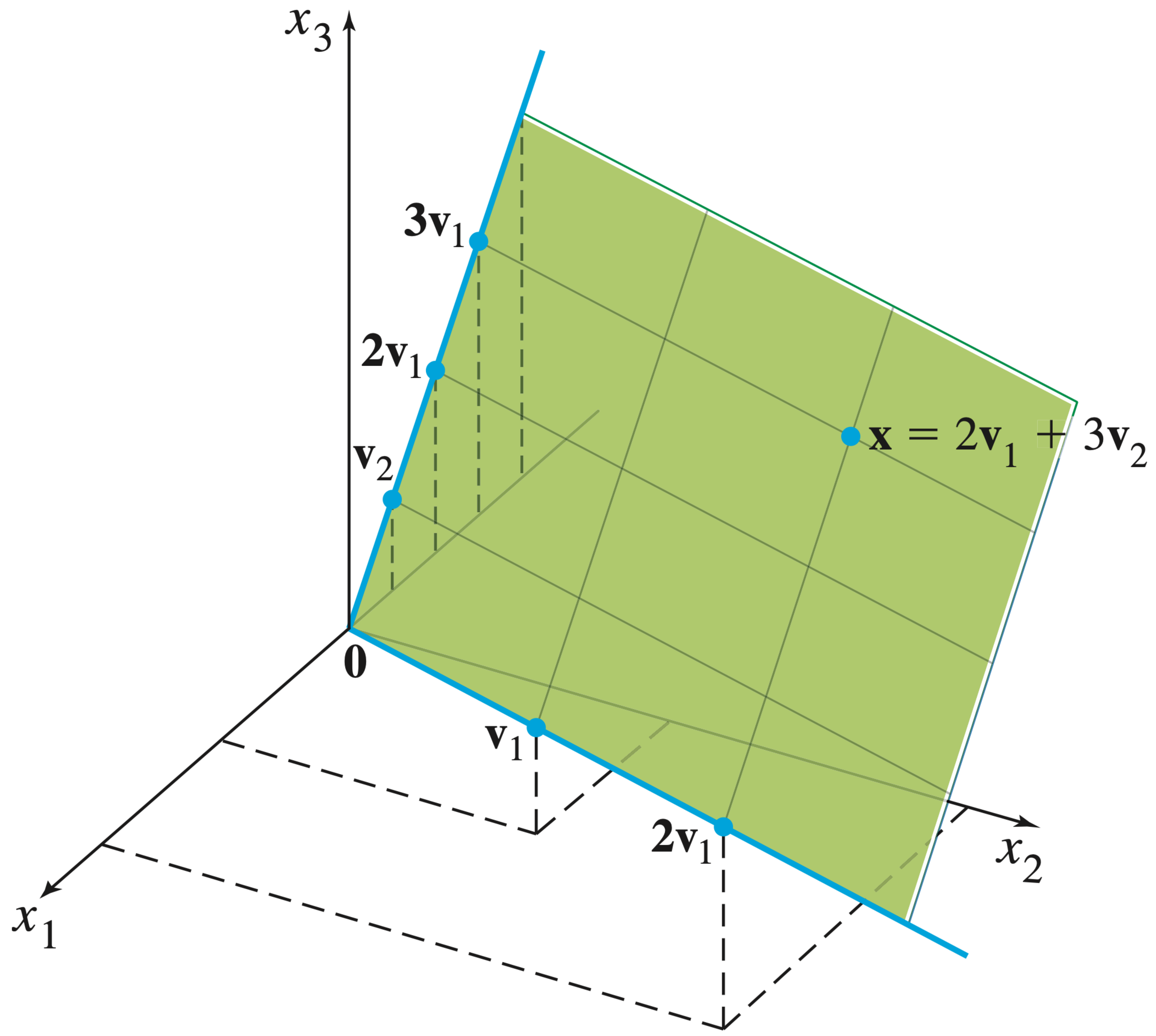
Find the coordinate vector for $\mathbf{u}$ relative to the basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ of a subspace H (of $\mathbb{R}^3$):

An Aside: Coordinates and one-to-one correspondences



An Aside: Coordinates and one-to-one correspondences

In the previous example $\mathbf{x} \mapsto [\mathbf{x}]_{\mathscr{B}}$ is a one-to-one correspondence from H to $\mathbb{R}^2$. This is also sometimes called an **isomorphism.**



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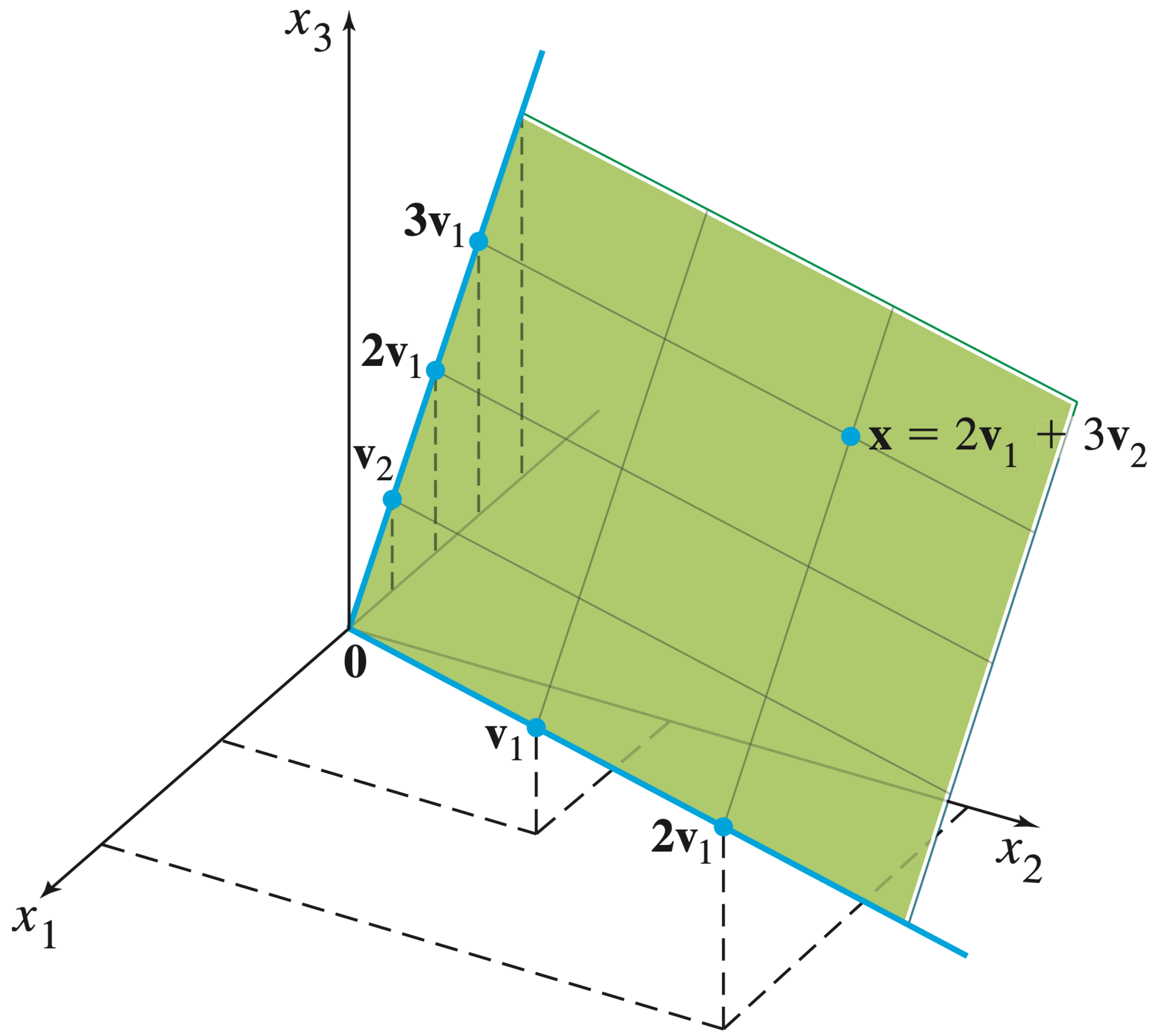

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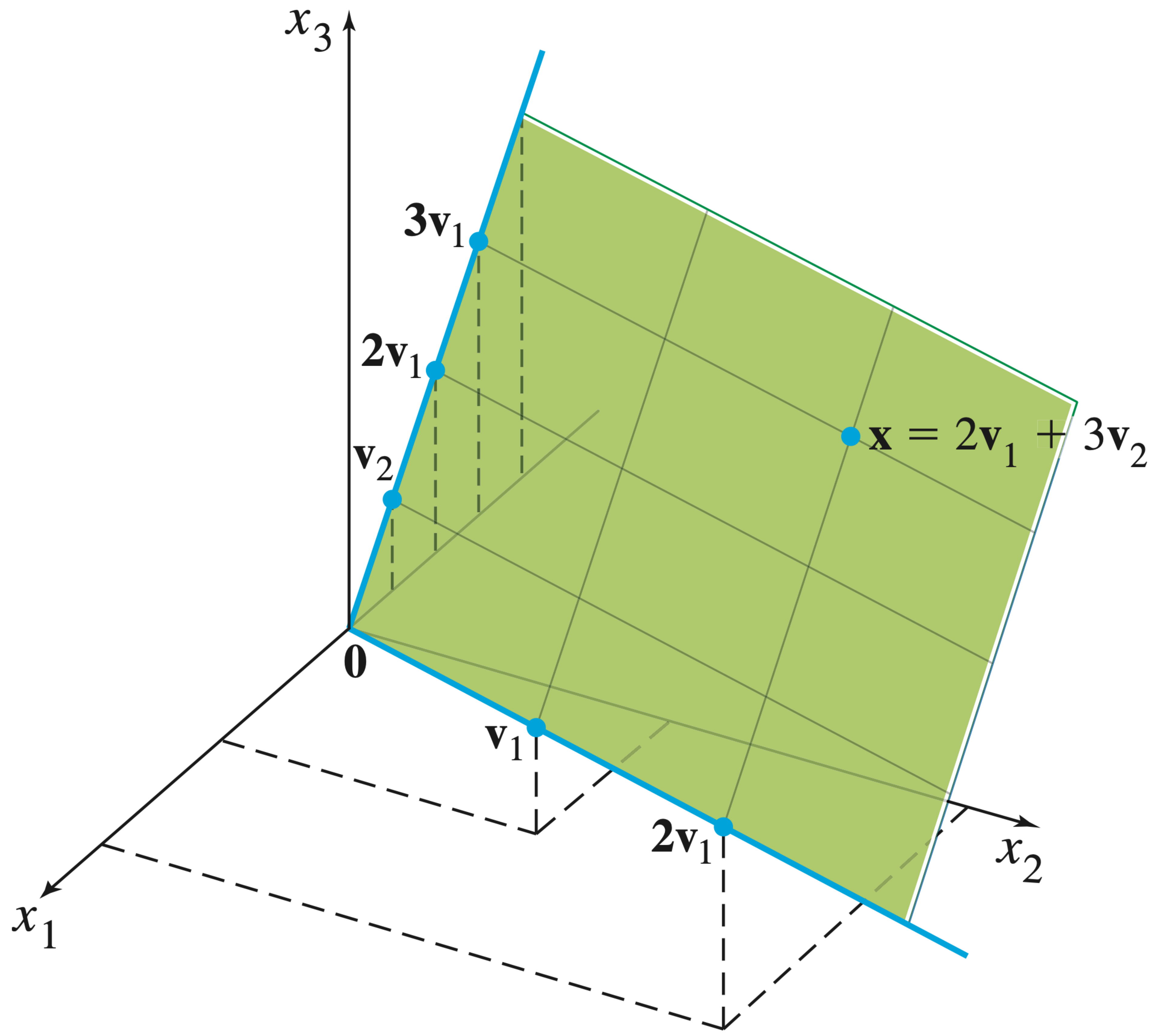
So $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is *isomorphic* to $\mathbb{R}^2$.



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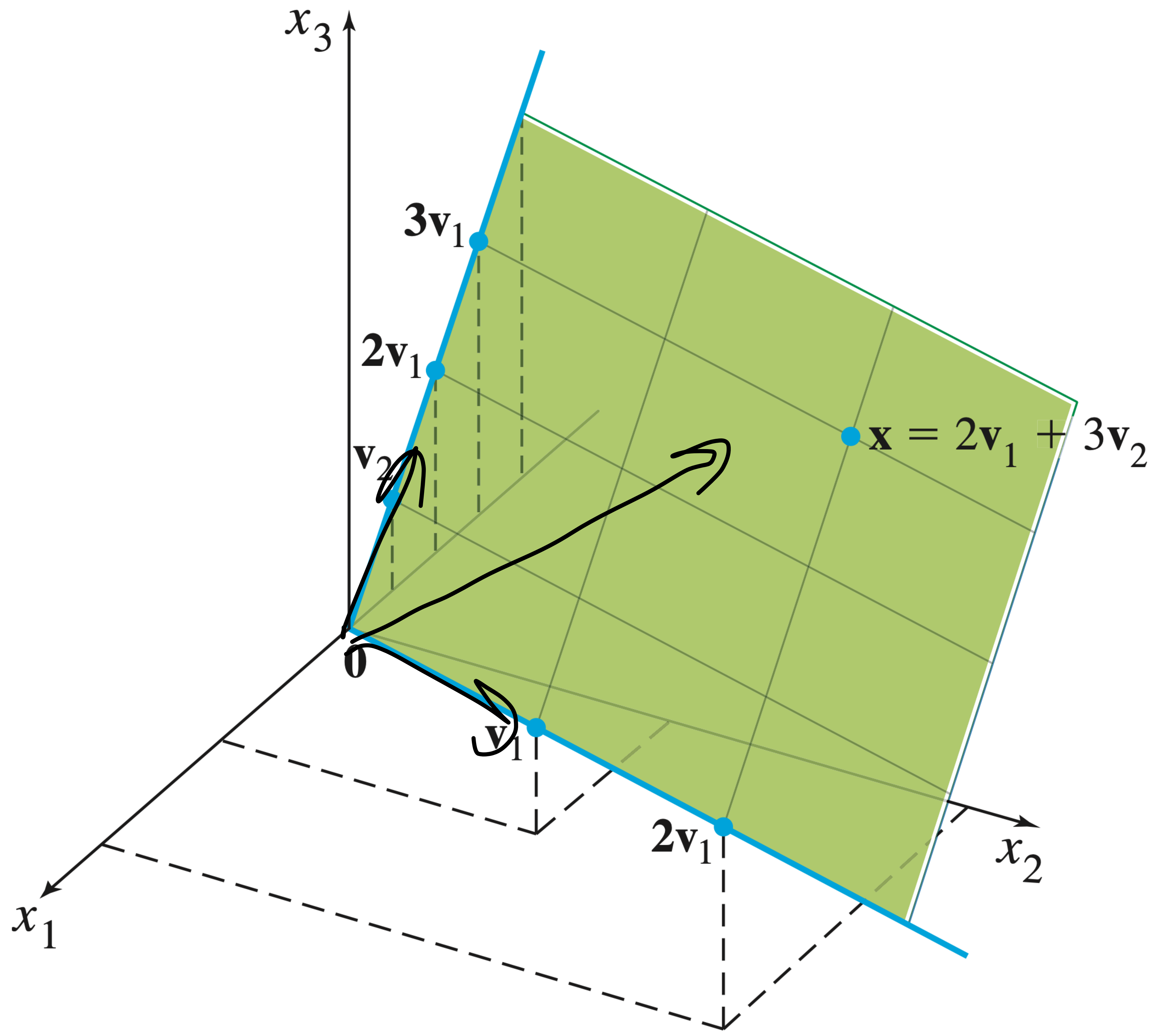
So $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is *isomorphic* to $\mathbb{R}^2$.

This is a formal way of saying that $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is a "copy of $\mathbb{R}^2$."



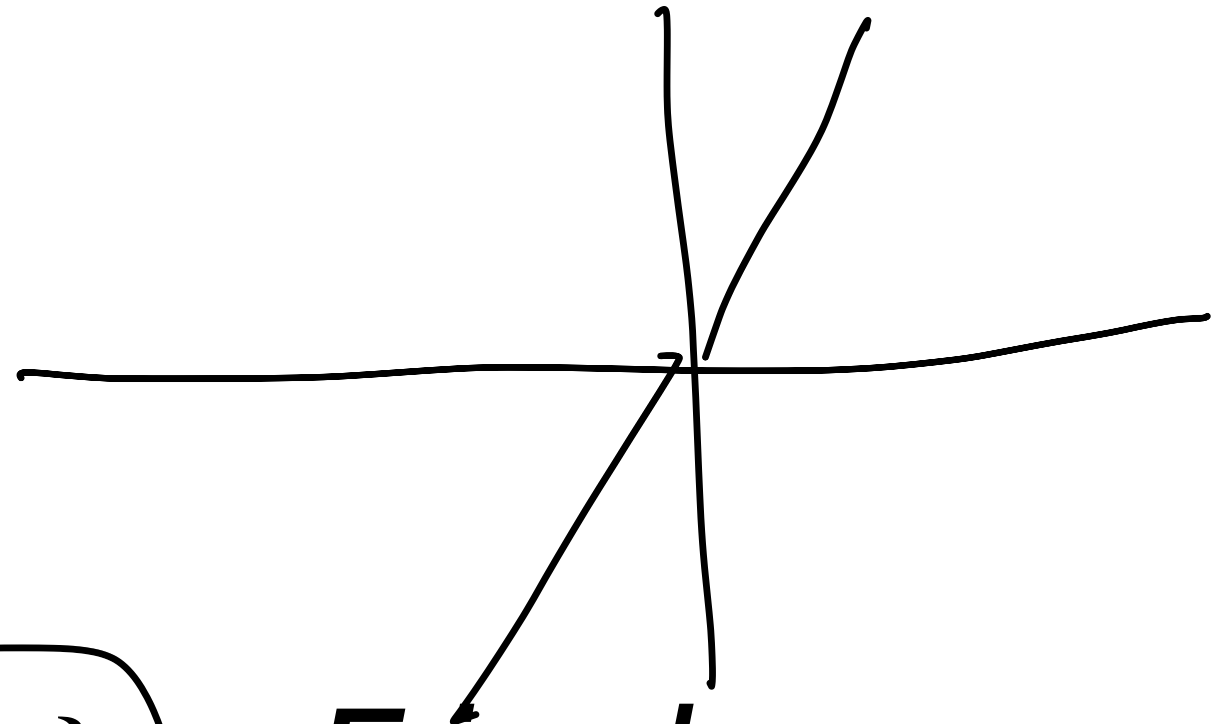
Linear Algebra and its Applications (Lay, Lay, McDonald)

Question

B is a basis of span$\{v_1, v_2\}$

$$\left[\begin{bmatrix} 1 \\ 6 \end{bmatrix}\right]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$



Suppose $[\mathbf{u}]_{\mathscr{B}} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$*, where* $\mathscr{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$*. Find* **u.**

coordinate vec. for $\vec{b}$ relative ~~to~~ $\vec{u}$

$$\vec{u} = 2\begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} + (-2)\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \\ 2 \end{bmatrix}$$

Answer

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad [\mathbf{u}]_{\mathscr{B}} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

Dimension and Rank

The Idea of Dimension

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Any more, we would have dependencies.

This number is a measure of how "large" H is.

Dimension

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Definition. The **dimension** of a subspace H of $\mathbb{R}^n$, written $\dim(H)$ or $\dim H$, is the *number* of vectors in <u>any</u> basis of H.

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<u>This should confirm our intuitions:</u>

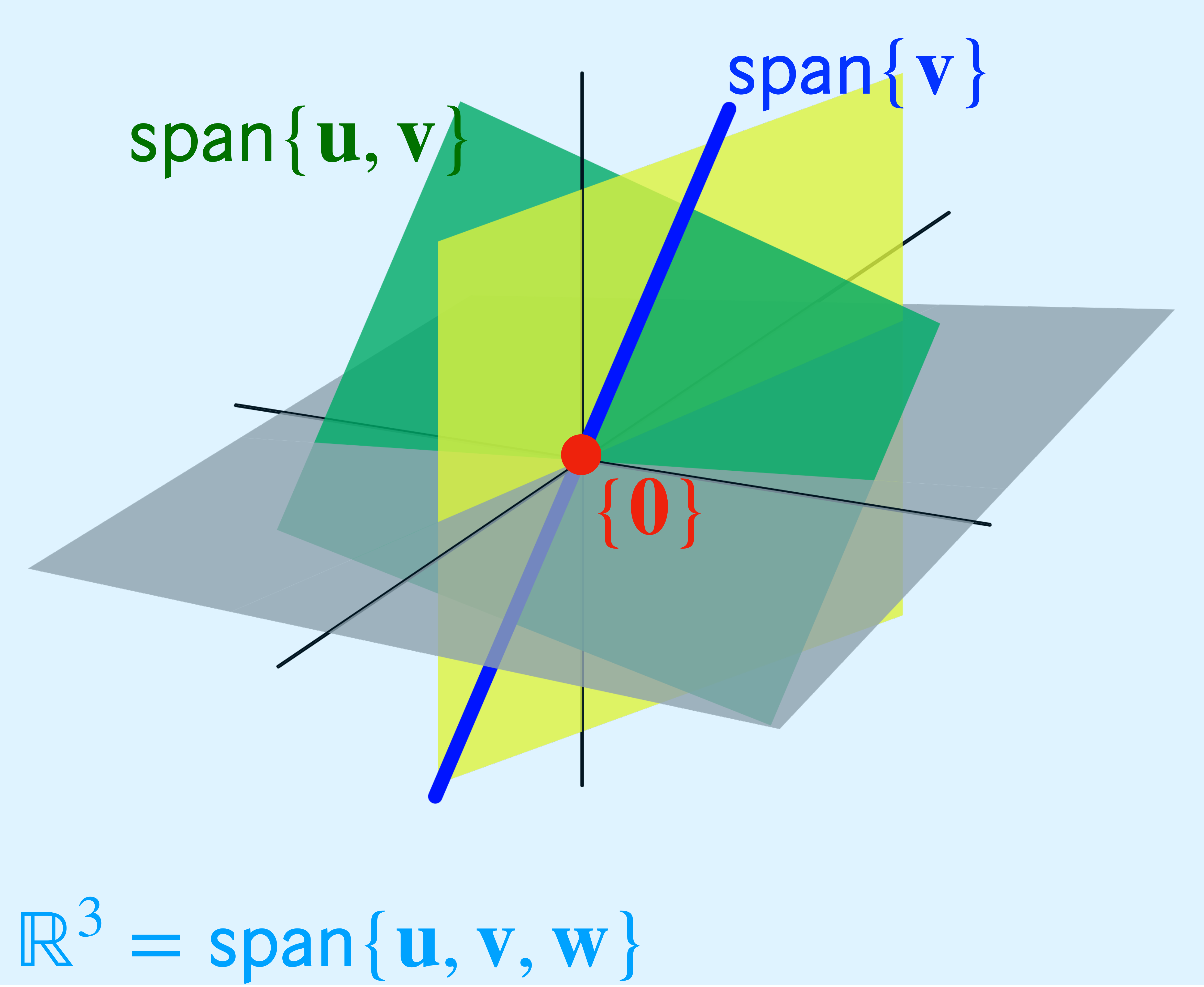
» a plane (through the origin) is a 2D subspace

» a line (through the origin) is a 1D subspace

Recall: Subspace in $\mathbb{R}^3$ (Geometrically)

There are only 4 kinds of subspaces of $\mathbb{R}^3$:

1. $\{\mathbf{0}\}$ just the origin
2. lines (through the origin)
3. planes (through the origin)
4. All of $\mathbb{R}^3$

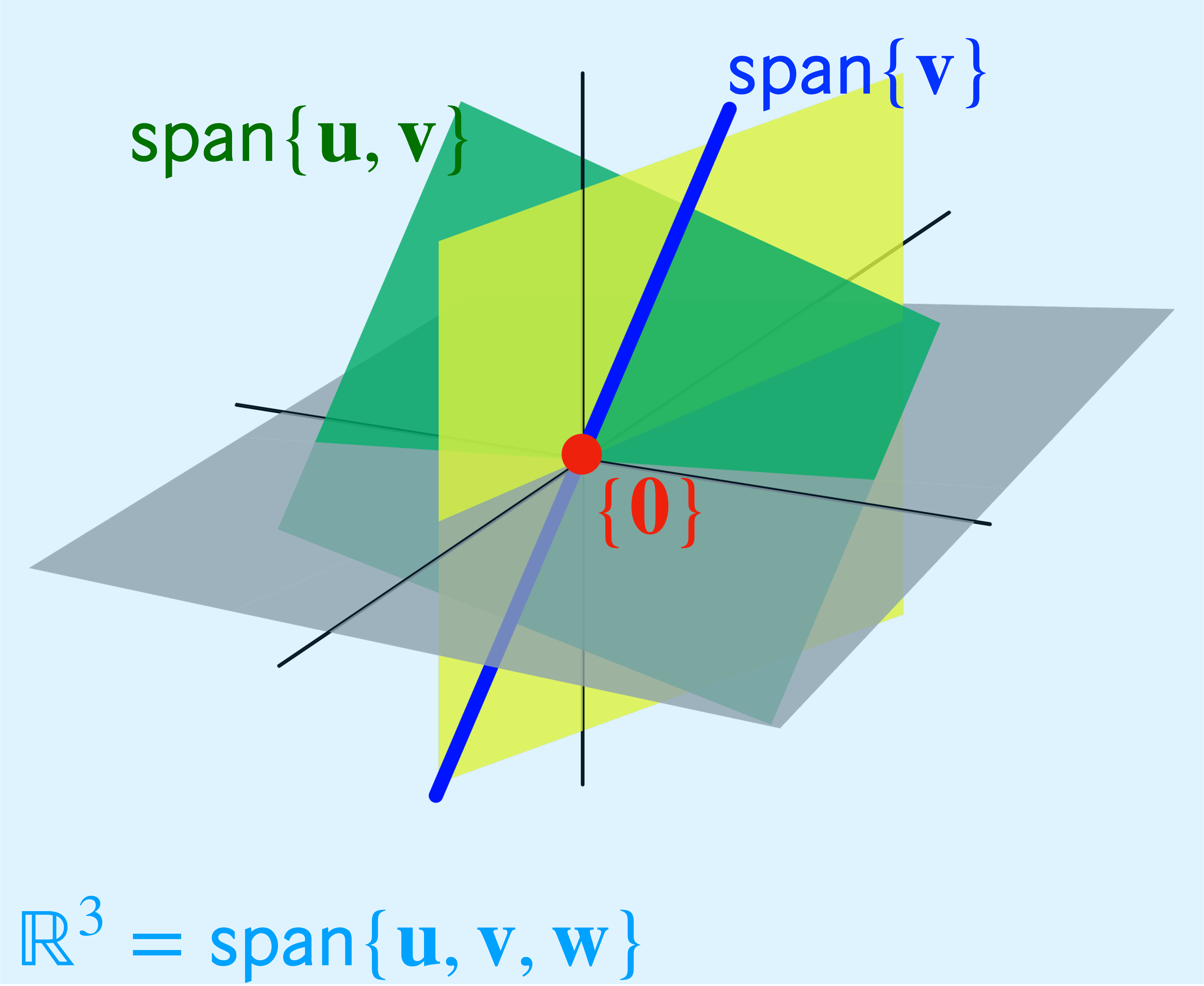


https://commons.wikimedia.org/wiki/File:Linear_subspaces_with_shading.svg

Recall: Subspace in $\mathbb{R}^3$ (Geometrically)

There are only 4 kinds of subspaces of $\mathbb{R}^3$:

1. 0-dimensional subspace
2. 1-dimensional subspaces
3. 2-dimensional subspaces
4. 3-dimensional subspace



https://commons.wikimedia.org/wiki/File:Linear_subspaces_with_shading.svg

How does this connect to null space and column space?

Recall: An Observation

The *number* of vectors in the basis we found is the same as the number of <u>free variables</u> in a general form solution.

$x_1 = 2x_2 + x_4 - 3x_5$

x_2 is free

$x_3 = (-2)x_4 + 2x_5$

x_4 is free

x_5 is free

$$\equiv \quad \begin{bmatrix} s \\ t \\ u \end{bmatrix} \mapsto \begin{bmatrix} 2s + t - 3u \\ s \\ (-2)t + 2u \\ t \\ u \end{bmatrix}$$

Dimension of the Null Space

The **dimension of** $\text{Nul}(A)$ is the number of <u>free variables</u> in a general form solution to $A\mathbf{x} = \mathbf{0}$.

$x_1 = 2x_2 + x_4 - 3x_5$

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Recall: An Observation

The *number* of vectors in the basis we found is the same as the number of basic variable or equivalently the number of pivot columns.

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4 \quad \mathbf{a}_5] \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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Dimension of the Column Space

The **dimension of** Col(A) is the number of basic variable in our solution, or equivalently the number of pivot columns of A.

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4 \quad \mathbf{a}_5] \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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The **dimension of** $\text{Col}(A)$ is the number of basic variable in our solution, or equivalently the number of pivot columns of A.

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Rank

Definition. The **rank** of a matrix A, written $\text{rank}(A)$ or $\text{rank}\, A$, is the dimension of $\text{Col}(A)$.

This is just terminology.

Rank-Nullity Theorem

Theorem. For an $m \times n$ matrix A,

$$\text{rank}(A) + \dim(\text{Nul}(A)) = n$$

Verify:

This is incredibly important.

Rank-Nullity Theorem

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nullity

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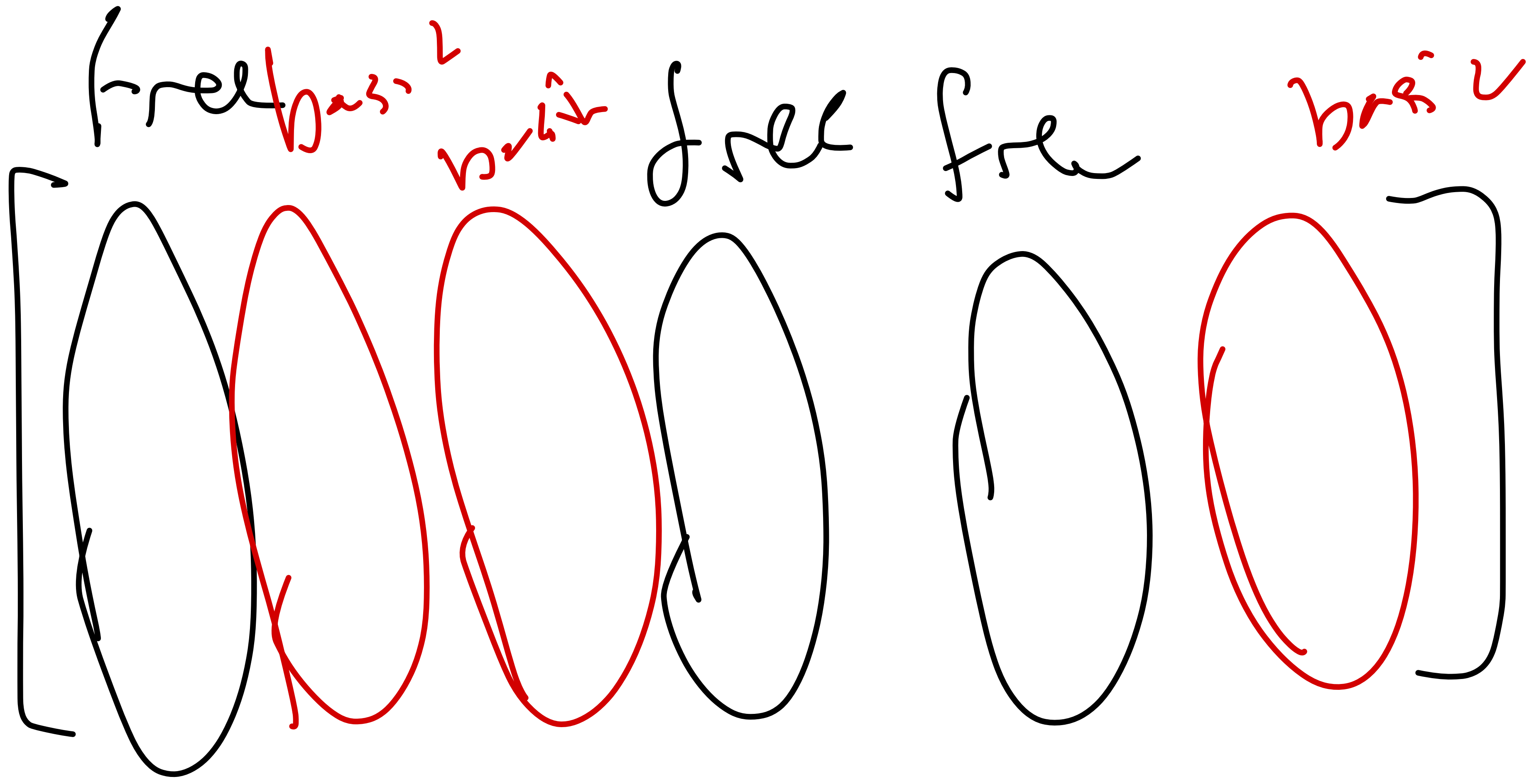
Verify:



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The Intuition

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For a $m \times n$ matrix A, its columns space $\text{Col}(A)$ *could* have n dimensions.

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Example. If a "line's worth of stuff" is pulled into the null space (and mapped to $\mathbf{0}$) then

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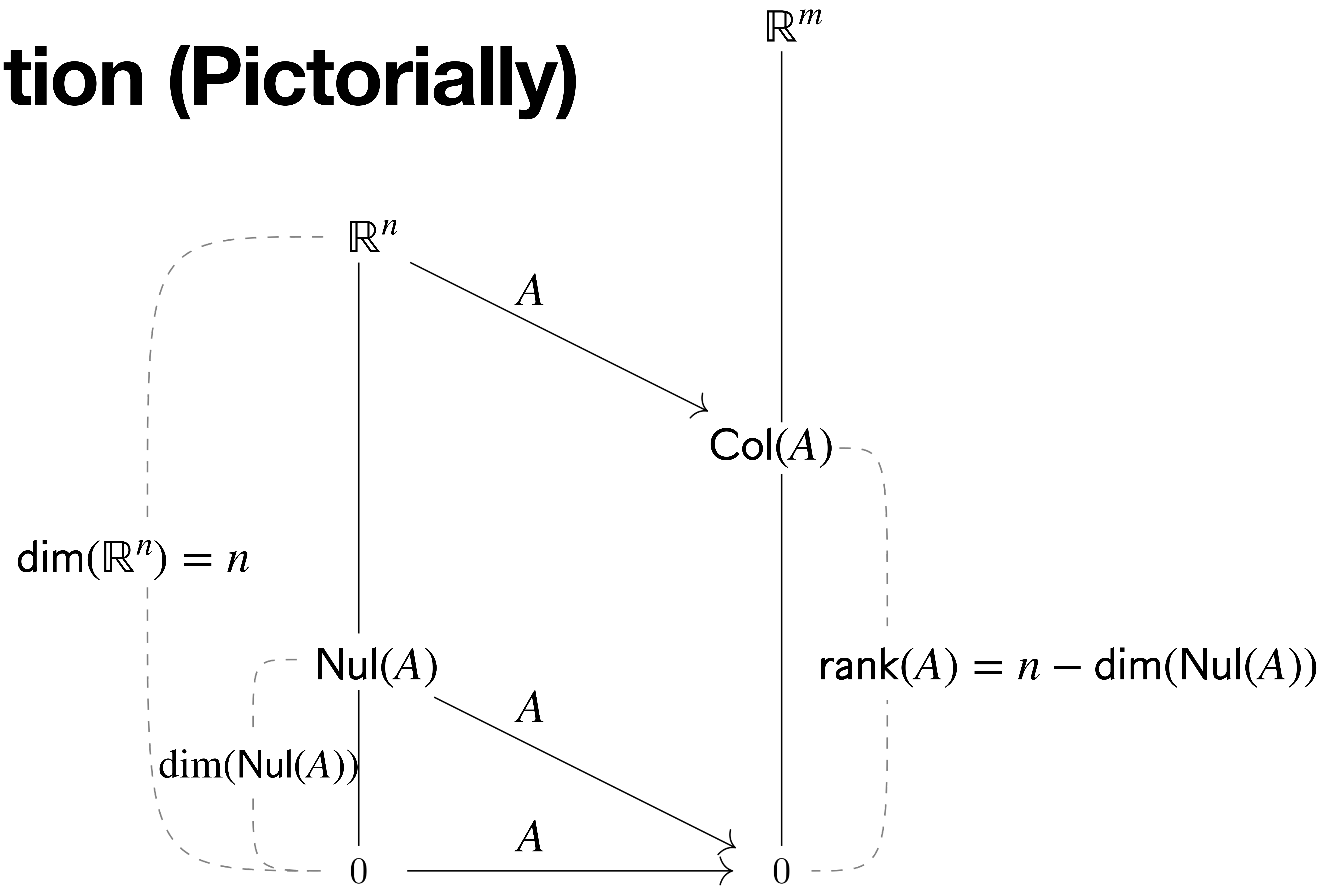
The null space "takes away" some of the dimensions of the column space.

The Intuition (Pictorially)



https://commons.wikimedia.org/wiki/File:Rank-nullity.svg

Question (Conceptual)

Let A *be a* 5×7 *matrix such that* $\dim(\text{Nul}(A)) = 3$*.*
What is the dimension of $\text{Col}(A)$*?*

Answer: 4

Extending the IMT

Theorem. For an $n \times n$ invertible matrix A, the following are logically equivalent (they must all by true or all by false.

- » $\text{Col}(A) = \mathbb{R}^n$
- » $\dim(\text{Col}(A)) = n$
- » $\text{rank}(A) = n$
- » $\text{Nul}(A) = \{\mathbf{0}\}$
- » $\dim(\text{Nul}(A)) = 0$

Summary

We can find bases for the column space and null space by looking at the reduced echelon form of a matrix.

Column vectors are written in terms of a coordinate system, which we can change.

Dimension is a measure of how large a space is.