

Dimension and Rank

Geometric Algorithms

Lecture 16

Introduction

Recap Problem

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$$

Consider the subspace H generated by \mathbf{v}_1 and \mathbf{v}_2 . Show that \mathbf{v}_3 and \mathbf{v}_4 form a basis for H .

Answer

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

Hint. Show that \mathbf{v}_1 and \mathbf{v}_2 are in the span of \mathbf{v}_3 and \mathbf{v}_4

Objectives

1. Learn techniques to find bases for the column space and the null space of a matrix
2. Briefly discuss the coordinate systems.
3. Introduce the fundamental notion of dimension, which quantifies how "large" a space is
4. Relate the dimension of the column space and the null space of a matrix

Keywords

basis

column space

null space

coordinate system

change of basis

dimension

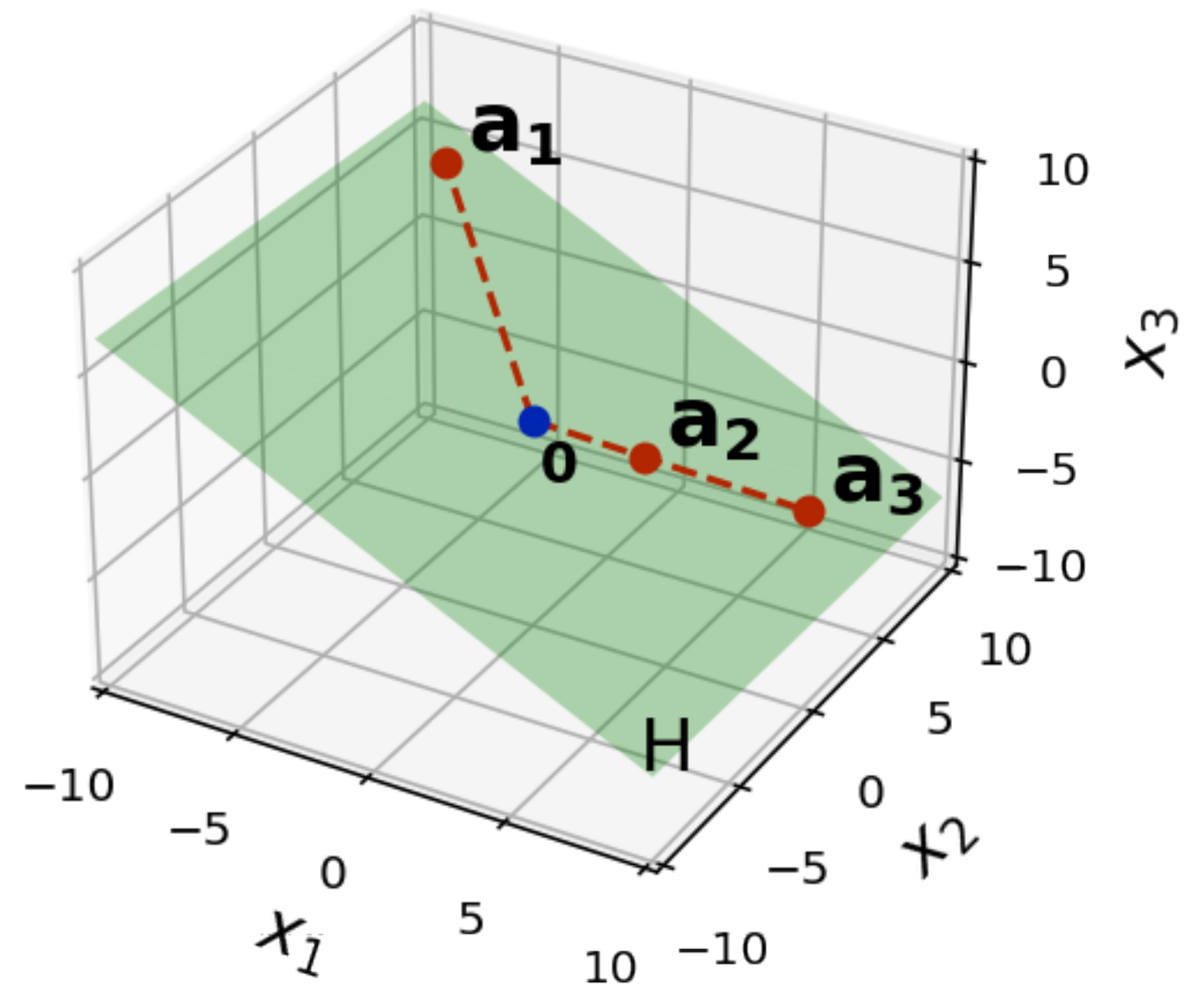
rank

rank theorem

invertible matrix theorem (extended)

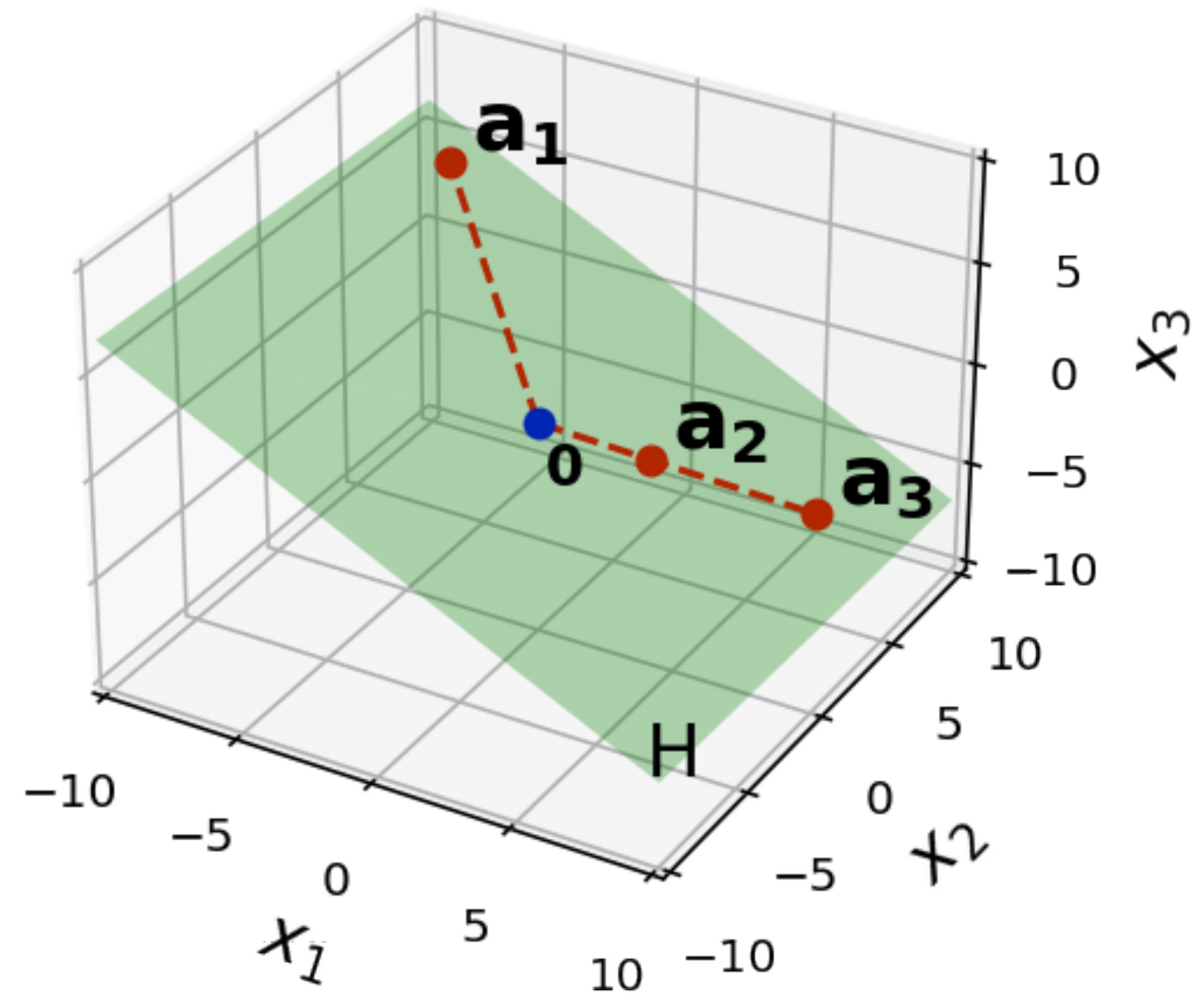
Recap

Recall: The Idea Behind Subspaces



Recall: The Idea Behind Subspaces

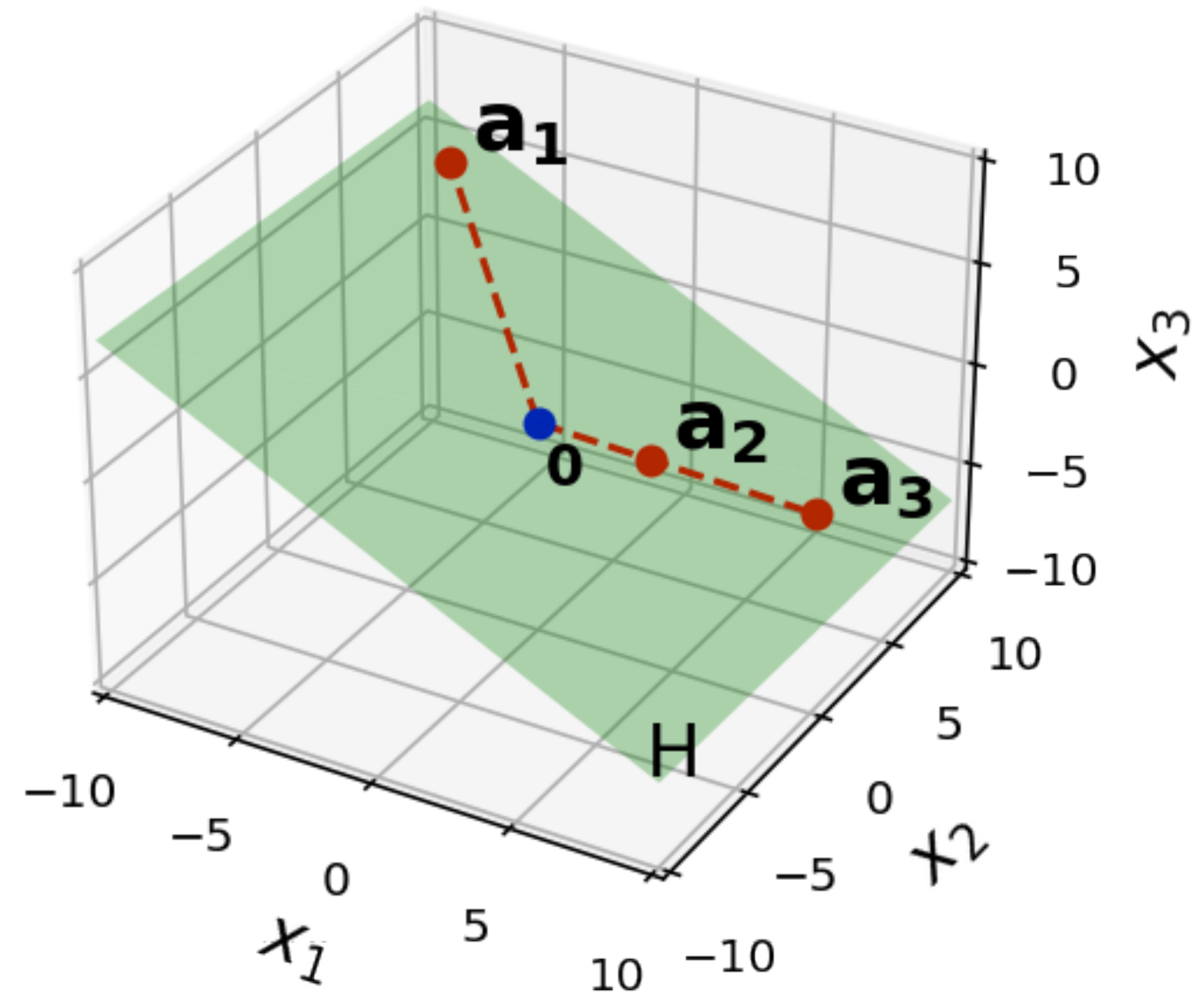
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A plane in \mathbb{R}^3 looks like a (possibly tilted) copy of \mathbb{R}^2

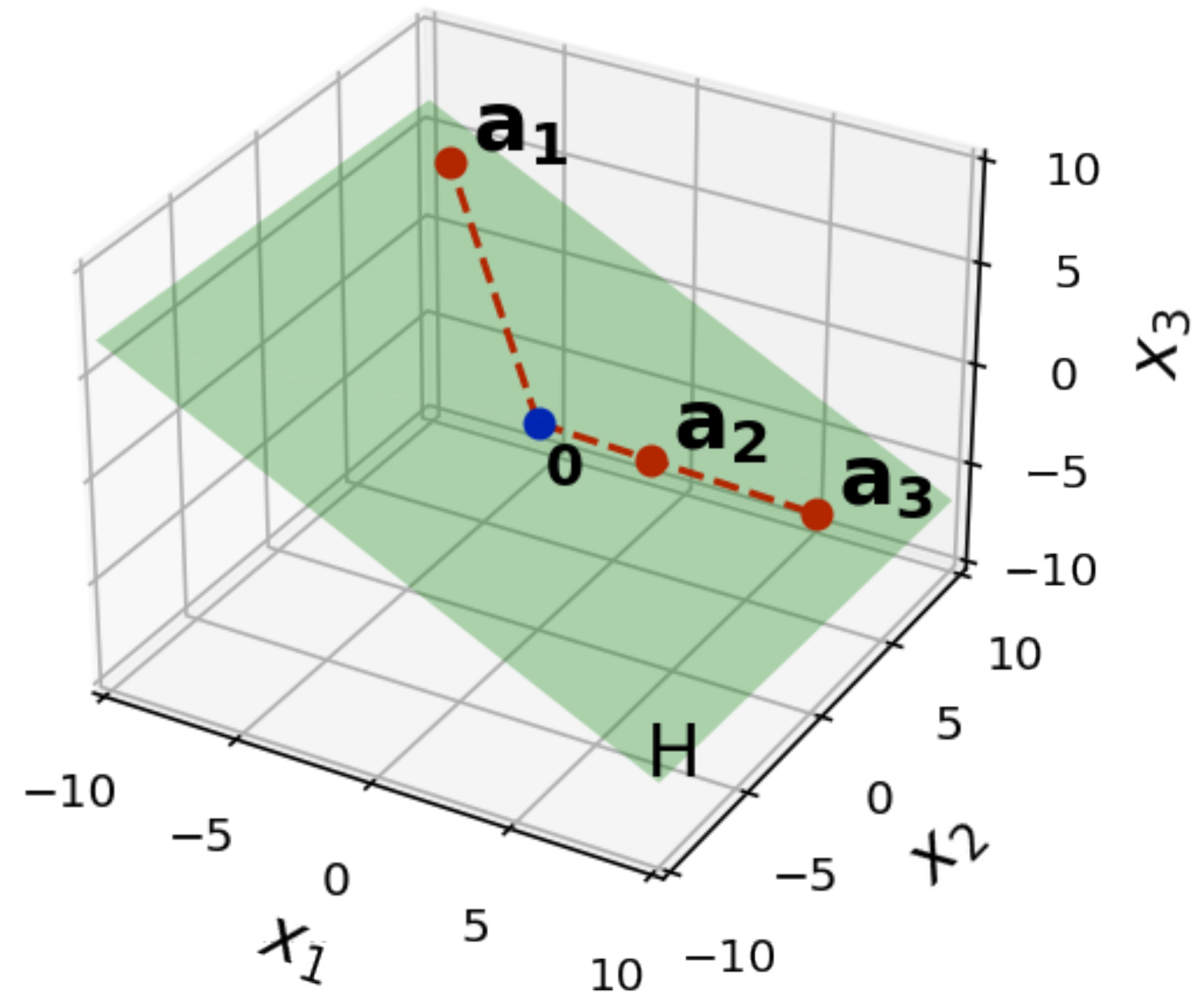


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Subspaces *generalize* of this idea.



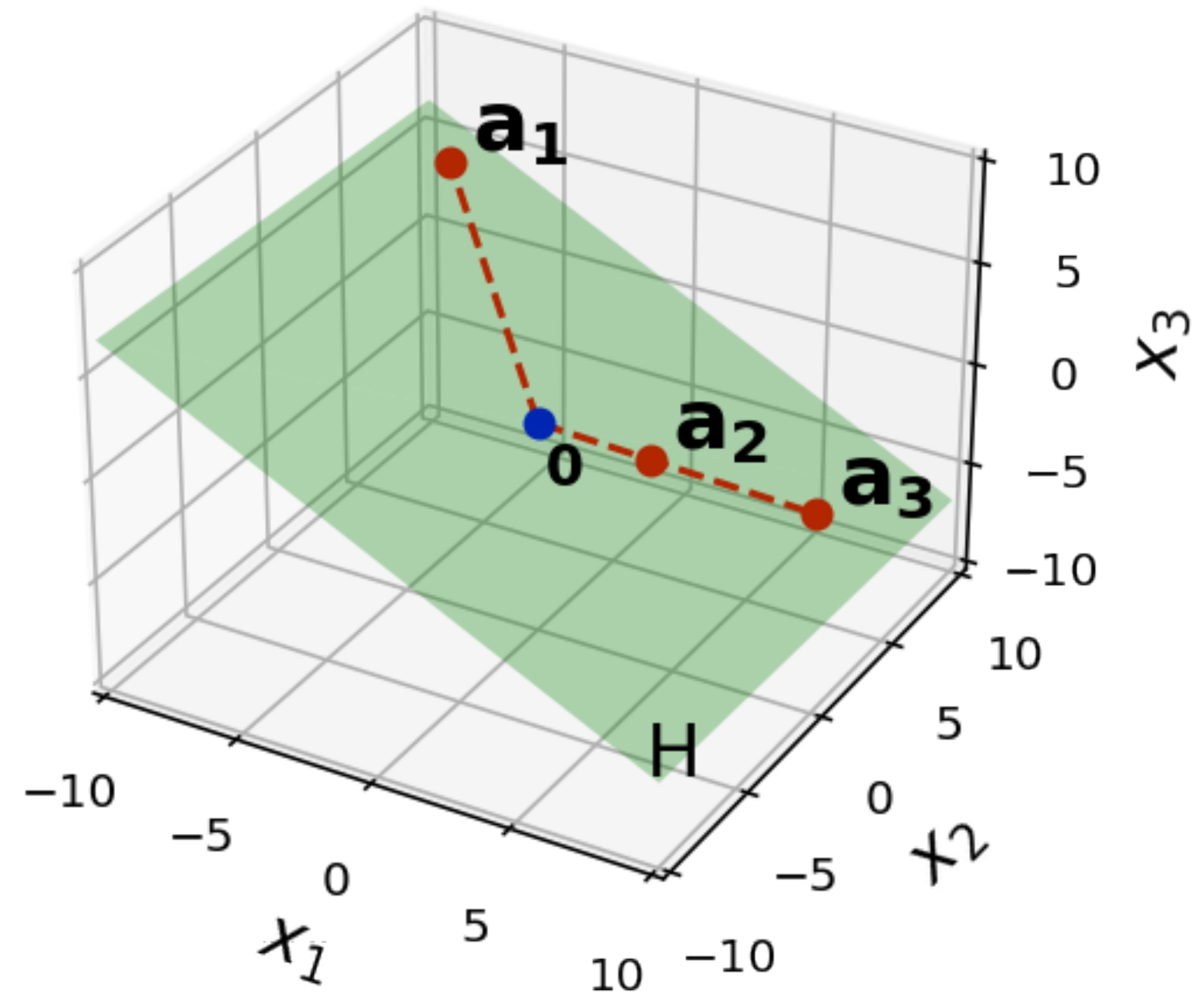
Recall: The Idea Behind Subspaces

"sub" means "part of" or "below"

A plane in \mathbb{R}^3 looks like a (possibly tilted) copy of \mathbb{R}^2

Subspaces *generalize* of this idea.

For example, there can be a "possibly tilted copy" of \mathbb{R}^3 sitting in \mathbb{R}^5



Recall: Subspace (Algebraic Definition)

Definition. A subspace of \mathbb{R}^n is a set H of vectors in \mathbb{R}^n such that

1. for every \mathbf{u} and \mathbf{v} in H , the vector $\mathbf{u} + \mathbf{v}$ is in H
2. for every \mathbf{u} in H and scalar c , the vector $c\mathbf{u}$ is in H

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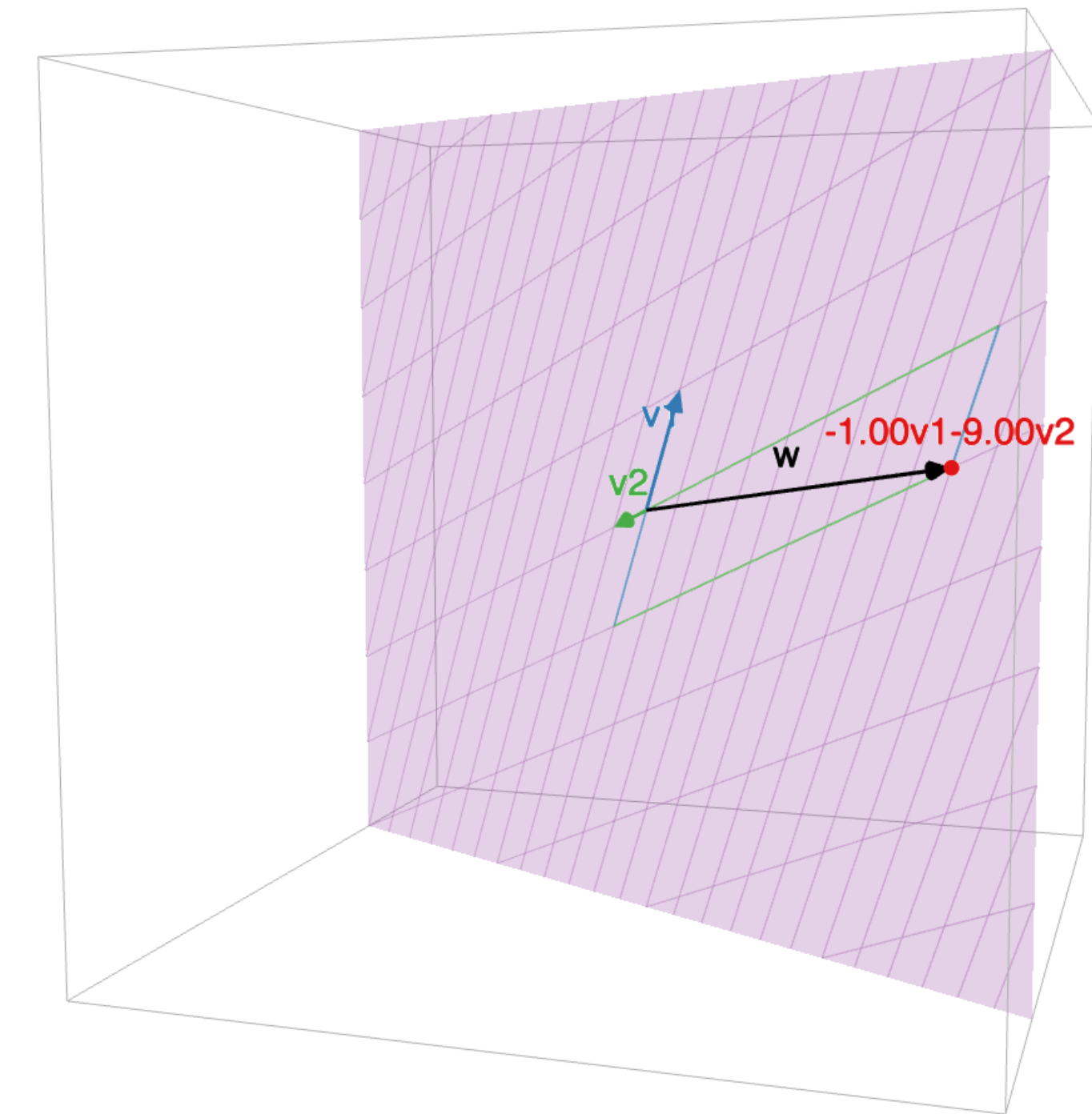
1. for every u and v in H , the vector $u+v$ is in H **H is closed under addition**
2. for every u in H and scalar c , the vector cu is in H **H is closed under scaling**

!! Subspaces must "live" somewhere !!

Recall: How to Think About this Definition

It's not possible to "leave" H by addition or scaling.

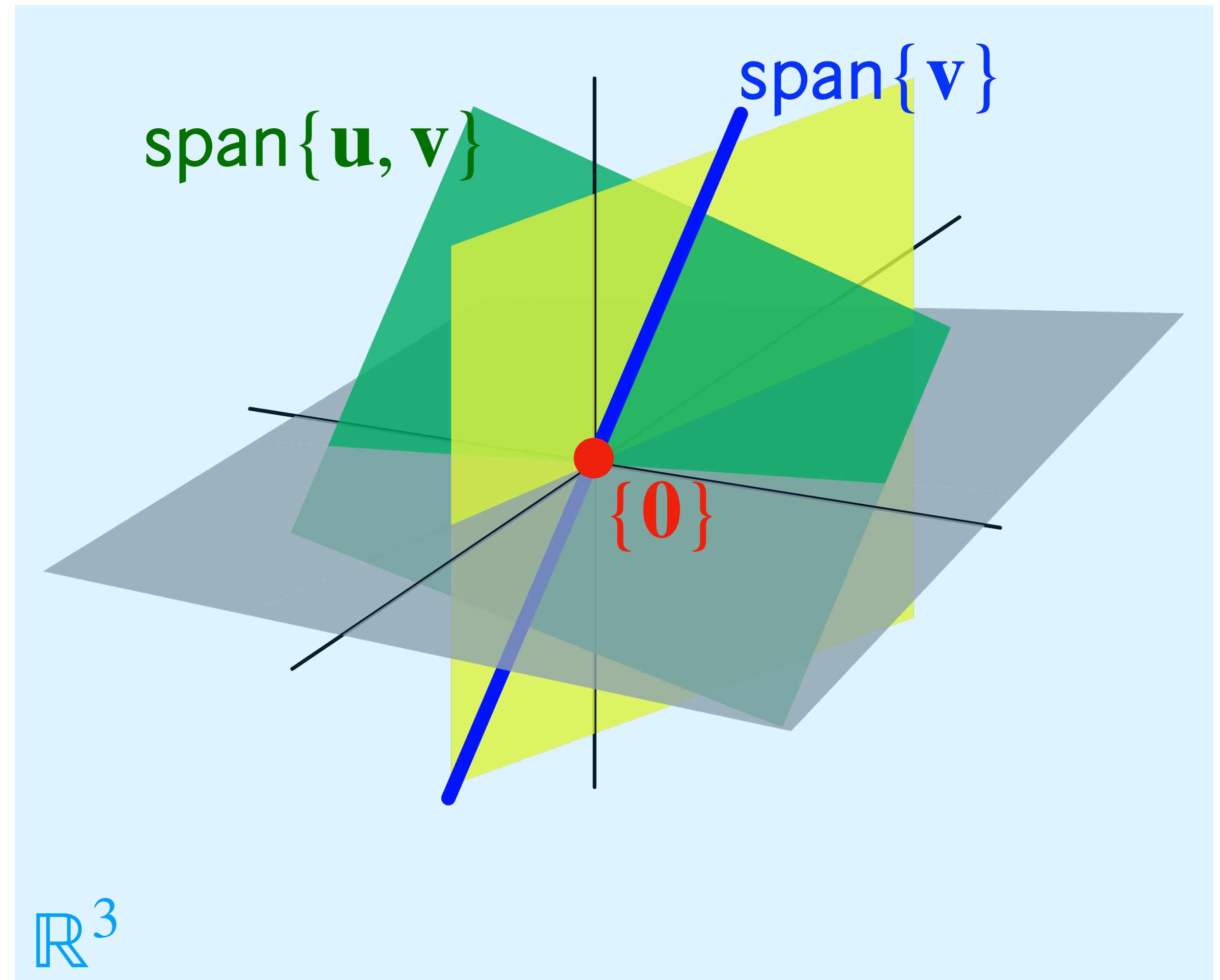
(recall this is also how we discussed spans)



Recall: Subspace in \mathbb{R}^3 (Geometrically)

There are only 4 kinds of subspaces of \mathbb{R}^3 :

1. $\{\mathbf{0}\}$ just the origin
2. lines (through the origin)
3. planes (through the origin)
4. All of \mathbb{R}^3



Column Space

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Definition. The **column space** of a matrix A , written $\text{Col}(A)$ or $\text{Col } A$, is the set of all linear combinations of the columns of A .

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The column space of a matrix is the span of its columns.

The column space of a matrix is the range of the linear transformation it implements.

Subspace of What?

$$m \left[\begin{array}{cccc} | & | & \dots & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_{n-1} & \mathbf{a}_n \\ | & | & \dots & | & | \end{array} \right]$$

$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \dots + c_n\mathbf{a}_n$ is a
vector in \mathbb{R}^m

$\text{Col}(A)$

is a subspace of

\mathbb{R}^m

Null Space

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Definition. The **null space** of a matrix A , written $\text{Nul}(A)$ or $\text{Nul } A$, is the set of all solutions to the homogenous equation

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$$A\mathbf{x} = \mathbf{0}$$

The null space of a matrix A is the set of all vectors that are mapped to the zero vector by A .

Subspace of What?

$$\begin{array}{c} \text{rows } m \\ \left| \begin{array}{c} \overbrace{A \mathbf{v}}^{n \text{ columns}} \\ \mathbf{v} \end{array} \right. = \mathbf{0} \\ \begin{array}{cc} m \times n & n \times 1 \\ & m \times 1 \end{array} \end{array}$$

v is a vector
in \mathbb{R}^n

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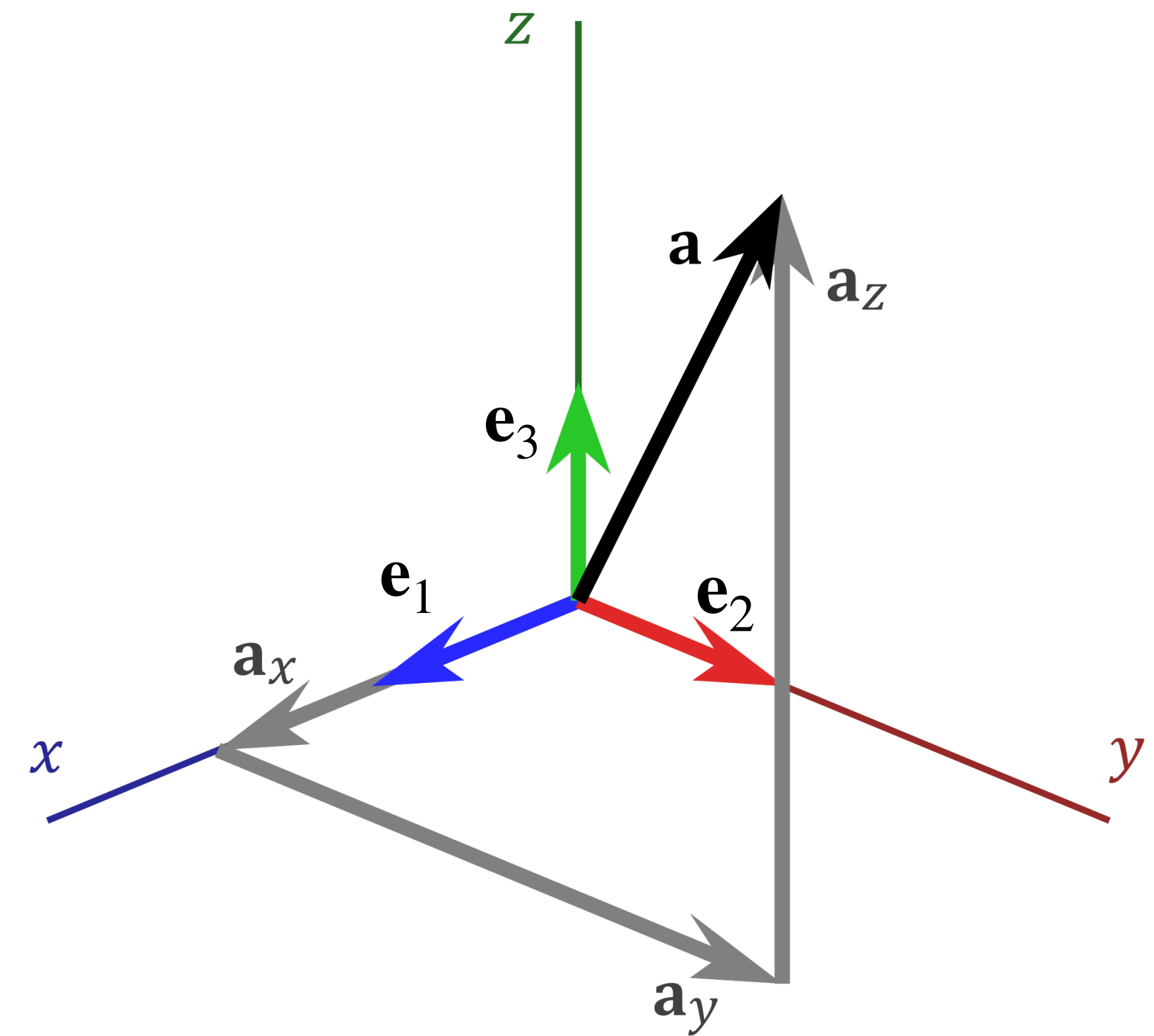
Recall: Basis

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Definition. A **basis** for a subspace H of \mathbb{R}^n is a linearly independent set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ of vectors that spans H (in symbols: $H = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$).

A **basis** is a *minimal* set of vectors which spans all of H .

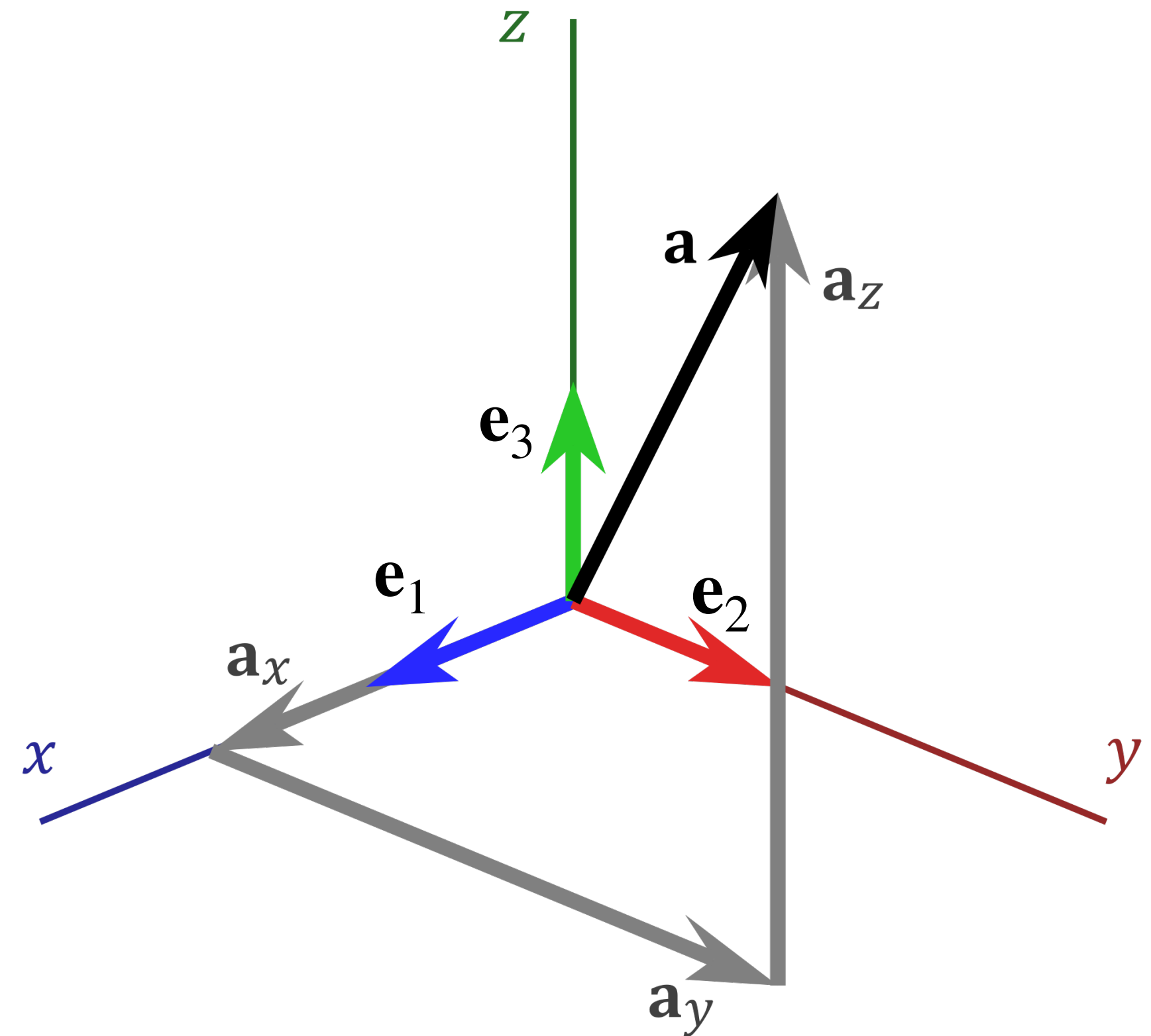
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The n standard basis vectors
in \mathbb{R}^n :

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- » span all of \mathbb{R}^n

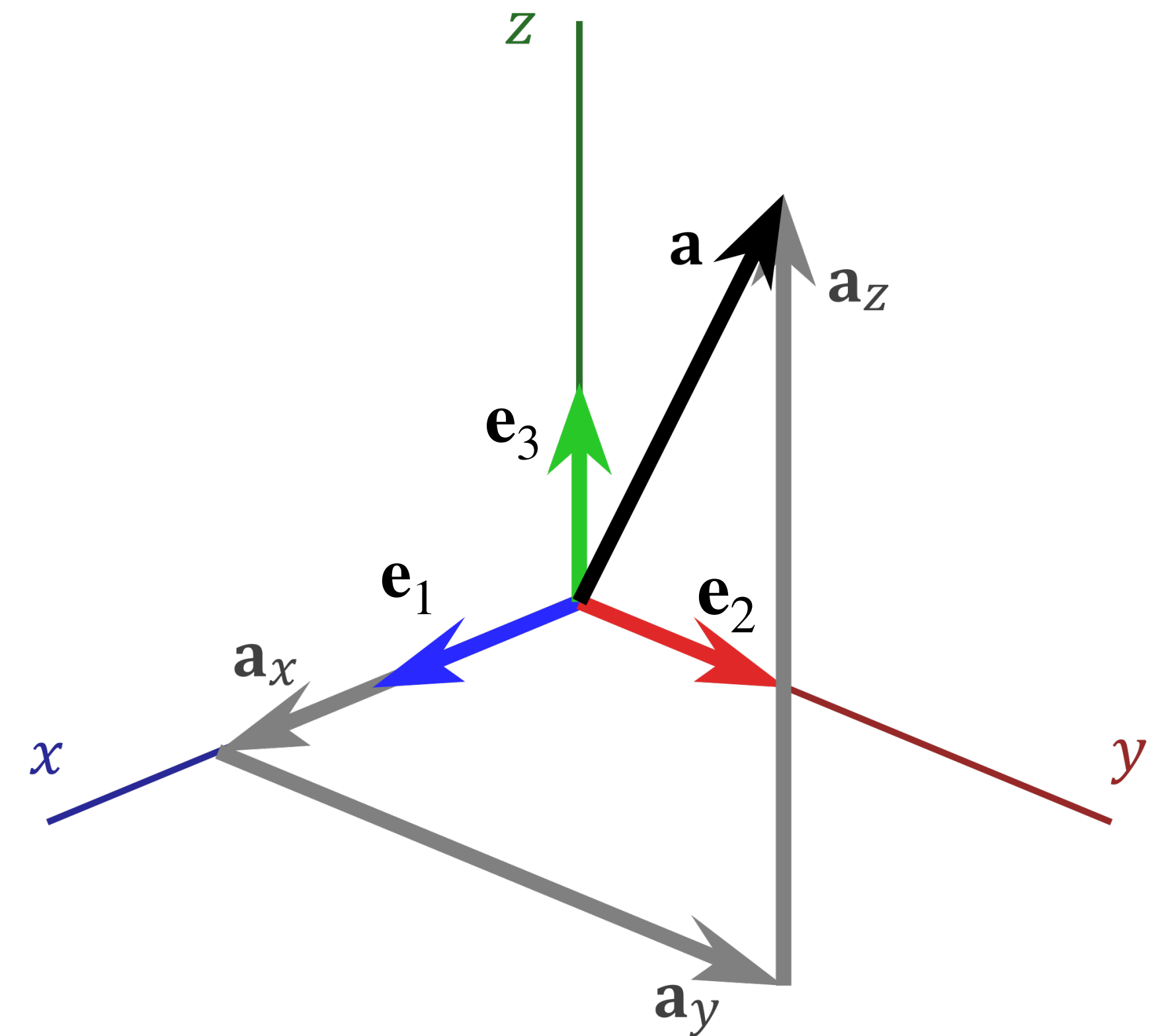


Recall: What's interesting about the standard basis?

The n standard basis vectors in \mathbb{R}^n :

- » are linearly independent
- » span all of \mathbb{R}^n

Their span is as "large" as possible while the set of vectors generating the span is as "small" as possible.



Recall: Example: Standard basis

The standard basis is a basis of \mathbb{R}^n .

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + \dots + v_n \mathbf{e}_n$$

Every column vector can be written in exactly one way as a linear combination of standard basis vectors

Recall: Example: Column Space of Invertible Matrices

Fact. The columns of an invertible $n \times n$ matrix form a basis of \mathbb{R}^n .

Verify:

Bases of Column Space and Null Space

The Goal of this Section

Determine how to find bases for the **column space** and the **null space** of a given matrix.

How to: Finding a basis for the null space

Question. Given a $m \times n$ matrix A find a basis for $\text{Nul}(A)$.

How to: Finding a basis for the null space

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The idea. Describe the solutions of $Ax = 0$ as linear combination of vectors

Example

$$A \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Suppose A has the above reduced echelon form.

Let's write down a general form solution for A :

Parametric Solutions

We can think of our general form solution as a (linear) transformation.

$$x_1 = 2x_2 + x_4 - 3x_5$$

x_2 is free

$$x_3 = (-2)x_4 + 2x_5$$

x_4 is free

x_5 is free

≡

"given values for x_2 , x_3 , and x_4 , I can give you a solution"

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$$\begin{bmatrix} s \\ t \\ u \end{bmatrix}$$

\mapsto

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Parametric Solutions

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!! in the case of homogeneous equations !!

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Example

$$\begin{bmatrix} s \\ t \\ u \end{bmatrix} \mapsto \begin{bmatrix} 2s + t - 3u \\ s \\ (-2)t + 2u \\ t \\ u \end{bmatrix}$$

Let's find the matrix implementing this linear transformation:

Example

$$\begin{bmatrix} 2 & 1 & -3 \\ 1 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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Every solution to $A\mathbf{x} = \mathbf{0}$ can be written as an *image* of this transformation.

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The columns of this matrix span $\text{Nul}(A)$.

Example

$$\begin{bmatrix} 2 & 1 & -3 \\ 1 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The columns of this matrix are linearly independent.

Verify:

Example

$$\begin{bmatrix} 2 & 1 & -3 \\ 1 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The columns of this matrix span $\text{Nul}(A)$.

The columns of this matrix are linearly independent.

The columns of this matrix form a basis for $\text{Nul}(A)$.

Example

Alternatively, we can think of writing a general form solution so that it is a linear combination of vectors with free variables as weights:

$$x_1 = 2x_2 + x_4 - 3x_5$$

x_2 is free

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Solution.

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Solution.

1. Find a general form solution for $A\mathbf{x} = \mathbf{0}$.
2. Write this solution as a linear combination of vectors where the free variables are the weights.
3. The resulting vectors form a basis for $\text{Nul}(A)$.

An Observation

The *number* of vectors in the basis we found is the same as the number of free variables in our general form solution.

$$x_1 = 2x_2 + x_4 - 3x_5$$

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$$x_3 = (-2)x_4 + 2x_5$$

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\equiv

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moving on to column space...

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So we also already know *some* subset of columns of A form a basis for $\text{Col}(A)$.

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We already know the columns of A span $\text{Col}(A)$.

So we also already know *some* subset of columns of A form a basis for $\text{Col}(A)$.

Which columns of A should we choose?

Column Space and Reduced Echelon form

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4 \quad \mathbf{a}_5] \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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The idea. What if we cover up the non-pivot columns?

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Then we see $[\mathbf{a}_1 \quad \mathbf{a}_3]$ has 2 pivots.

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The idea. What if we cover up the non-pivot columns?

Then we see $[\mathbf{a}_1 \quad \mathbf{a}_3]$ has 2 pivots.

So the pivot columns are linearly independent.

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Column Space and Reduced Echelon form

$$\begin{bmatrix} \overset{2}{\mathbf{a}_1} & \overset{1}{\mathbf{a}_2} & \blacksquare & \blacksquare & \blacksquare \end{bmatrix} \sim \begin{bmatrix} \overset{2}{1} & \overset{1}{-2} & \blacksquare & \blacksquare & \blacksquare \\ 0 & 0 & \blacksquare & \blacksquare & \blacksquare \\ 0 & 0 & \blacksquare & \blacksquare & \blacksquare \end{bmatrix}$$

Observation. $[2 \ 1 \ 0 \ 0 \ 0]^T$ is a solution to the system $A\mathbf{x} = \mathbf{0}$.

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In general, every non-pivot column of A can be written as a linear combination pivots in front of it.

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In general, every non-pivot column of A can be written as a linear combination pivots in front of it.

This tells us that \mathbf{a}_1 and \mathbf{a}_3 span $\text{Col}(A)$.

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The takeaway. The pivot columns of A form a basis for $\text{Col}(A)$.

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!! IMPORTANT !!

Choose the columns of A .

(e_1 and e_2 do not necessarily form a basis for $\text{Col}(A)$)

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Solution.

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Solution.

1. Find the pivot columns in an echelon form of A .
2. The associated columns in A form a basis for $\text{Col}(A)$.

An Observation

The *number* of vectors in the basis we found is the same as the number of basic variable or equivalently the number of pivot columns.

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Question

$$A = \begin{bmatrix} 1 & -2 & 19 & 0 & -4 \\ 1 & 0 & 9 & 1 & 1 \\ 1 & -1 & 14 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 9 & 0 & 0 \\ 0 & 1 & -5 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Find a bases for the column space and null space of A .

Answer

$$A = \begin{bmatrix} 1 & -2 & 19 & 0 & -4 \\ 1 & 0 & 9 & 1 & 1 \\ 1 & -1 & 14 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 9 & 0 & 0 \\ 0 & 1 & -5 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

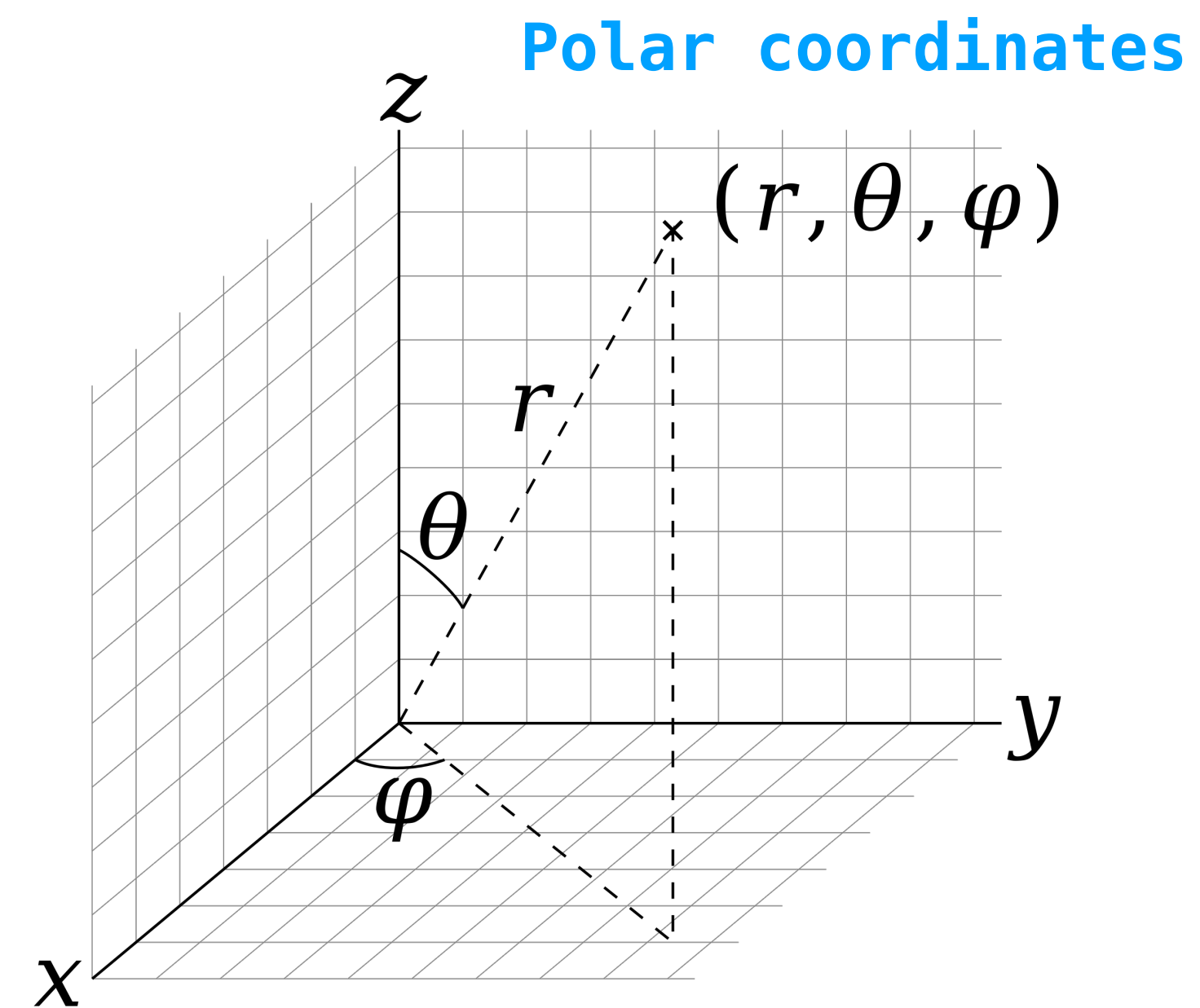
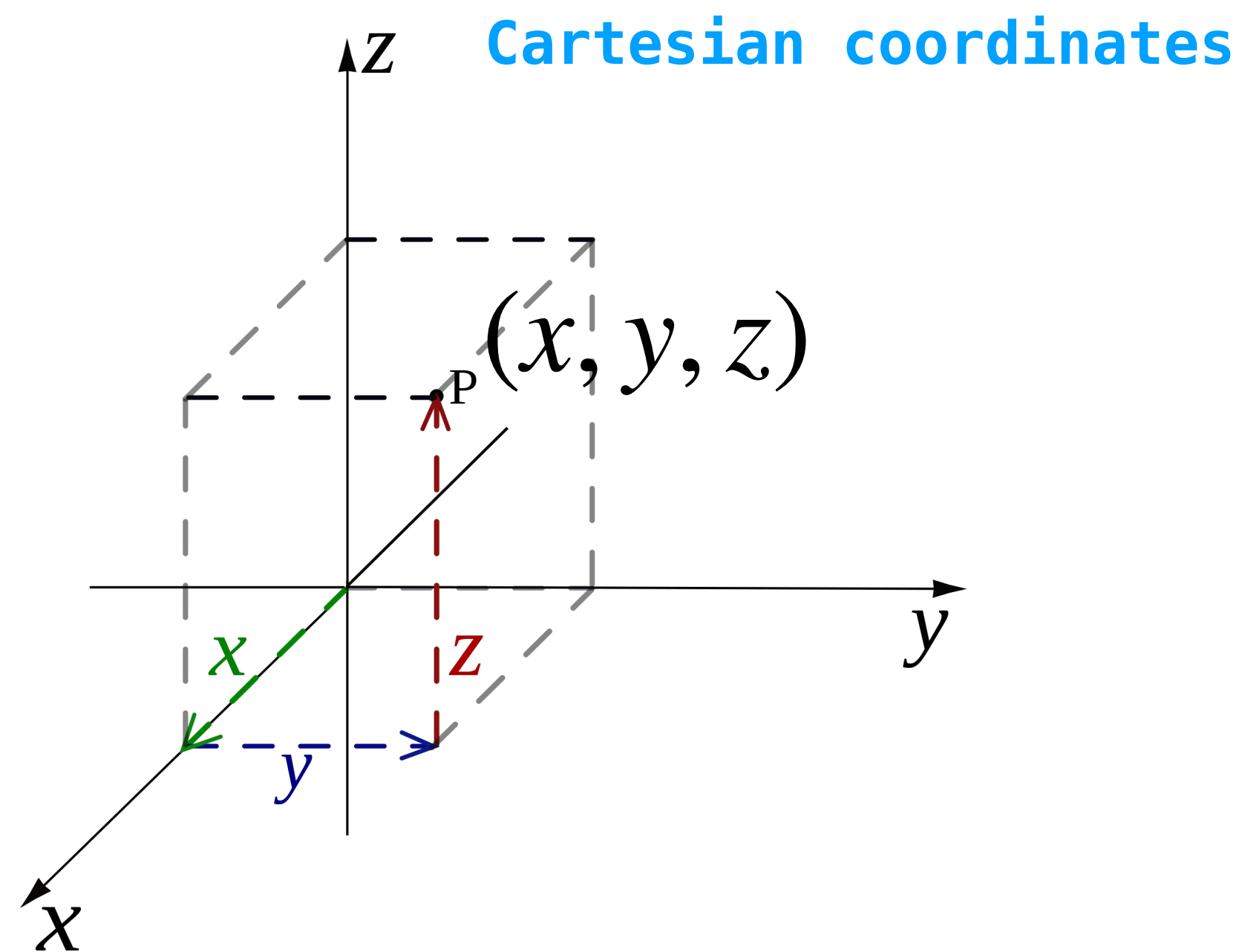
moving on . . .

Coordinate Systems

At a High Level

A coordinate system is a way of representing positions in terms of a sequence of numbers.

Examples.



Question (Conceptual)*

*And a bit of a trick question

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Is (2.3, 0.01, 5) a polar coordinate or a cartesian coordinate?

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Is (2.3, 0.01, 5) a polar coordinate or a cartesian coordinate?

This question is non-sensical.

It's just a sequence of numbers. We need to be *told* if it should be interpreted in the **polar** coordinate system or the **Cartesian** coordinate system.

Bases define Coordinate Systems

Given a basis \mathcal{B} of a subspace H , there is **exactly one way** to write every vector in H as a linear combination of vectors in \mathcal{B} .

Verify:

Bases define Coordinate Systems

Given a basis \mathcal{B} of a subspace H , there is **exactly one way** to write every vector in H as a linear combination of vectors in \mathcal{B} .

Every basis provides a way to write down *coordinates* of a vector.

And every time we write down a vector, we are **assuming a coordinate system**.

what do we mean by this?

A Thought Experiment

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Imagine doing this whole class from the beginning, but never saying *what vectors are*.

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Then one day, you get tired of talking about "abstract" vectors, you want to work with *numbers*.

A Thought Experiment

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$$\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ \vdots \\ -0.1 \end{bmatrix}$$

and then choose those weights as a representation of \mathbf{v} as a sequence of numbers

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Every basis defined a different coordinate system

Standard Basis

The standard basis defines the Cartesian coordinate system for \mathbb{R}^n .

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + \dots + v_n \mathbf{e}_n$$

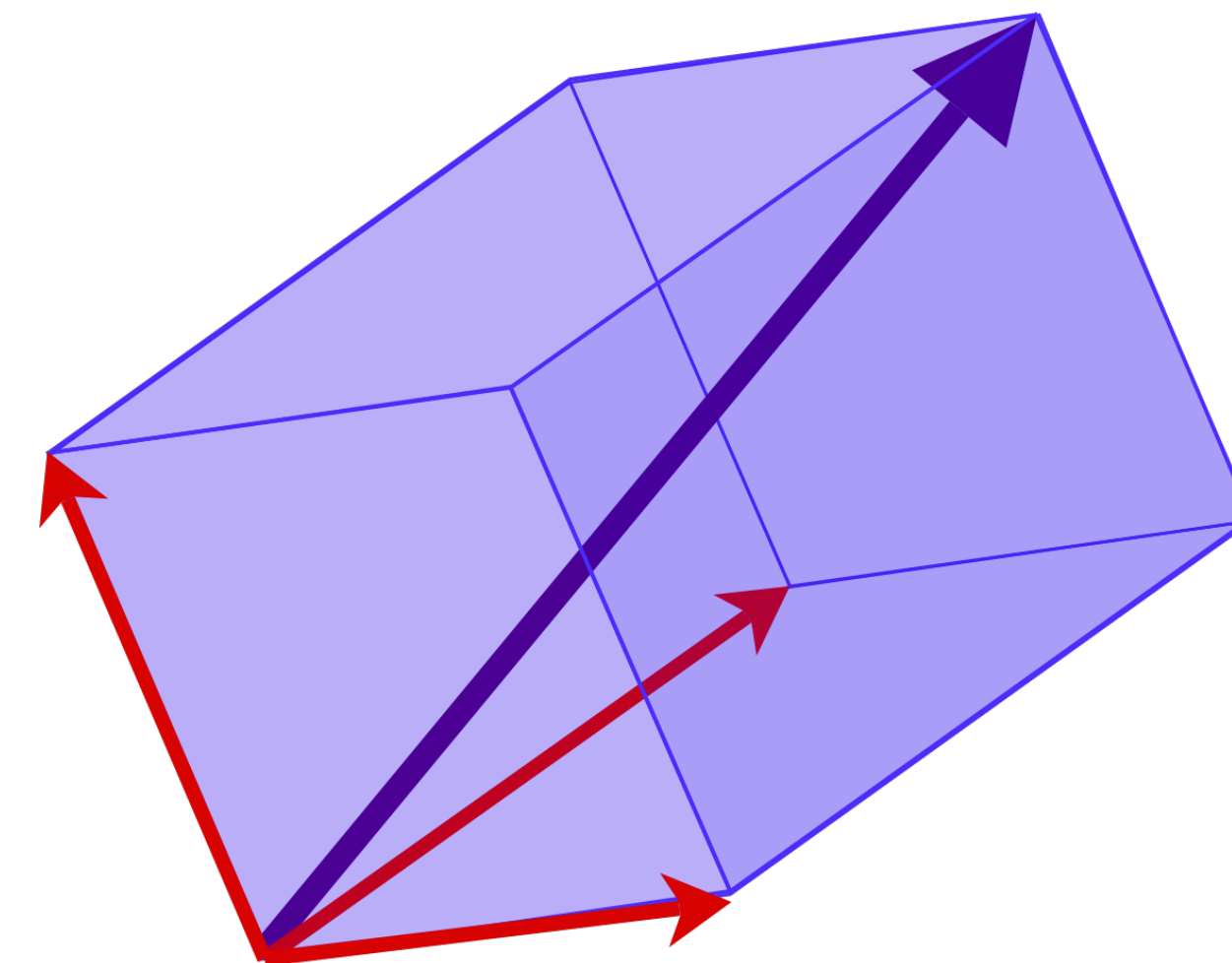
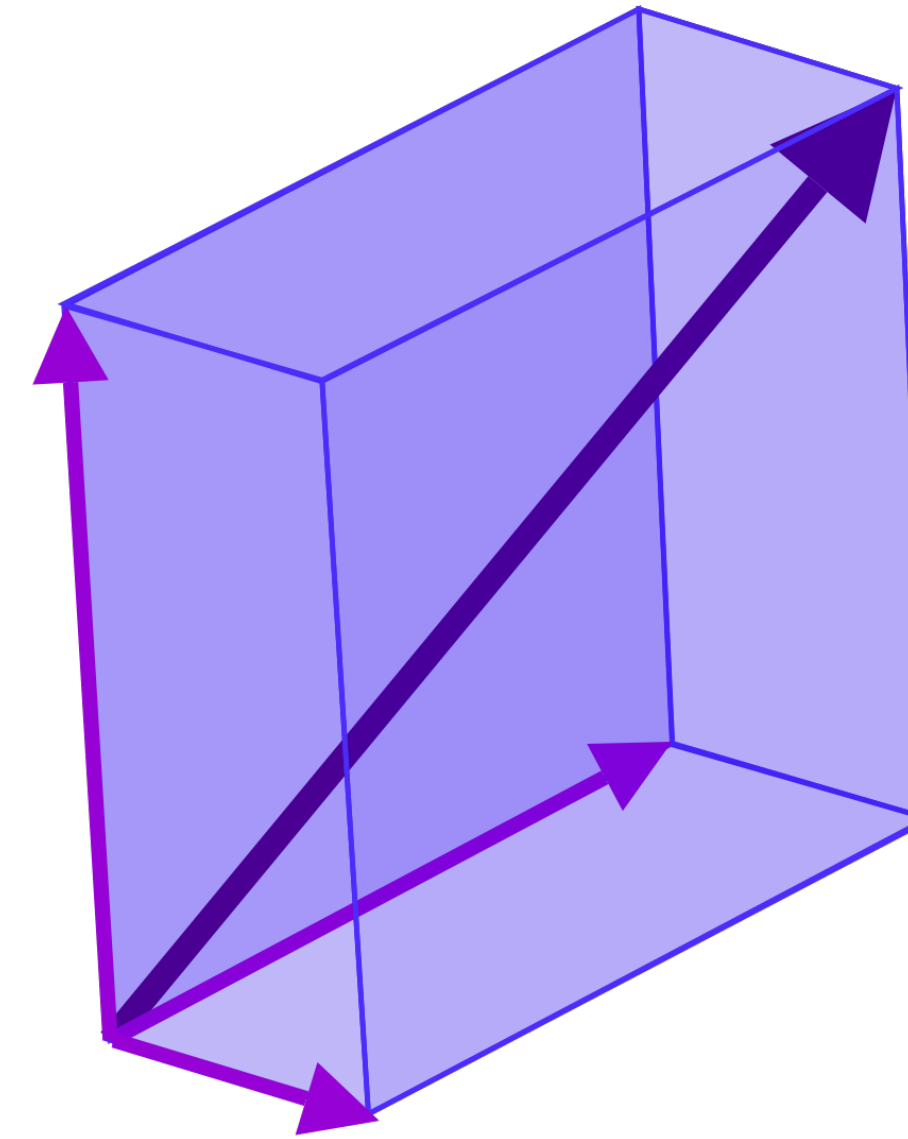
Column vectors are just weights for a linear combination of the standard basis

but we can also use
different coordinate systems

How to think about this

Changing the coordinate system "warps space".

The question is: how do we represent a vector v in the warped space if we wanted it to "be in the same place"?



Coordinate Vectors

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Let \mathbf{v} be a vector in a subspace H of \mathbb{R}^n and let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_k\}$ be a basis of H where

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$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix}$$

Coordinate Vectors and the Standard Basis

When we write down a vector \mathbf{v} in \mathbb{R}^n , we're really writing down a coordinate vector **relative to the standard basis \mathcal{E}** .

$$[\mathbf{v}]_{\mathcal{E}} = \mathbf{v}$$

How do we find coordinate vectors?

For an arbitrary basis \mathcal{B} , to determine $[\mathbf{v}]_{\mathcal{B}}$, we need to find weights a_1, \dots, a_k such that

$$a_1 \mathbf{b}_1 + \dots + a_k \mathbf{b}_k = \mathbf{v}$$

This is just solving a vector equation.

Example: 2D Case

Write the coordinate vector for $\begin{bmatrix} 1 \\ 6 \end{bmatrix}$ relative to the basis $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ for \mathbb{R}^2

Example: 2D Case (Geometrically)

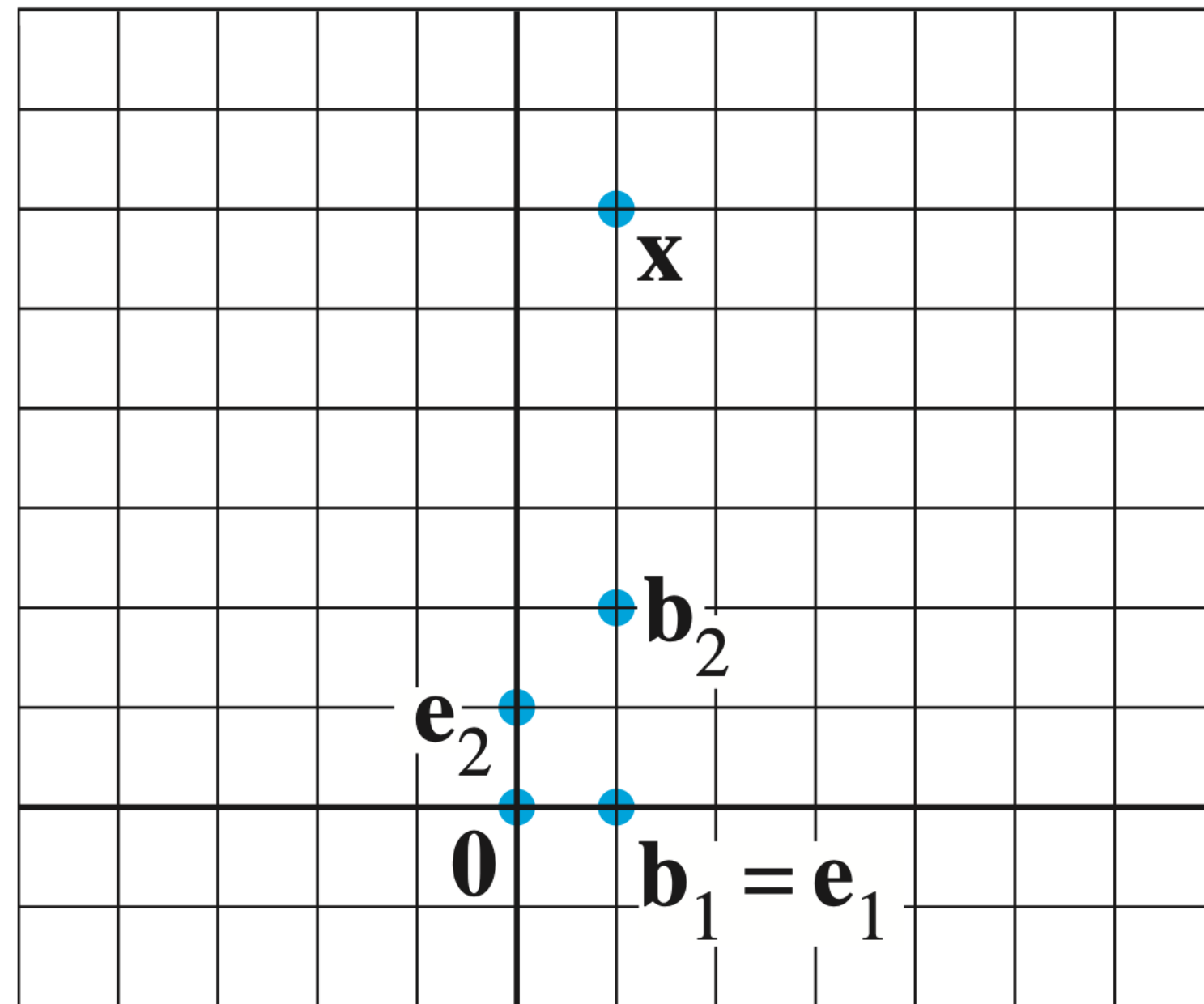


FIGURE 1 Standard graph paper.

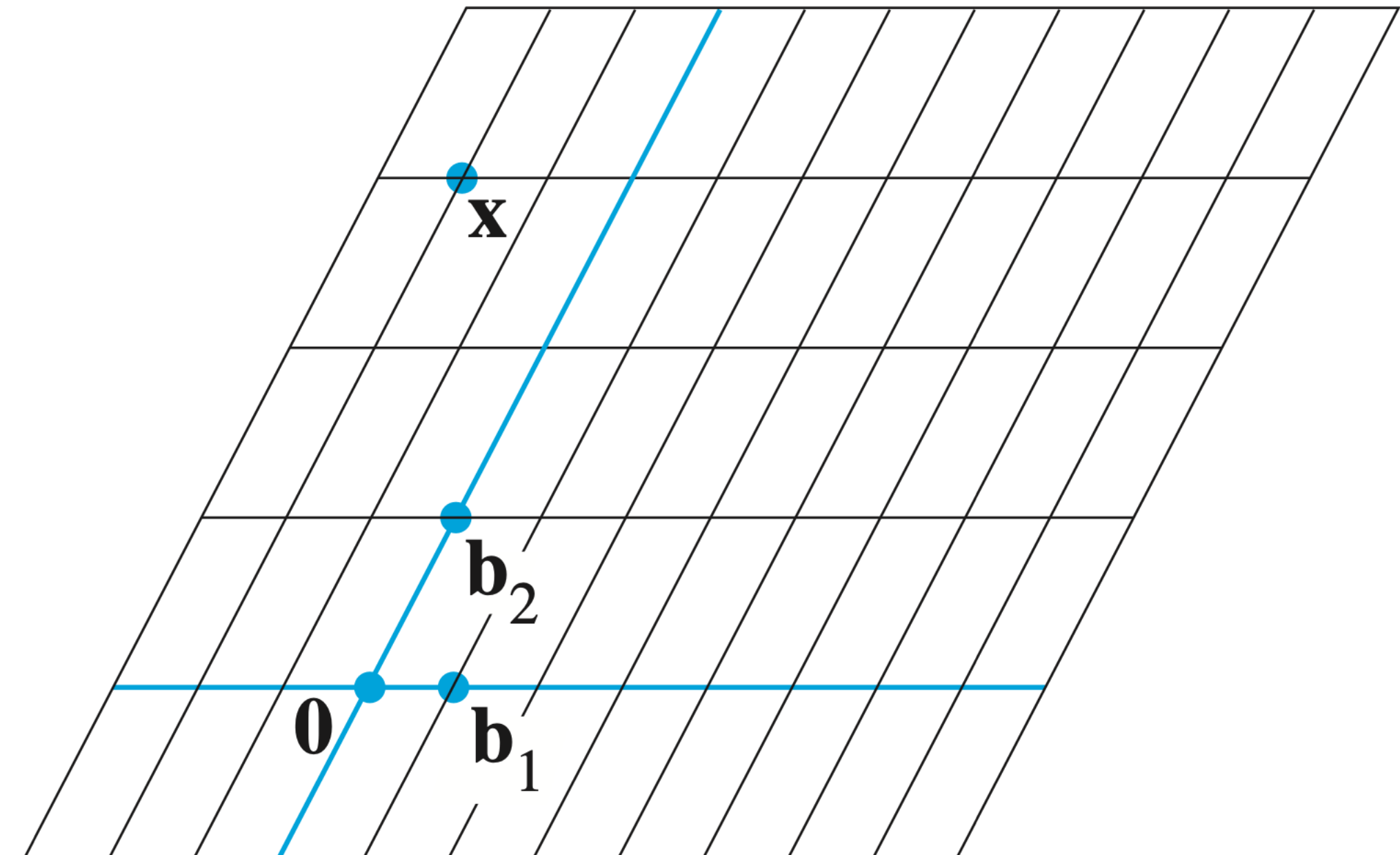


FIGURE 2 \mathcal{B} -graph paper.

\mathcal{B} defines a "different grid for our graph paper"

How To: Coordinate Vectors

Question. Find the coordinate vector for \mathbf{v} in the subspace H relative to the basis $\mathbf{b}_1, \dots, \mathbf{b}_k$.

Solution. Solve the vector equation

$$x_1 \mathbf{b}_1 + \dots + x_k \mathbf{b}_k = \mathbf{v}$$

A solution (a_1, \dots, a_k) means

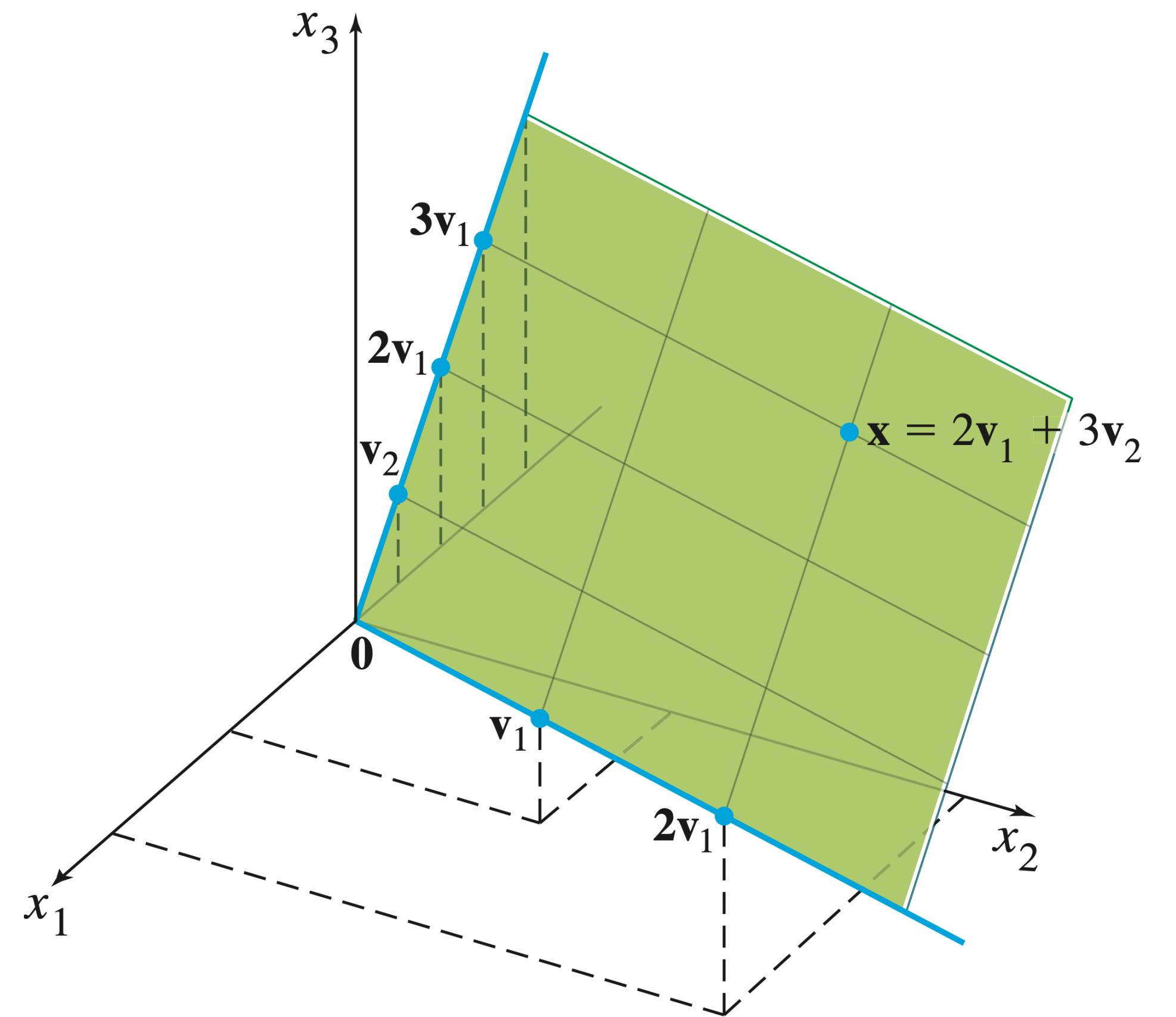
$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix}$$

Example: 3D Case

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{u} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$$

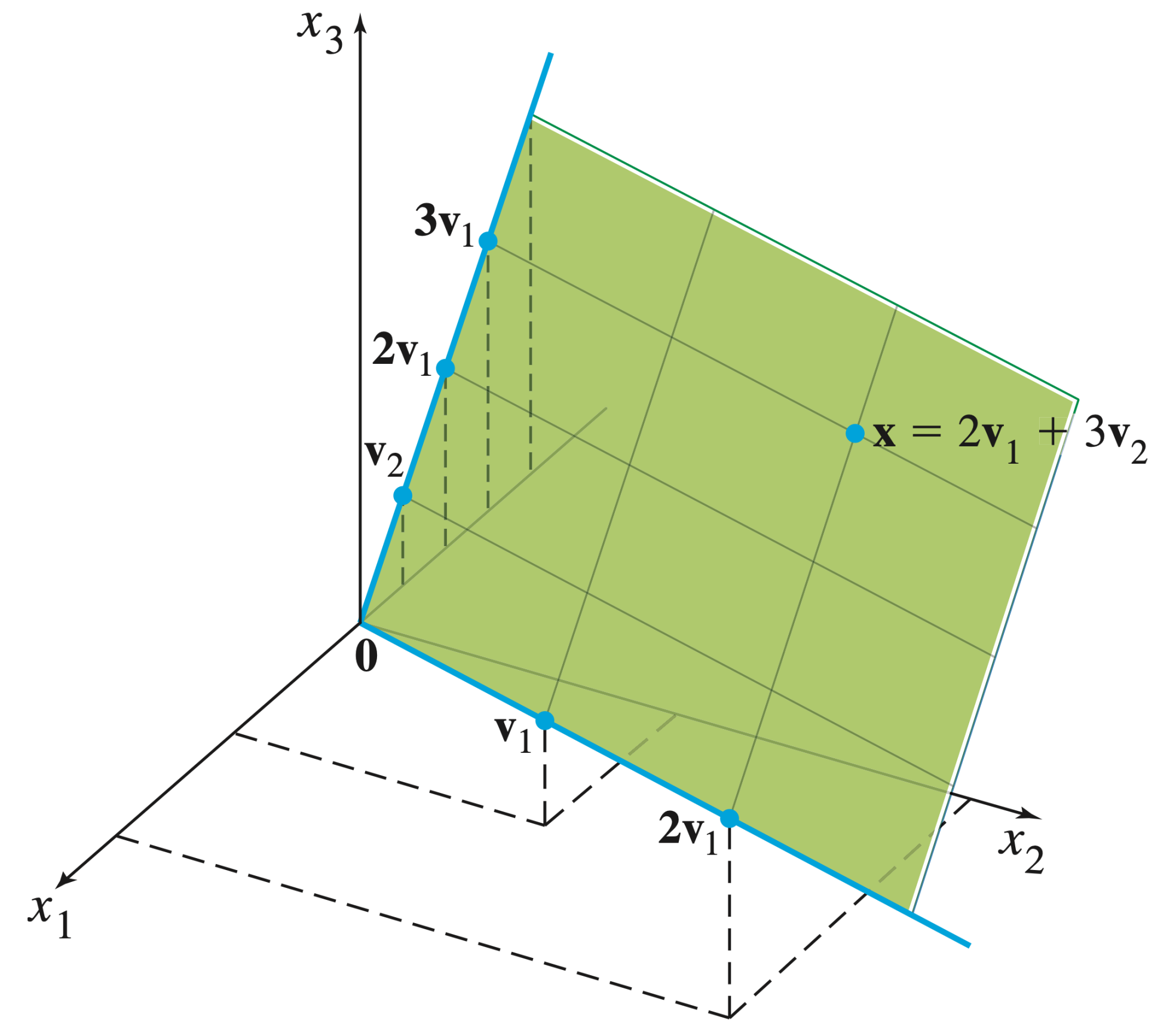
Find the coordinate vector for \mathbf{u} relative to the basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ of a subspace H (of \mathbb{R}^3):

An Aside: Coordinates and one-to-one correspondences



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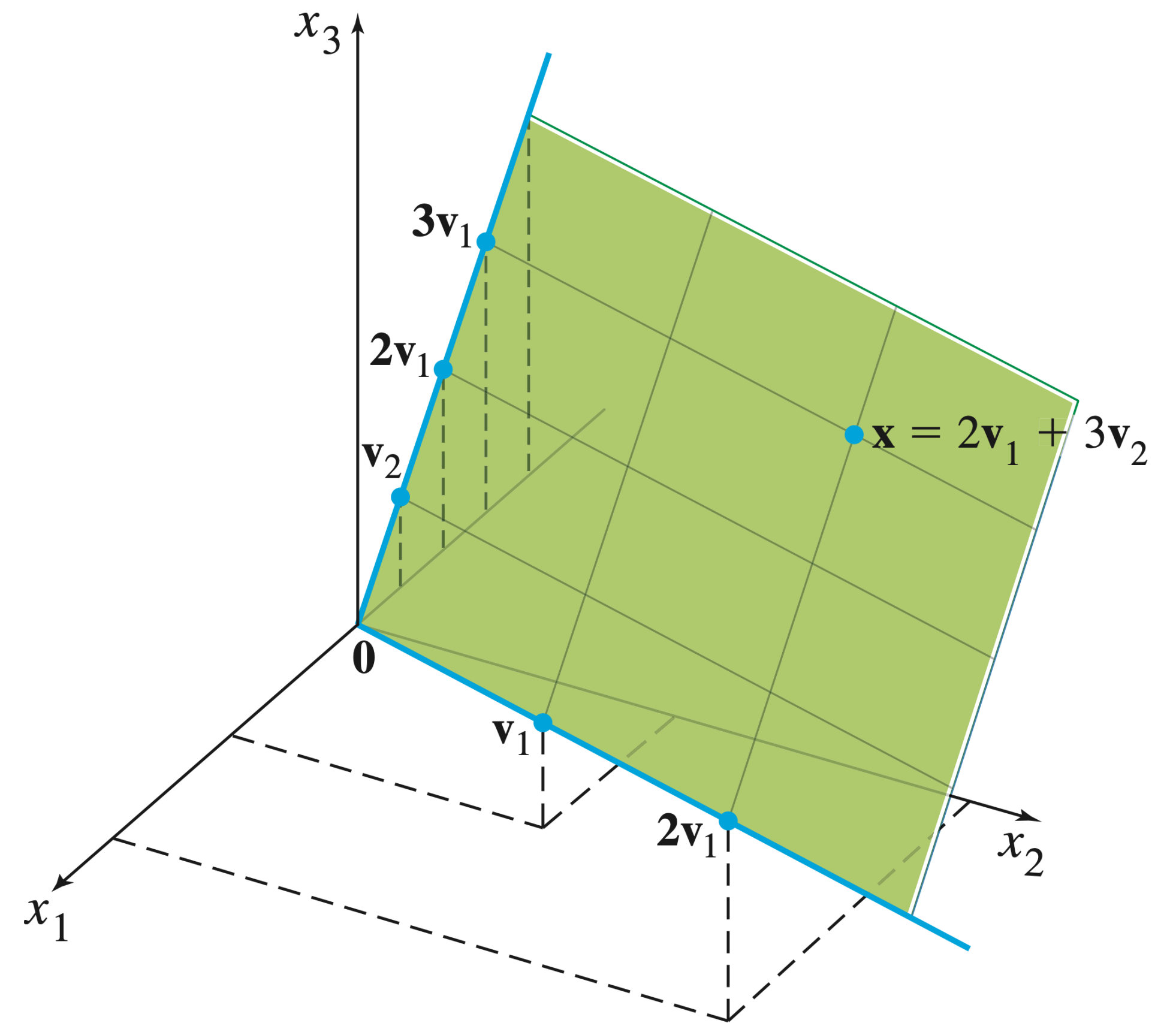
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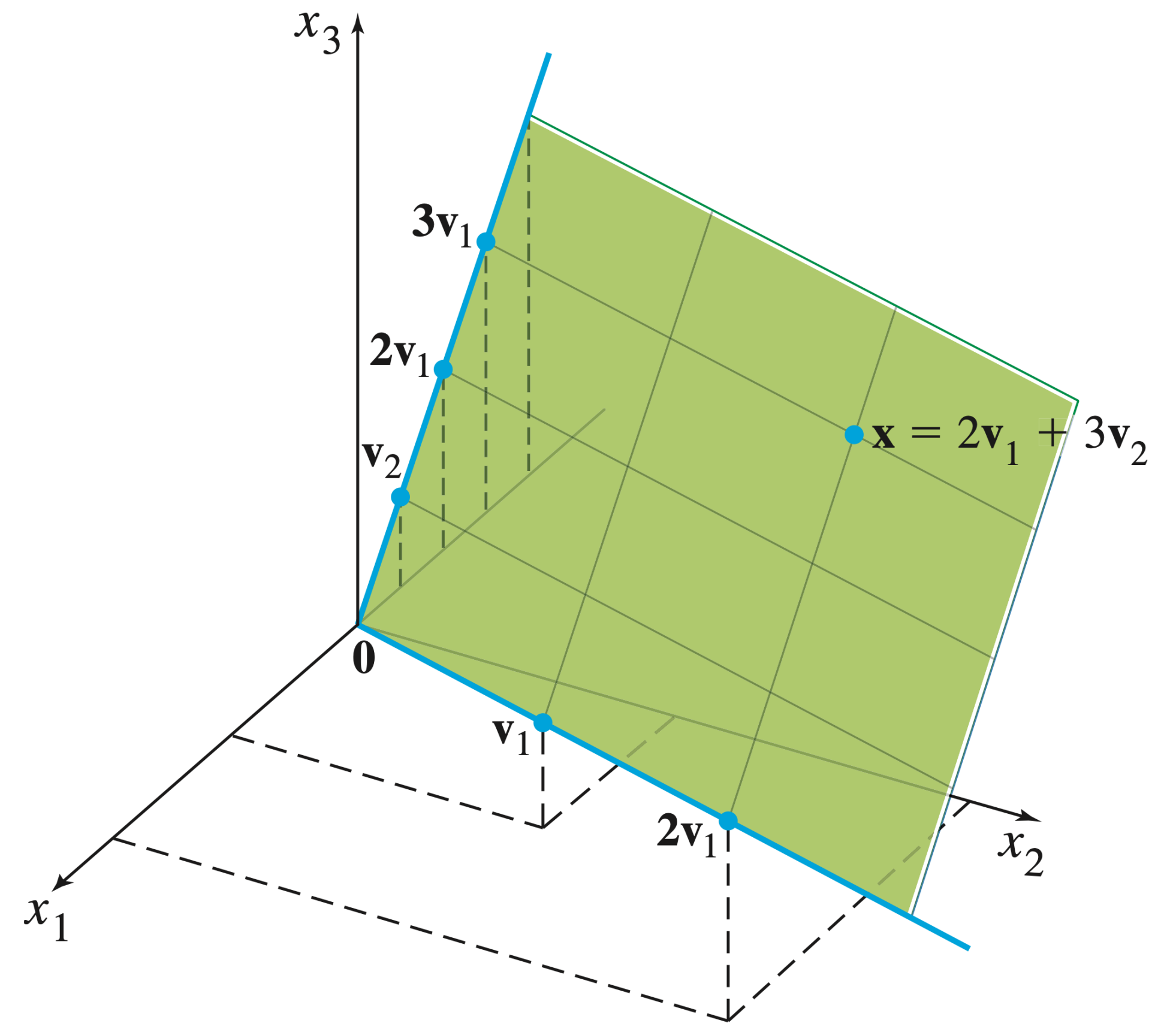


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So $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is *isomorphic* to \mathbb{R}^2 .



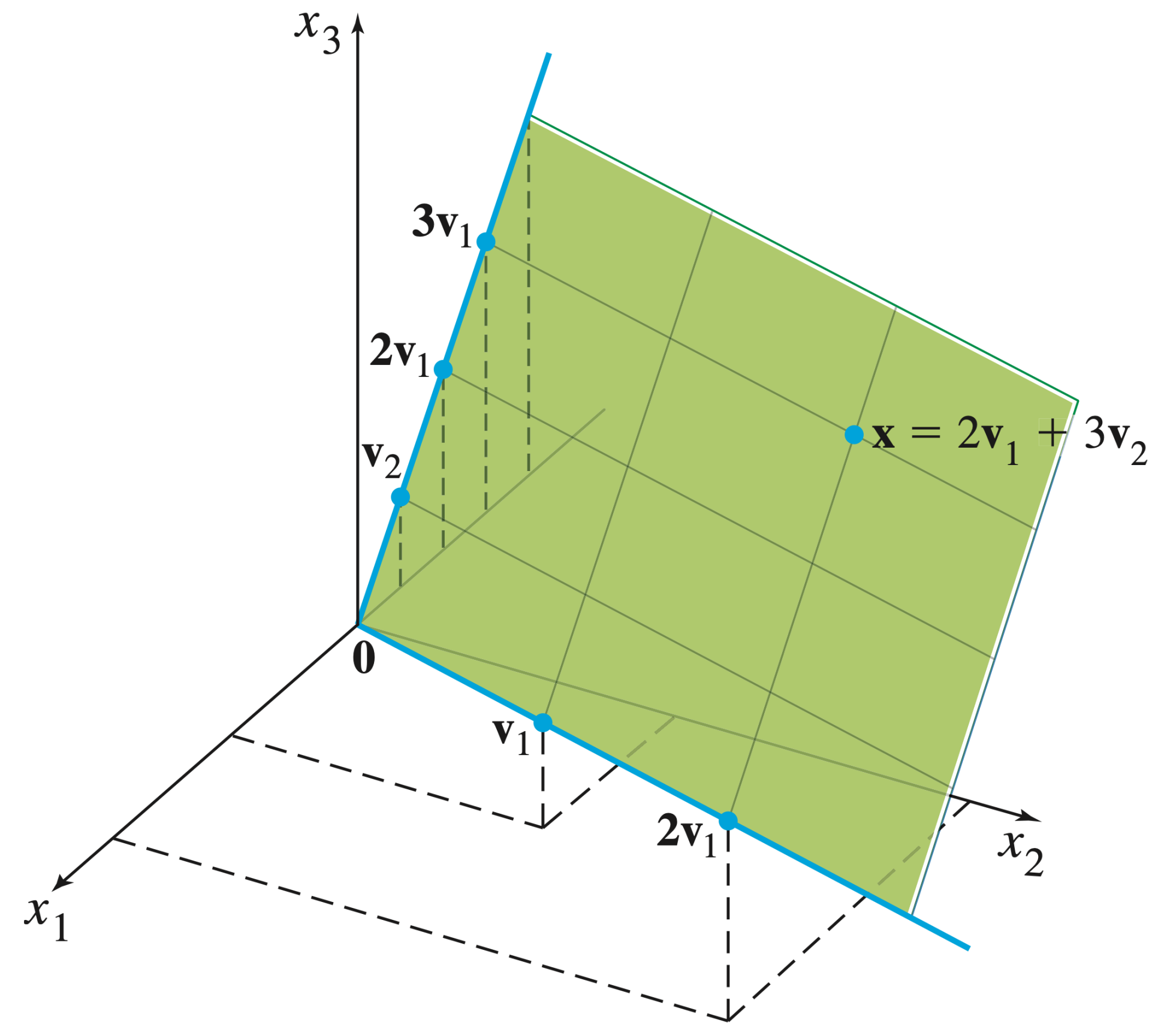
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So $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is *isomorphic* to \mathbb{R}^2 .

This is a formal way of saying that $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is a "copy of \mathbb{R}^2 ."



Question

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Suppose $[\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$, where $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$. Find \mathbf{u} .

Answer

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$$

$$\mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$[\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

Dimension and Rank

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This number is a measure of how "large" H is.

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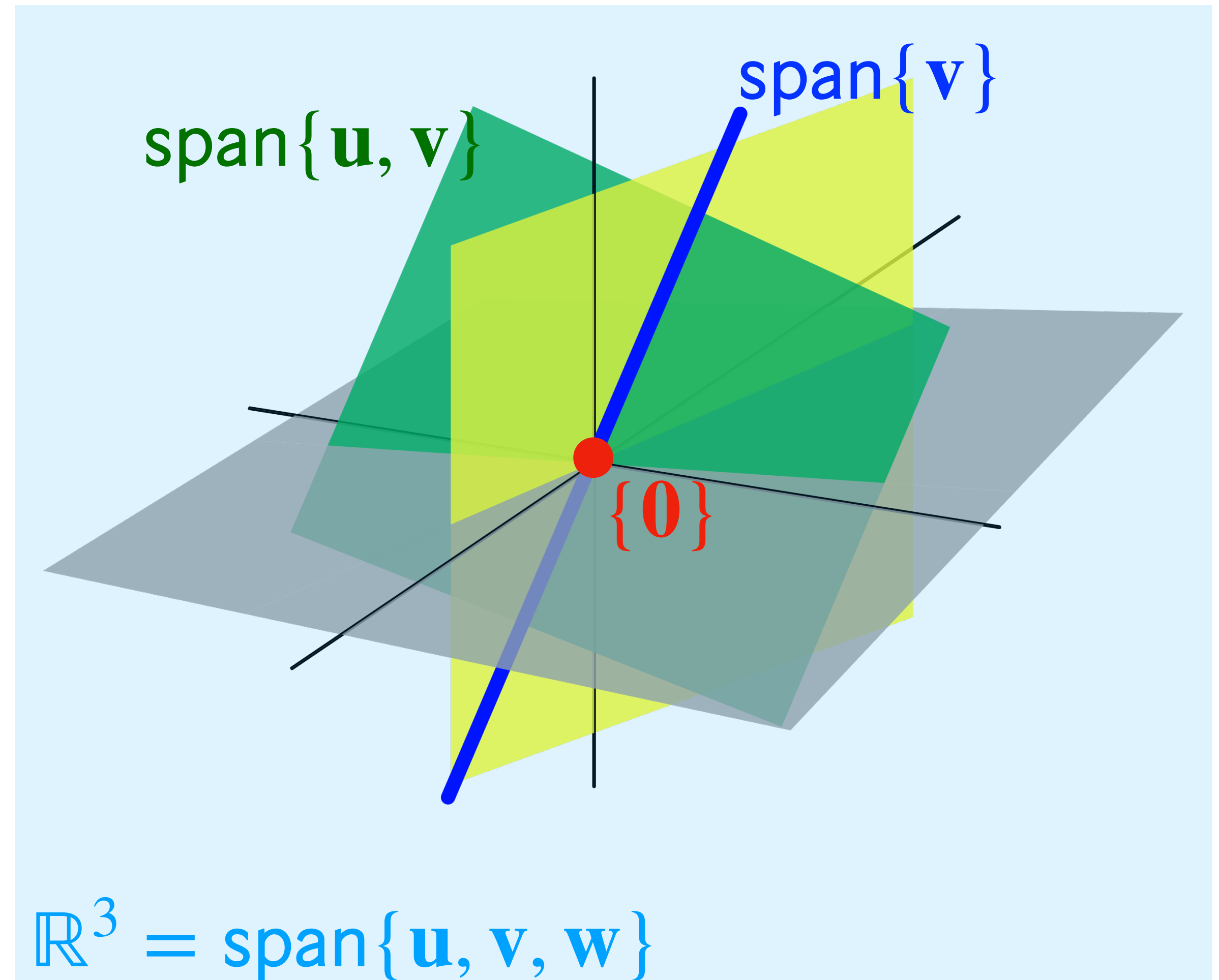
This should confirm our intuitions:

- » a plane (through the origin) is a 2D subspace
- » a line (through the origin) is a 1D subspace

Recall: Subspace in \mathbb{R}^3 (Geometrically)

There are only 4 kinds of subspaces of \mathbb{R}^3 :

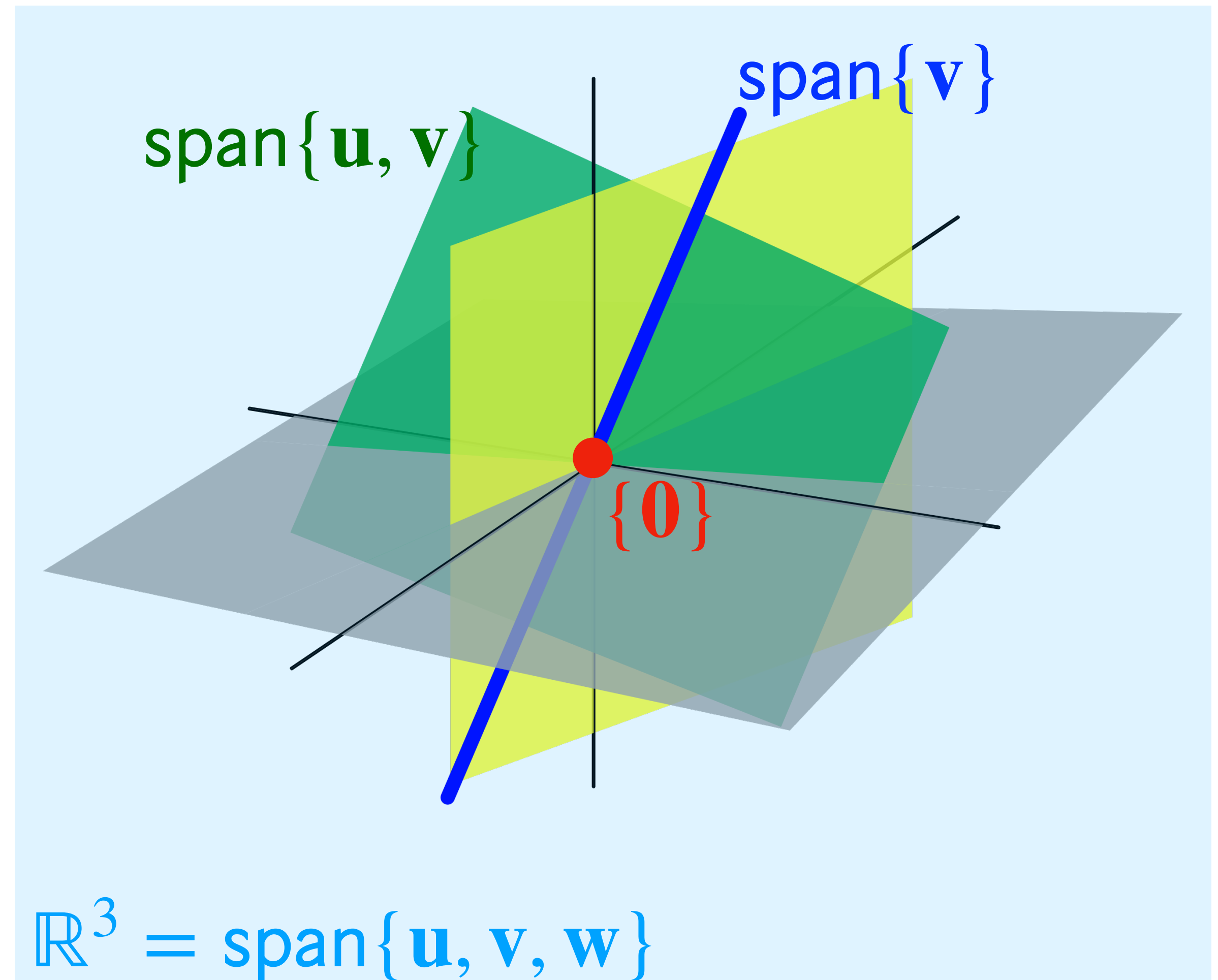
1. $\{\mathbf{0}\}$ just the origin
2. lines (through the origin)
3. planes (through the origin)
4. All of \mathbb{R}^3



Recall: Subspace in \mathbb{R}^3 (Geometrically)

There are only 4 kinds of subspaces of \mathbb{R}^3 :

1. 0-dimensional subspace
2. 1-dimensional subspaces
3. 2-dimensional subspaces
4. 3-dimensional subspace



How does this connect to
null space and column space?

Recall: An Observation

The *number* of vectors in the basis we found is the same as the number of free variables in a general form solution.

$$x_1 = 2x_2 + x_4 - 3x_5$$

x_2 is free

$$x_3 = (-2)x_4 + 2x_5$$

x_4 is free

x_5 is free

\equiv

$$\begin{bmatrix} s \\ t \\ u \end{bmatrix}$$

\mapsto

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Dimension of the Null Space

The **dimension** of $\text{Nul}(A)$ is the number of free variables in a general form solution to $A\mathbf{x} = \mathbf{0}$.

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The *number* of vectors in the basis we found is the same as the number of basic variable or equivalently the number of pivot columns.

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4 \quad \mathbf{a}_5] \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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The **dimension** of $\text{Col}(A)$ is the number of basic variable in our solution, or equivalently the number of pivot columns of A .

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Rank

Definition. The rank of a matrix A , written $\text{rank}(A)$ or $\text{rank } A$, is the dimension of $\text{Col}(A)$.

This is just terminology.

Rank-Nullity Theorem

Theorem. For an $m \times n$ matrix A ,

$$\text{rank}(A) + \dim(\text{Nul}(A)) = n$$

Verify:

This is incredibly important.

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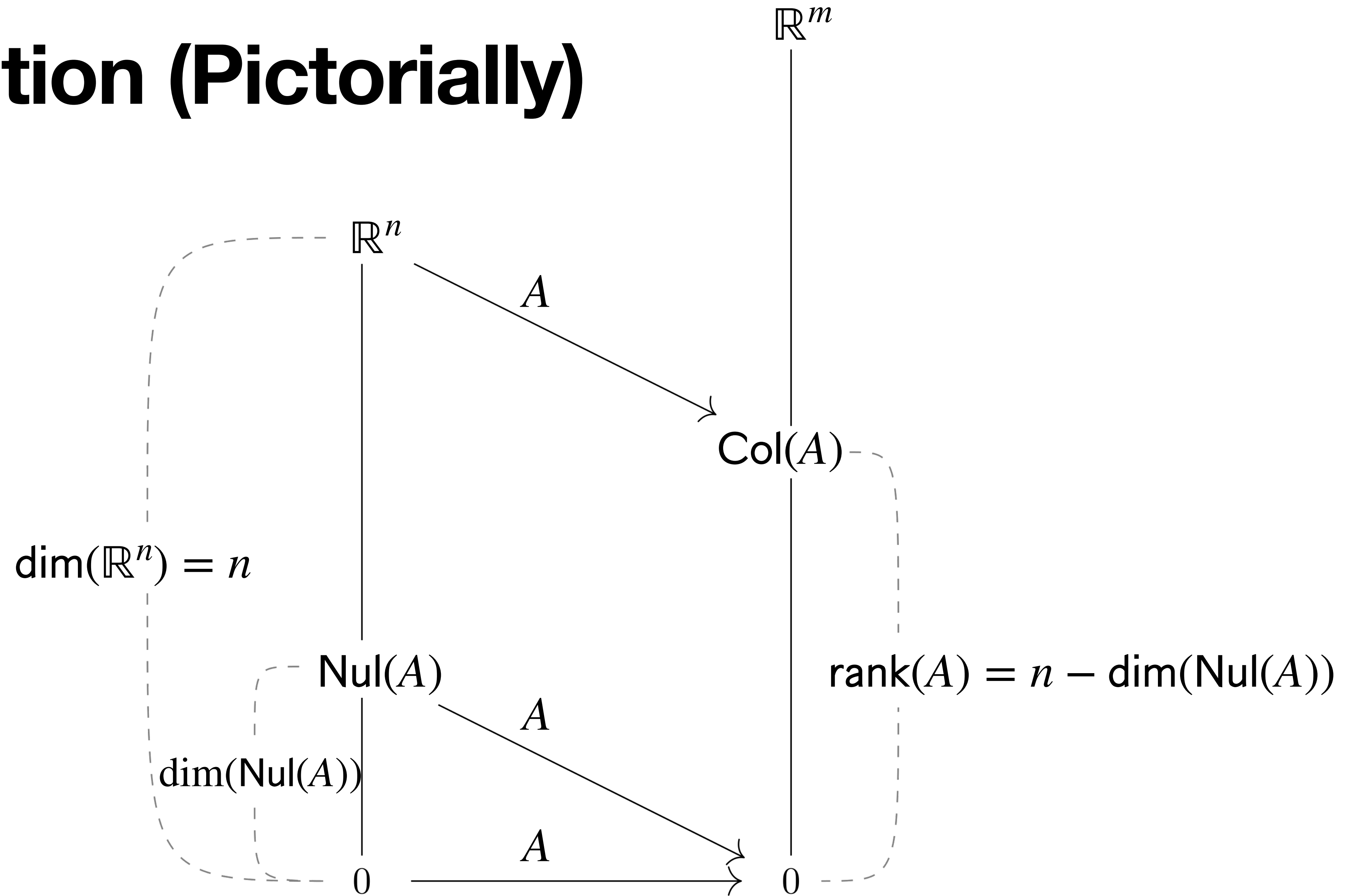
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The null space "takes away" some of the dimensions of the column space.

The Intuition (Pictorially)



Question (Conceptual)

*Let A be a 5×7 matrix such that $\dim(\text{Nul}(A)) = 3$.
What is the dimension of $\text{Col}(A)$?*

Answer: 4

Extending the IMT

Theorem. For an $n \times n$ invertible matrix A , the following are logically equivalent (they must all be true or all be false).

- » $\text{Col}(A) = \mathbb{R}^n$
- » $\dim(\text{Col}(A)) = n$
- » $\text{rank}(A) = n$
- » $\text{Nul}(A) = \{\mathbf{0}\}$
- » $\dim(\text{Nul}(A)) = 0$

Summary

We can find bases for the column space and null space by looking at the reduced echelon form of a matrix.

Column vectors are written in terms of a coordinate system, which we can change.

Dimension is a measure of how large a space is.