Eigenvalues and Eigenvectors

Geometric Algorithms
Lecture 17

CAS CS 132

Introduction

Recap Problem

Show that the set

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} : x_2 = x_3 \right\}$$

is a subspace of $\mathbb{R}^4$.

Answer

Objectives

1. Motivate and introduce the fundamental notion of eigenvalues and eigenvectors.
2. Determine how to verify eigenvalues and eigenvectors.
3. Look at the **subspace** generated by eigenvectors.
4. Apply the study of eigenvectors to dynamical linear systems.

Keyword

Eigenvalues

Eigenvectors

Null Space

Eigenspace

Linear Dynamical Systems

Closed-Form Solutions

Motivation

demo

How can matrices transform vectors?*

In 2D and 3D we've seen:

» rotations
» projections
» shearing
» reflection
» scaling/stretching
» ...

* square matrices

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How can matrices transform vectors?*

In 2D and 3D we've seen:

- rotations
- projections
- shearing
- reflection
- scaling/stretching
- ... **Today's focus**

All matrices do some combination of these things

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What's special about scaling?

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We don't need a whole matrix to scaling

$$\mathbf{x} \mapsto c\mathbf{x}$$

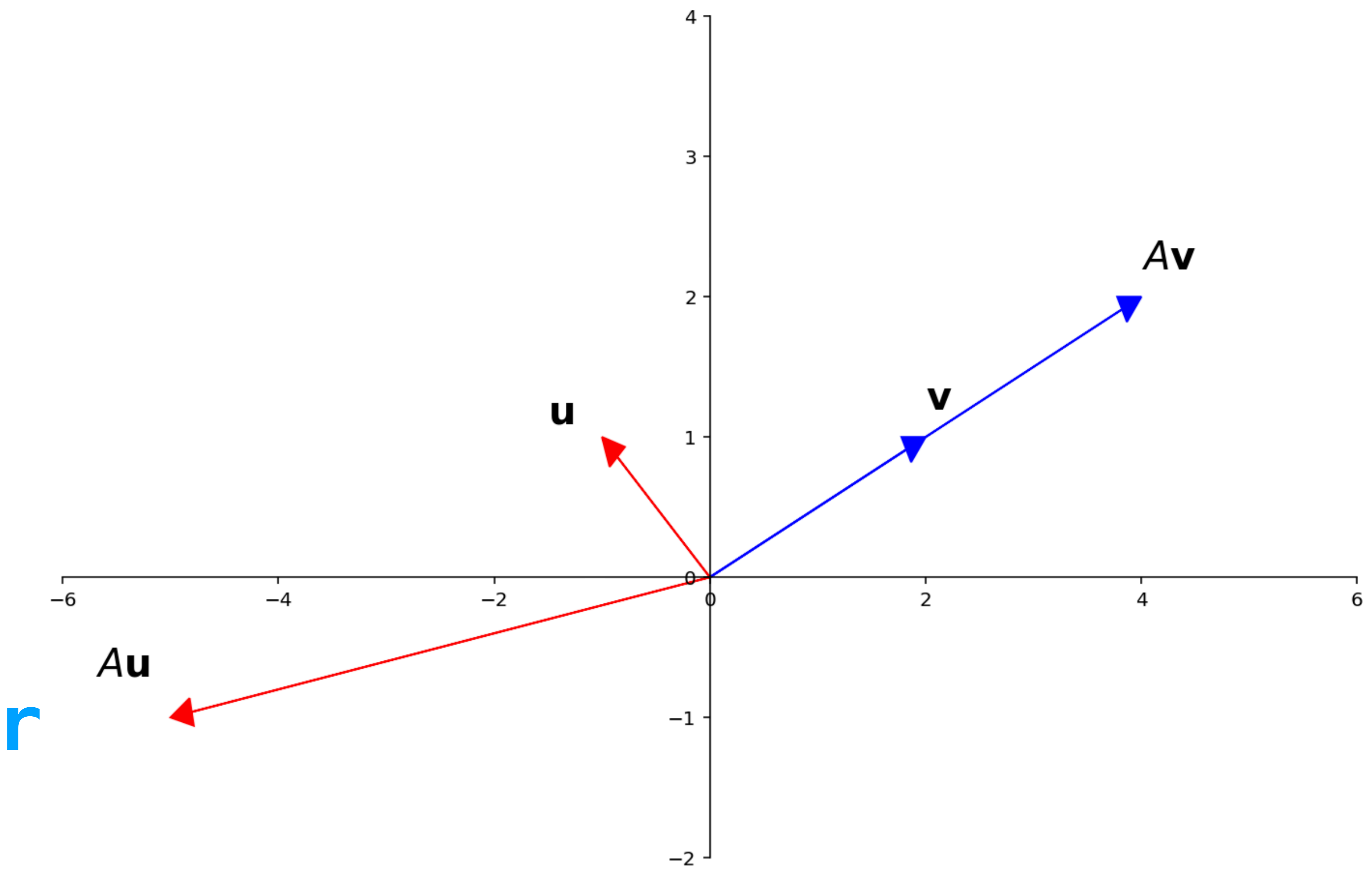
So if $A\mathbf{v} = c\mathbf{v}$ then it's "easy to describe" what A does to $\mathbf{v}$.

Eigenvectors (Informal)

eigenvalue

$$A\mathbf{v} = \lambda\mathbf{v}$$

eigenvector



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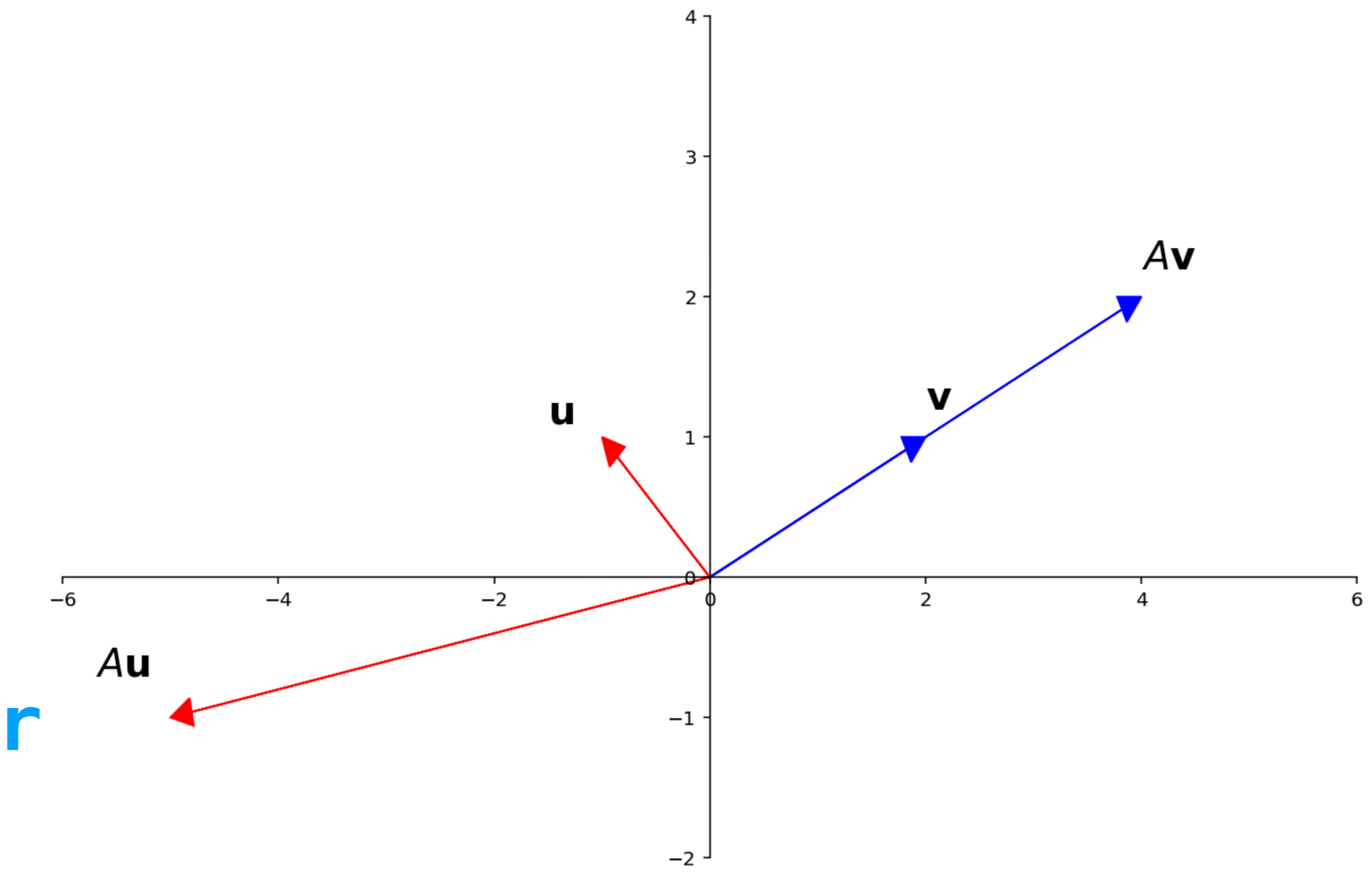

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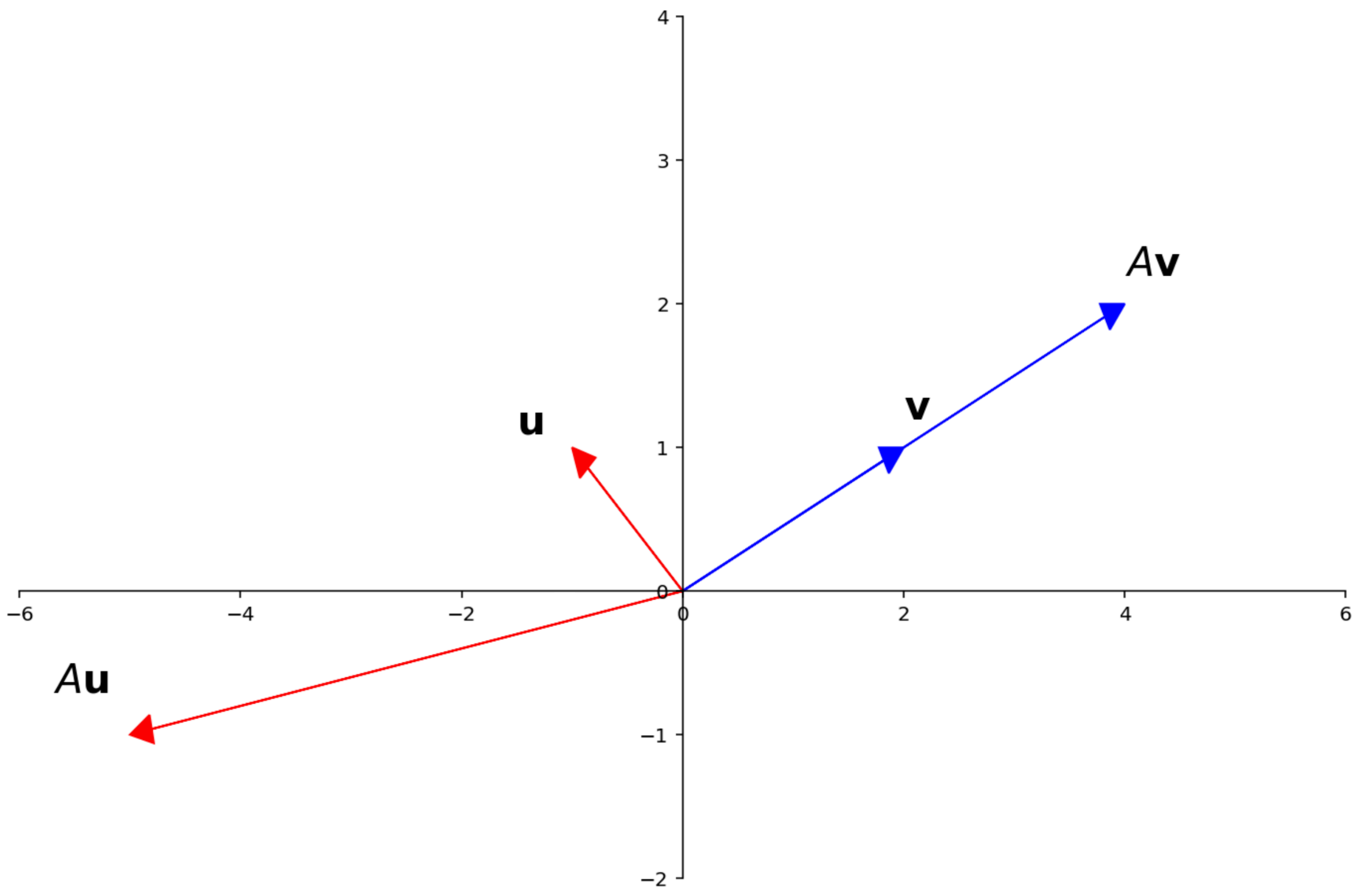


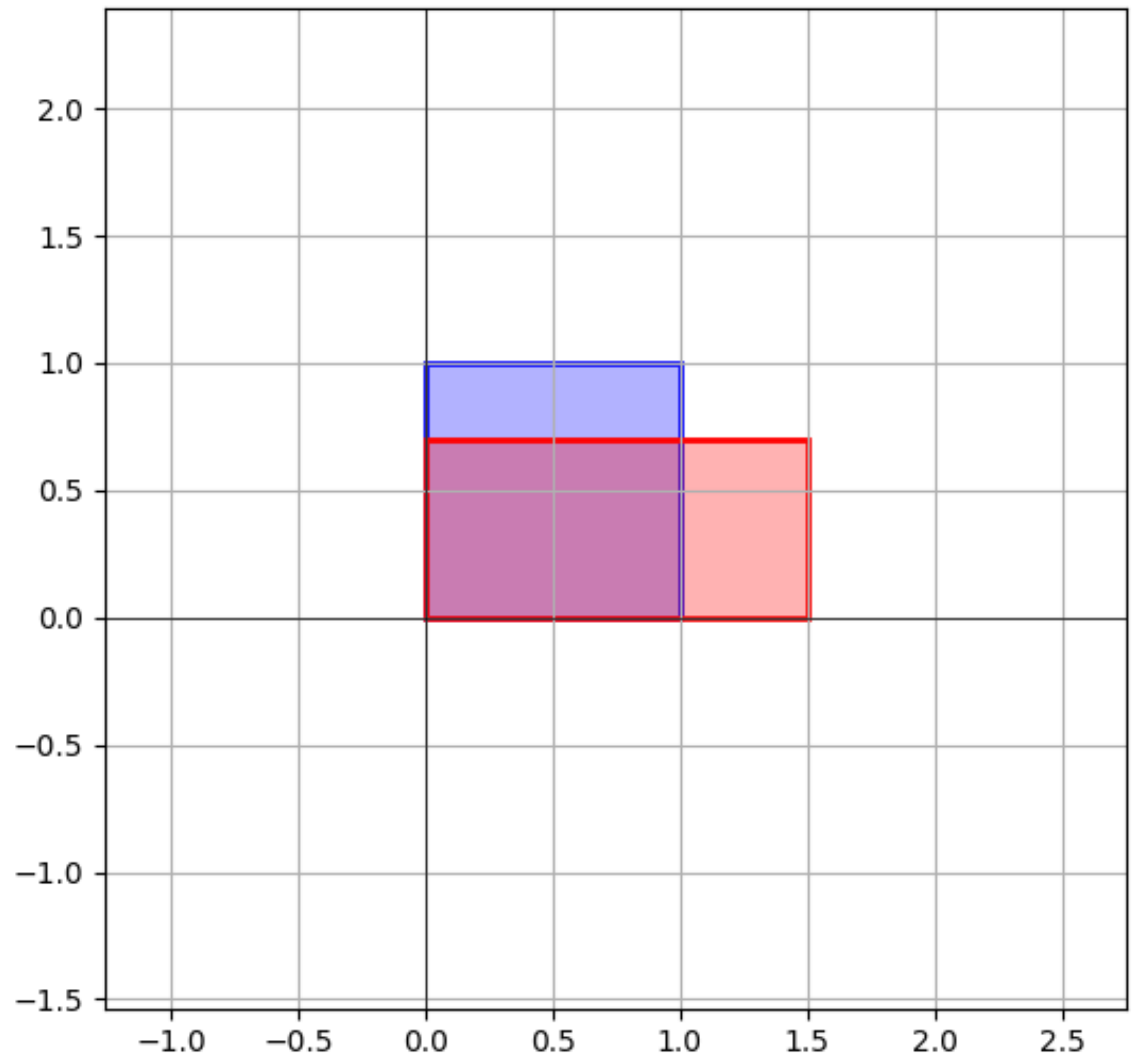
Eigenvectors of A are stretched by A without changing their direction.

The amount they are stretched is called the **eigenvalue.**

Example: Unequal Scaling

It's "easy to describe" how unequal scaling transforms vectors.

It transforms each entry individually and then combines them.



$$\begin{bmatrix} 1.5 & 0 \\ 0 & 0.7 \end{bmatrix}$$

Eigenbases (Informal)

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Imagine if $\mathbf{v} = 2\mathbf{b}_1 - \mathbf{b}_2 - 5\mathbf{b}_3$ and $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ are *eigenvectors of* A. Then

$$A\mathbf{v} = 2\lambda_1\mathbf{b}_1 - \lambda_2\mathbf{b}_2 - 5\lambda_3\mathbf{b}_3$$

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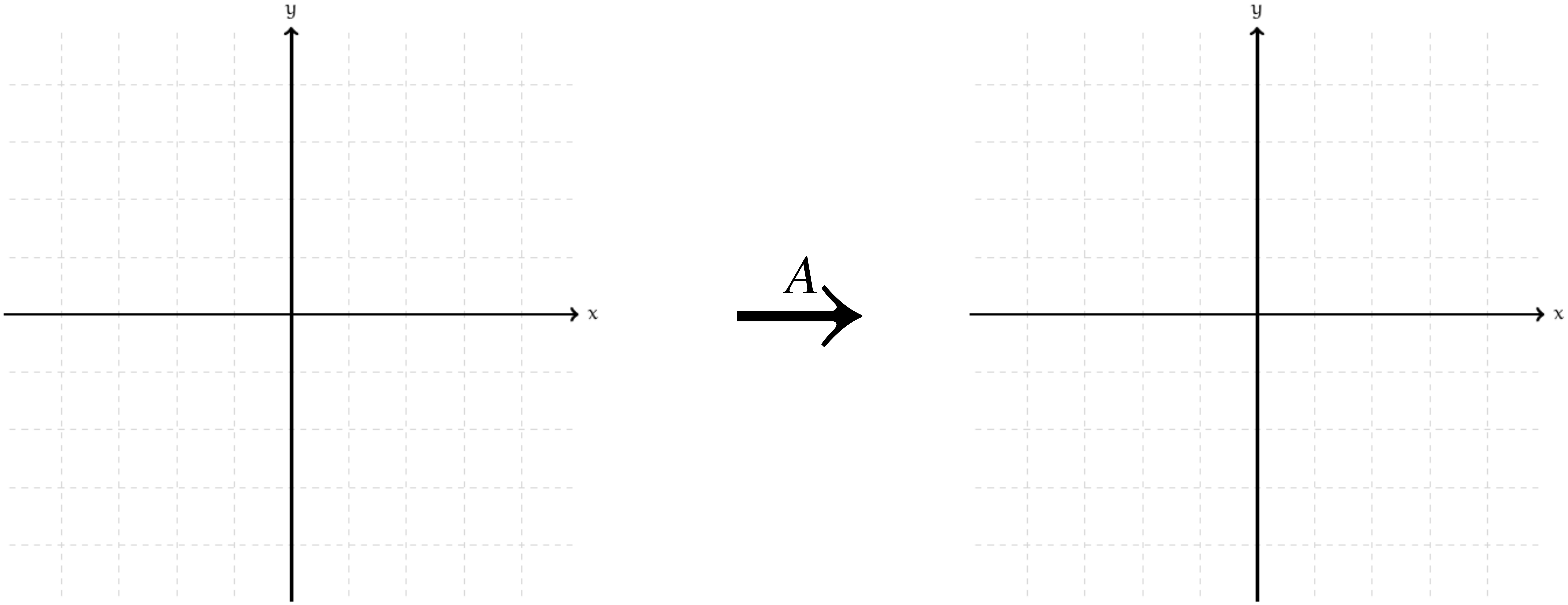
It's "easy to describe" how A transforms $\mathbf{v}$.

It *transforms each "component" individually and then combines them.*

Verify:

Eigenbases (Pictorially)



Fundamental Questions

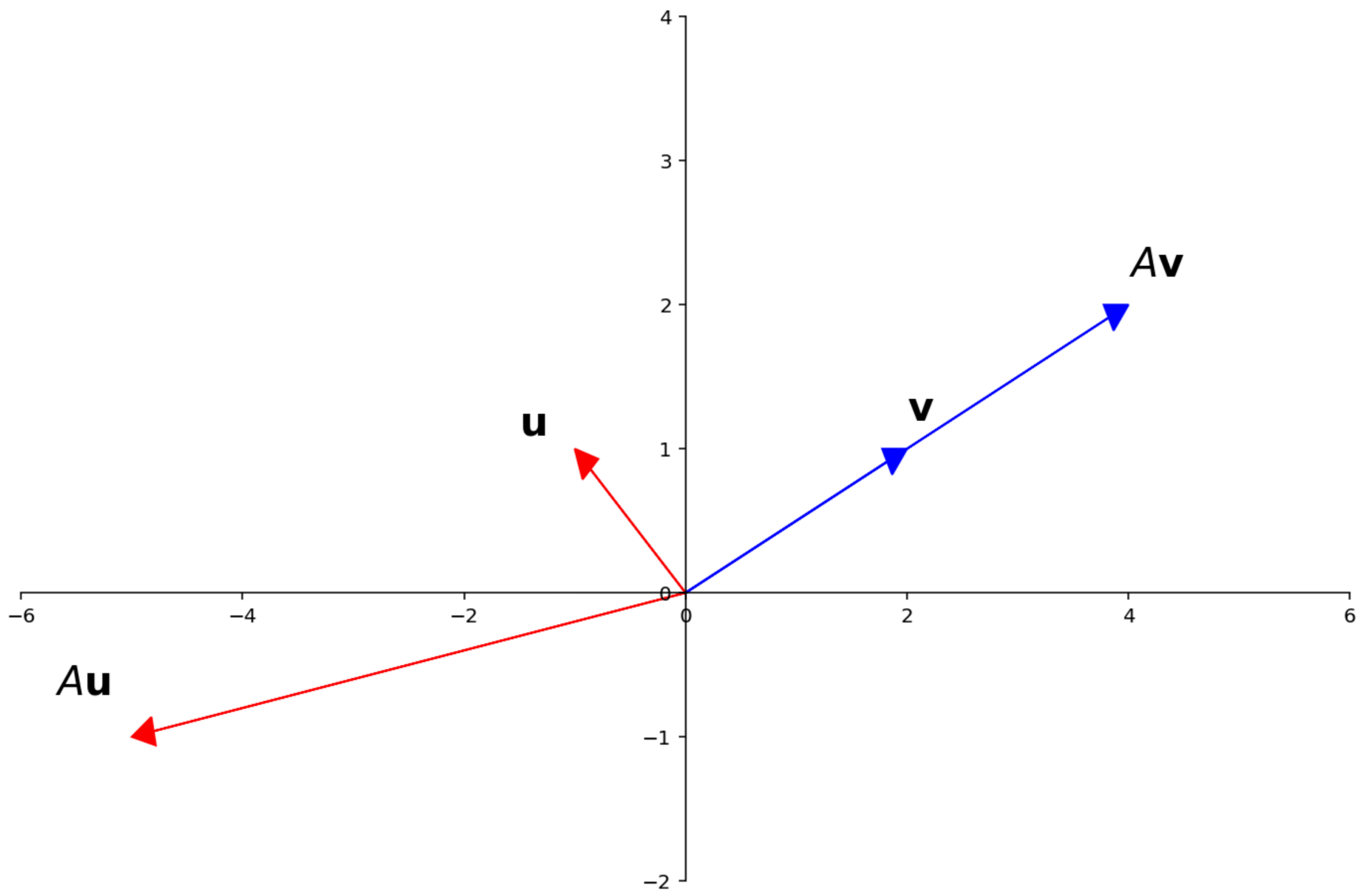
How do we understand the effect of a matrix on a vector?

When is this effect "easy to describe"?

Which vectors are "just stretched" by a matrix?

Eigenvalues and Eigenvectors

Formal Definition



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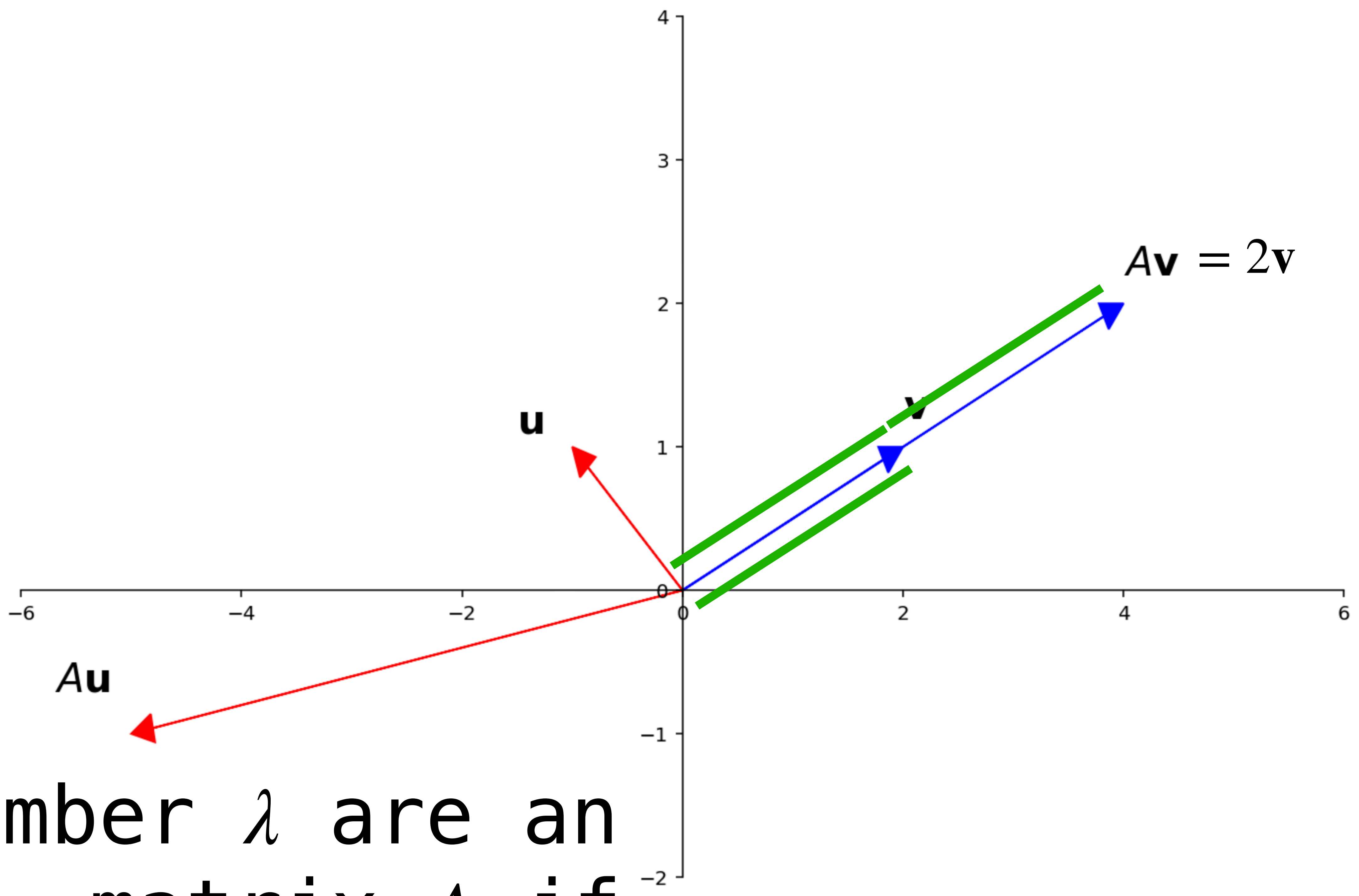

A *nonzero* vector $\mathbf{v}$ in $\mathbb{R}^n$ and real number λ are an **eigenvector and eigenvalue** for a $n \times n$ matrix A if

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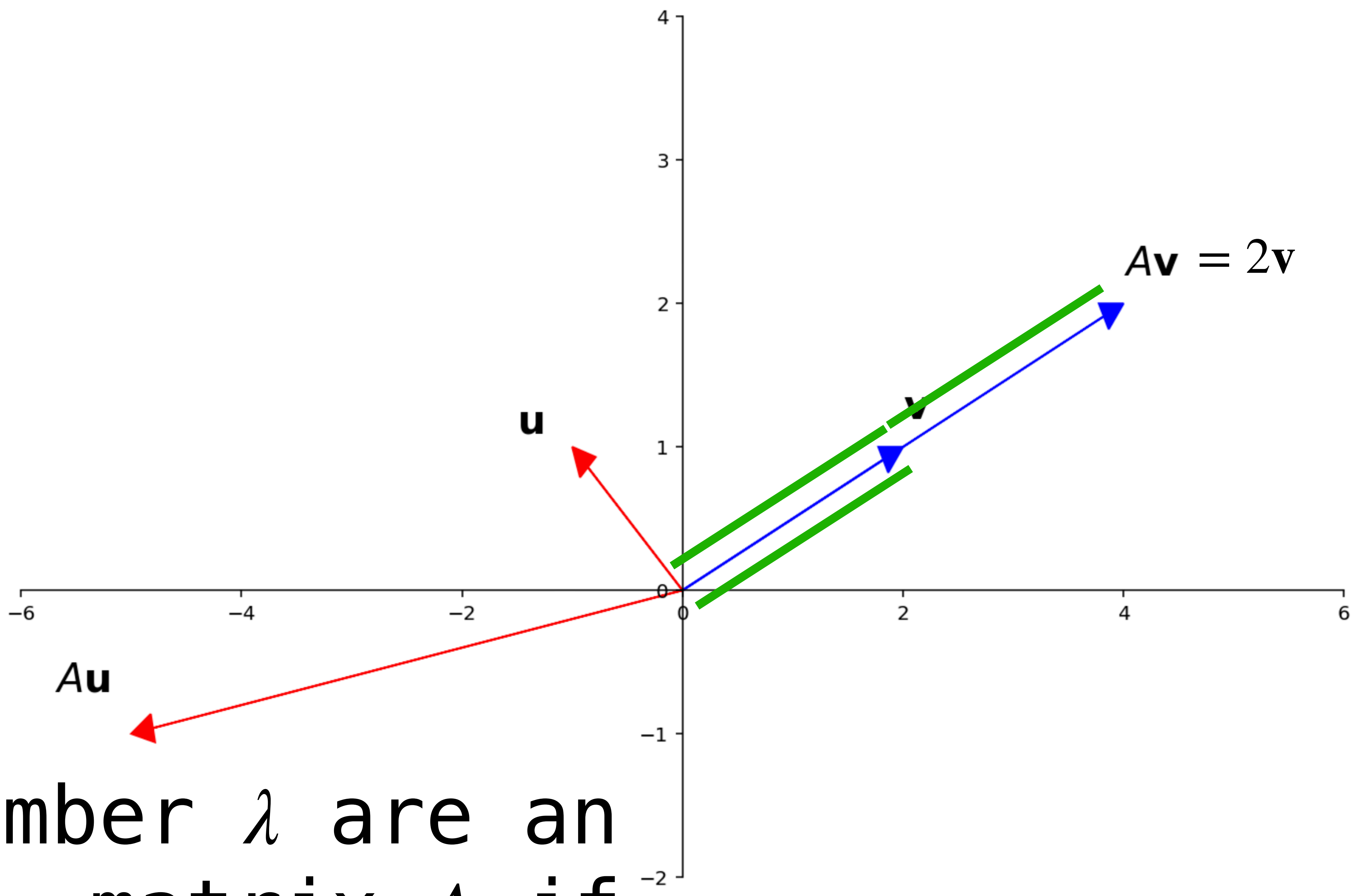

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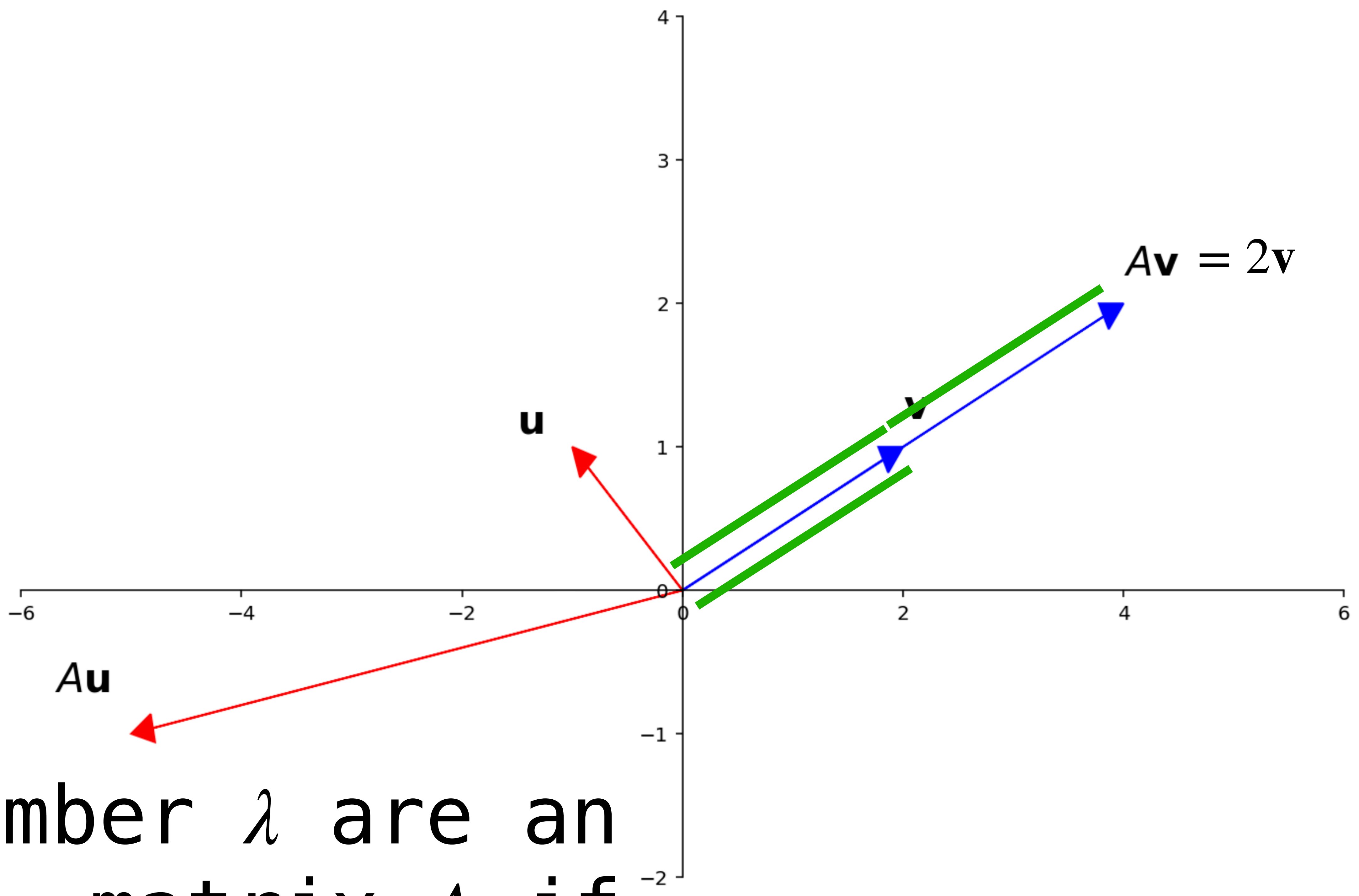

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Note. Eigenvectors <u>must</u> be nonzero, but it is possible for 0 to be an eigenvalue.

What if 0 is an eigenvalue?

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If A has the eigenvalue 0 with the eigenvector $\mathbf{v}$, then

$$A\mathbf{v} = 0\mathbf{v} = \mathbf{0}$$

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In other words,

- $\mathbf{v} \in \text{Nul}(A)$
- $\mathbf{v}$ is a nontrivial solution to $A\mathbf{v} = \mathbf{0}$

Extending the IMT (Again)

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Theorem. A $n \times n$ matrix is invertible if and only if it does not have 0 as an eigenvalue.

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To reiterate. An eigenvalue 0 implies

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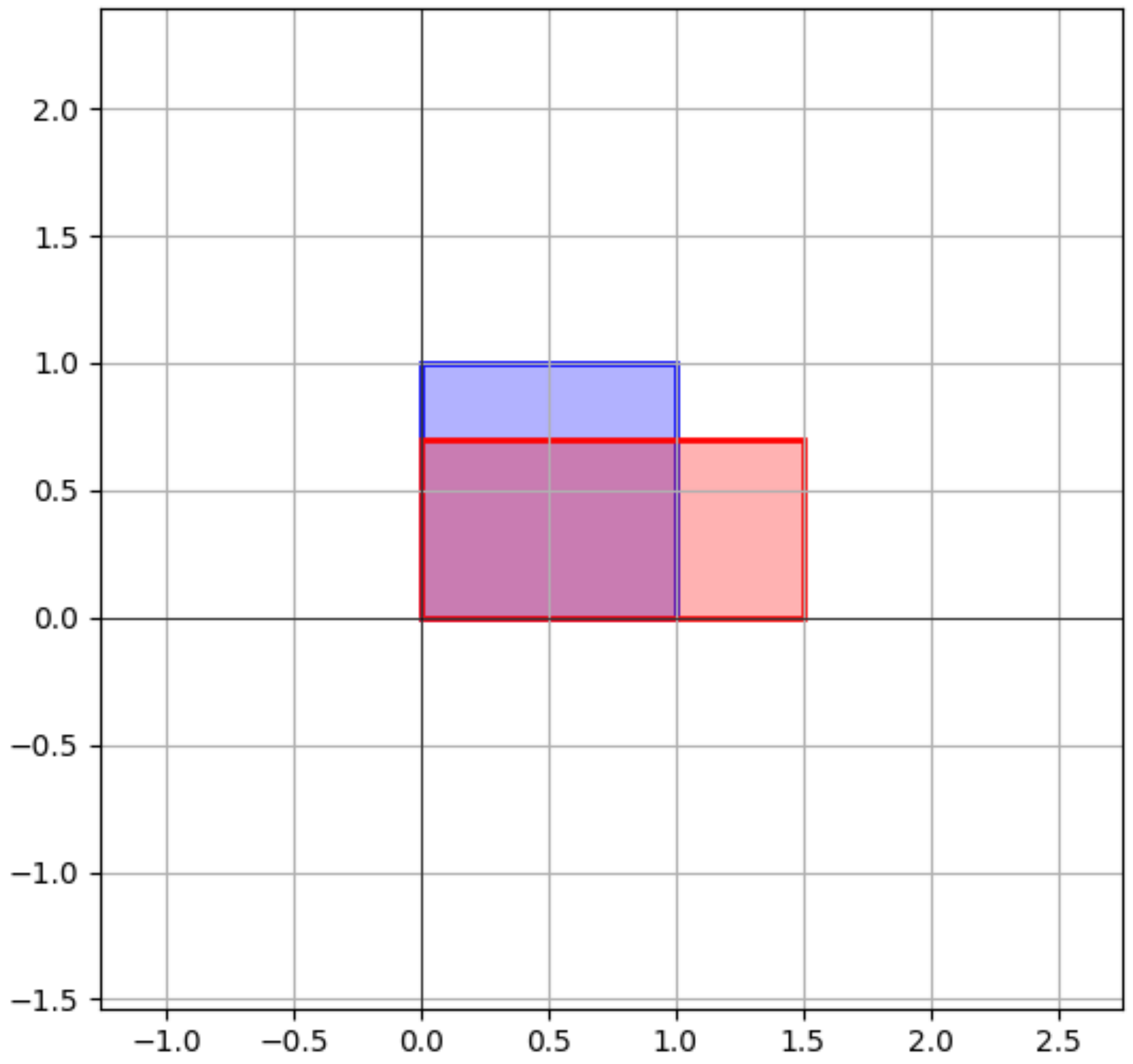
Theorem. A $n \times n$ matrix is invertible if and only if it does not have 0 as an eigenvalue.

To reiterate. An eigenvalue 0 implies

- » $A\mathbf{x} = \mathbf{0}$
- » the columns of A are linearly dependent
- » $\text{Col}(A) \neq \mathbb{R}^n$
- » ...

Example: Unequal Scaling

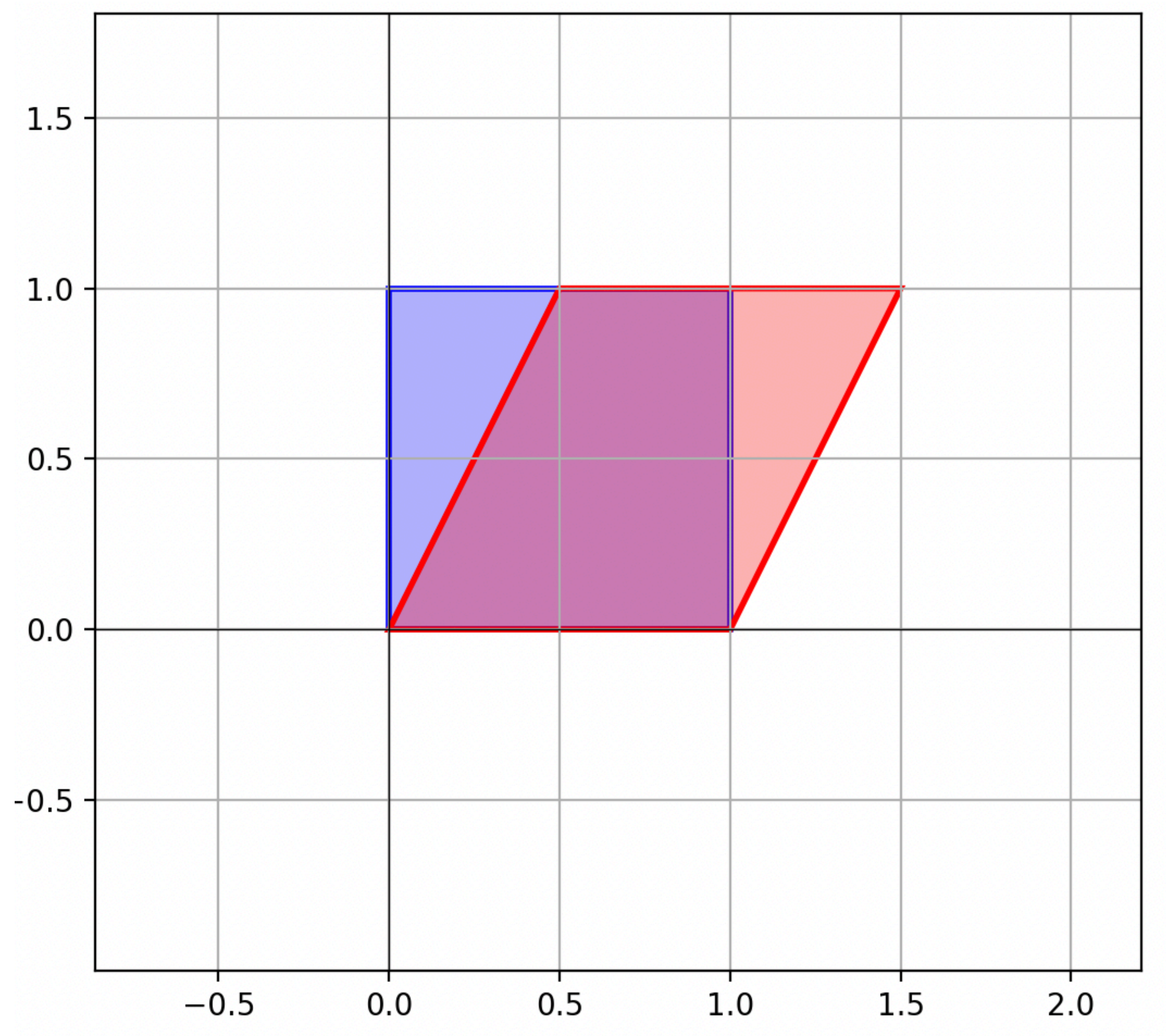
Let's determine it's eigenvalues and eigenvectors:



$$\begin{bmatrix} 1.5 & 0 \\ 0 & 0.7 \end{bmatrix}$$

Example: Shearing

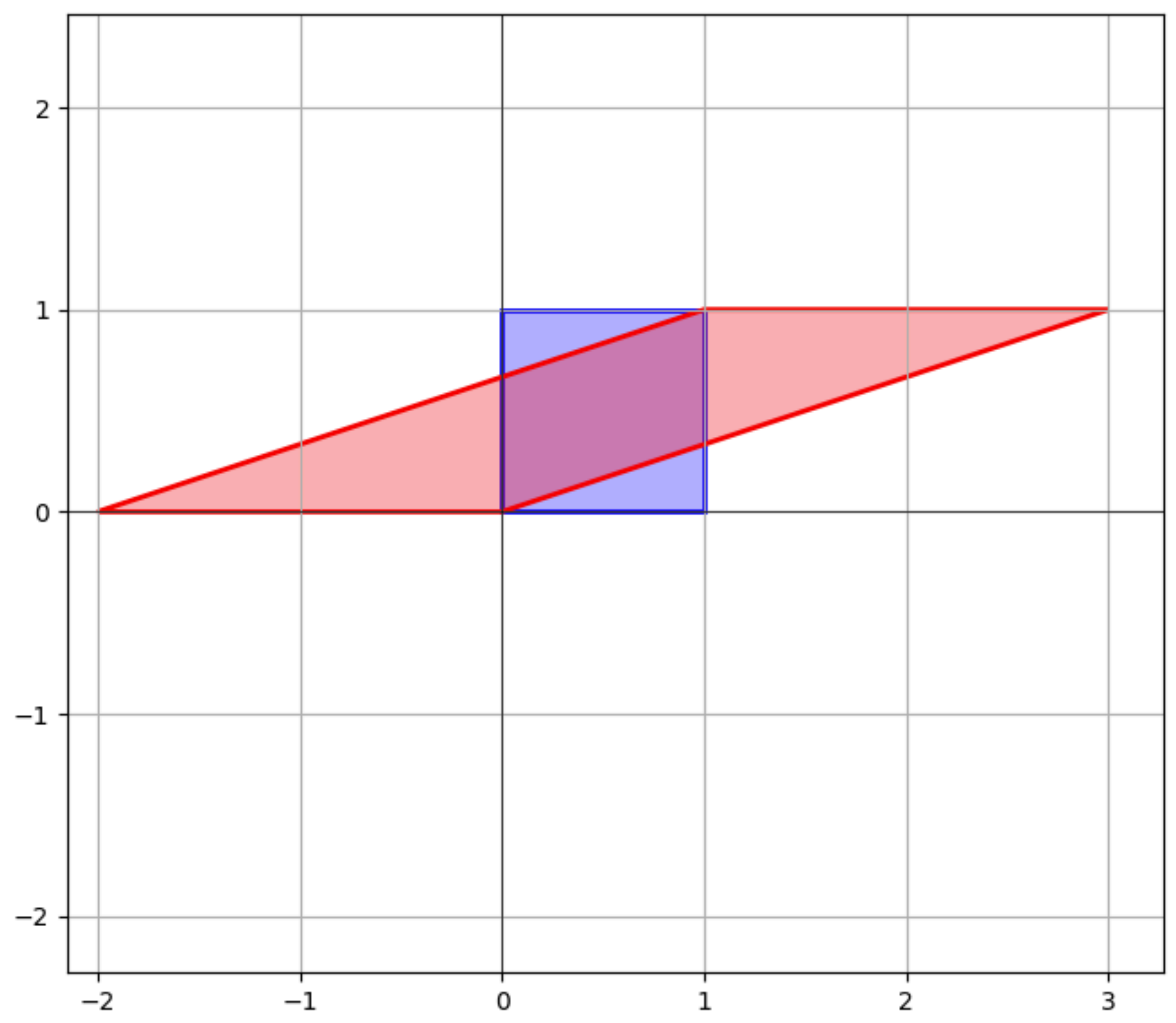
Let's determine it's eigenvalues and eigenvectors:



$$\begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}$$

Example (Algebraic)

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \quad \mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$



How do we verify eigenvalues and eigenvectors?

Verifying Eigenvectors

Verifying Eigenvectors

Question. *Determine if* $\begin{bmatrix} 6 \\ -5 \end{bmatrix}$ *or* $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ *are eigenvectors of* $\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ *and determine the corresponding eigenvalues.*

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Solution. Easy. Work out the matrix-vector multiplication.

Verifying Eigenvectors

$$\begin{bmatrix} 6 \\ -5 \end{bmatrix} \quad \begin{bmatrix} 3 \\ -2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$$

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Before we go over how to do this...

Verifying Eigenvalues (Warm Up)

Question. Verify that 1 is an eigenvalue of

$$\begin{bmatrix} 0.1 & 0.7 \\ 0.9 & 0.3 \end{bmatrix}$$

Hint. Recall our discussion of Markov Chains.

Solution:

Steady-States and Eigenvectors

Steady–state vectors of stochastic matrices are eigenvectors corresponding to the eigenvalue 1.

How did we find steady–state vectors?:

Steady-States and Eigenvectors

$$\mathbf{v} \text{ is a steady-state vector}^* \equiv \mathbf{v} \in \text{Nul}(A - I)$$

$*$It must also be a probability vector

Verifying Eigenvalues

This is harder...

Question. Show that λ is an eigenvalue of A.

Solution:

Verifying Eigenvalues

$\mathbf{v}$ `is an eigenvector for` $\lambda \quad \equiv \quad \mathbf{v} \in \mathsf{Nul}(A - \lambda I)$

Verifying Eigenvalues

This is harder...

Question. Show that 7 is an eigenvalue of $\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$.

Solution:

Problem

Verify that 2 *is an eigenvalue of* $\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$

Answer

How many eigenvectors can a matrix have?

Linear Independence of Eigenvectors

Theorem.* If $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are eigenvectors for distinct eigenvalues, then they are linearly independent.

So an $n \times n$ matrix can have at most n eigenvalues.

Why?:

*We won't prove this.

Eigenspace

Fact. The set of eigenvectors for a eigenvalue λ of $A \in \mathbb{R}^{n \times n}$ form a subspace of $\mathbb{R}^n$.

Verify:

Eigenspace

Definition. The set of eigenvectors for a eigenvalue λ of A is called the **eigenspace** of A corresponding to λ.

It is the same as $\text{Nul}(A - \lambda I)$.

How To: Basis of an Eigenspace

Question. Find a basis for the eigenspace of A corresponding to λ.

Solution. Find a basis for $\text{Nul}(A - \lambda I)$.

We know how to do this.

How do we find eigenvalues?

How do we find eigenvalues?

We'll cover this next time...

Eigenvalues of Triangular Matrices

Theorem. The eigenvalues of a triangular matrix are its entries along the diagonal.

Verify:

Problem

Determine the eigenvalues of the following matrix

$$\begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}$$

Then find eigenvectors for those eigenvalues.

Answer

Linear Dynamical Systems

Recall: Linear Dynamical Systems

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A tells us how our system evolves over time.

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Recall: State Vectors

$$
\begin{aligned}
\mathbf{v}_1 &= A\mathbf{v}_0 \\
\mathbf{v}_2 &= A\mathbf{v}_1 = A(A\mathbf{v}_0) \\
\mathbf{v}_3 &= A\mathbf{v}_2 = A(AA\mathbf{v}_0) \\
\mathbf{v}_4 &= A\mathbf{v}_3 = A(AAA\mathbf{v}_0) \\
\mathbf{v}_5 &= A\mathbf{v}_4 = A(AAAA\mathbf{v}_0) \\
&\vdots
\end{aligned}
$$

The state vector $\mathbf{v}_k$ tells us what the system looks like after a number k time steps.

This is also called a *recurrence relation* or a *linear difference function.*

Recall: State Vectors

$$\mathbf{v}_1 = A\mathbf{v}_0$$

$$\mathbf{v}_2 = A\mathbf{v}_1 = A(A\mathbf{v}_0)$$

$$\mathbf{v}_k = A^k\mathbf{v}_0$$

$$\mathbf{v}_5 = A\mathbf{v}_4 = A(AAAA\mathbf{v}_0)$$

$$\vdots$$

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It's defined in terms of A itself, which doesn't tell us much about how the system behaves.

It's also difficult computationally because matrix multiplication is expensive.

(Closed-Form) Solutions

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A **(closed-form) solution** of a linear dynamical system $\mathbf{v}_{i+1} = A\mathbf{v}_i$ is an expression for $\mathbf{v}_k$ which is is not defined in terms of A or previously defined terms.

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A **(closed–form) solution** of a linear dynamical system $\mathbf{v}_{i+1} = A\mathbf{v}_i$ is an expression for $\mathbf{v}_k$ which is is not defined in terms of A or previously defined terms.

In other word, it does not depend on A and is not **recursive**.

Solutions with Eigenvectors as Initial States

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It's easy to give a solution if the initial state is an eigenvector:

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$$\mathbf{v}_k = A^k \mathbf{v}_0 = \lambda^k \mathbf{v}_0$$

<u>The Key Point.</u> This is still true of sums of eigenvectors.

Solutions in terms of eigenvectors

Let's simplify $A^k\mathbf{v}$, given we have eigenvectors $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4$ for A which span all of $\mathbb{R}^4$:

Eigenvectors and Growth in the Limit

Theorem. For a linear dynamical system A with initial state $\mathbf{v}_0$, if $\mathbf{v}_0$ can be written in terms of eigenvectors $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k$ of A with eigenvalues

$$\lambda_1 \geq \lambda_2 \ldots \geq \lambda_k$$

then $\mathbf{v}_k \sim \lambda_1^k c_1 \mathbf{b}_1$ for some constant c_1 (in other words, in the long term, the system grows <u>exponentially in λ_1</u>).

Verify:

Eigenbases

Definition. An **eigenbasis** of $\mathbb{R}^n$ for a $n \times n$ matrix A is a basis of $\mathbb{R}^n$ made up entirely of eigenvectors of A.

We can represent vectors as ***unique*** *linear combinations of eigenvectors.*

Not all matrices have eigenbases.

Eigenbases and Growth in the Limit

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for some constant c_1, where where λ_1 is the **largest eigenvalue of A and $\mathbf{b}_1$ is its eigenvalue.**

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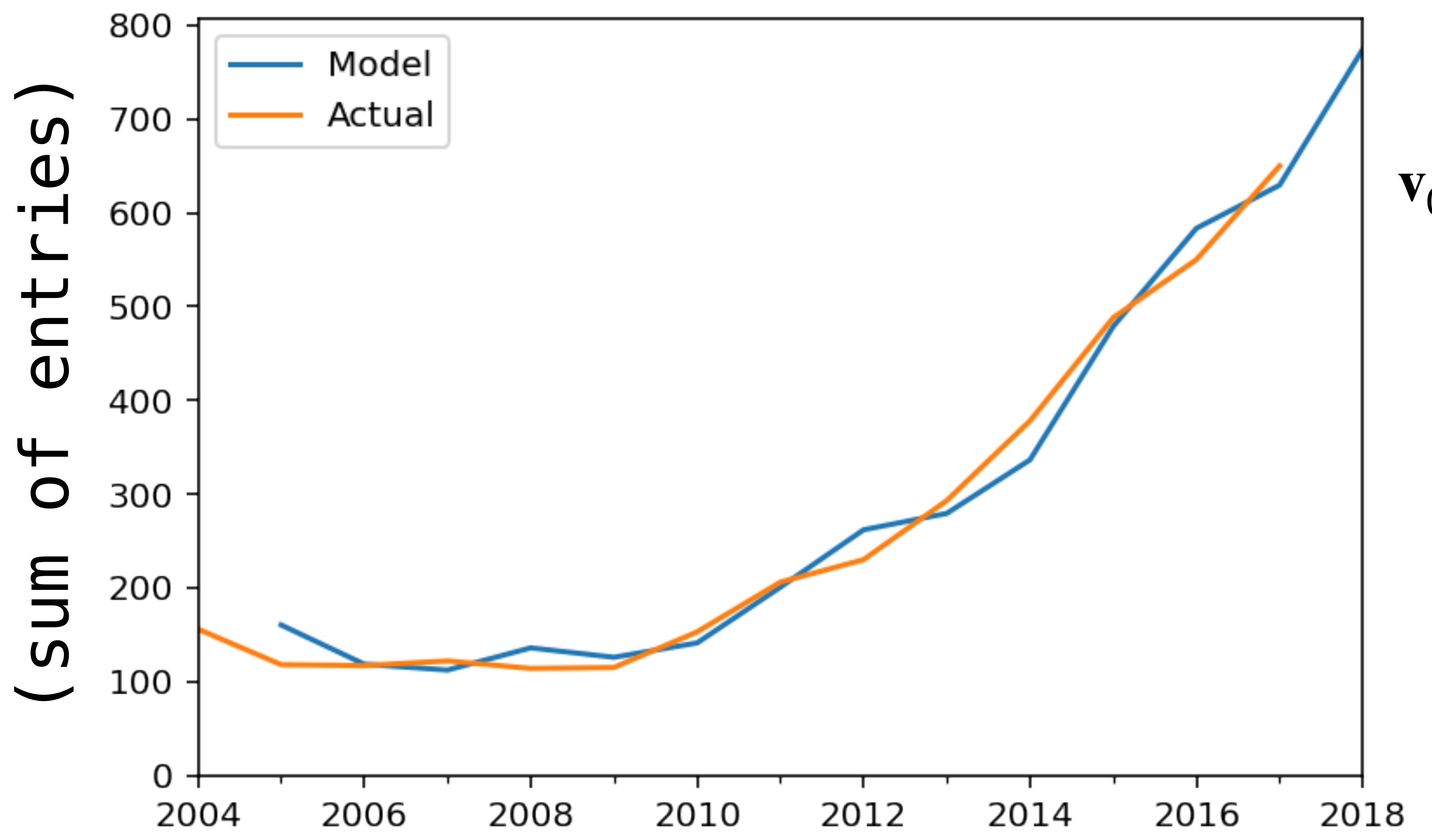
for some constant c_1, where where λ_1 is the **largest eigenvalue of A and $\mathbf{b}_1$ is its eigenvalue.**

The largest eigenvalue describes the long-term exponential behavior of the system.

Example: CS Major Growth

see the notes for more details



$$\mathbf{v}_0 = \begin{bmatrix} v_{0,1} \\ v_{0,2} \\ v_{0,3} \\ v_{0,4} \end{bmatrix} = \begin{bmatrix} \text{\# of year 1 students enrolled in 2024} \\ \text{\# of year 2 students enrolled in 2024} \\ \text{\# of year 3 students enrolled in 2024} \\ \text{\# of year 4 students enrolled in 2024} \end{bmatrix}$$

$$\mathbf{v}_k = A^k \mathbf{v}_0$$

(A is determined by least squares)

This is clearly exponential. If we want to "extract" the exponent, we need to look at the <u>largest eigenvalue</u>.

Extended Example: Golden Ratio

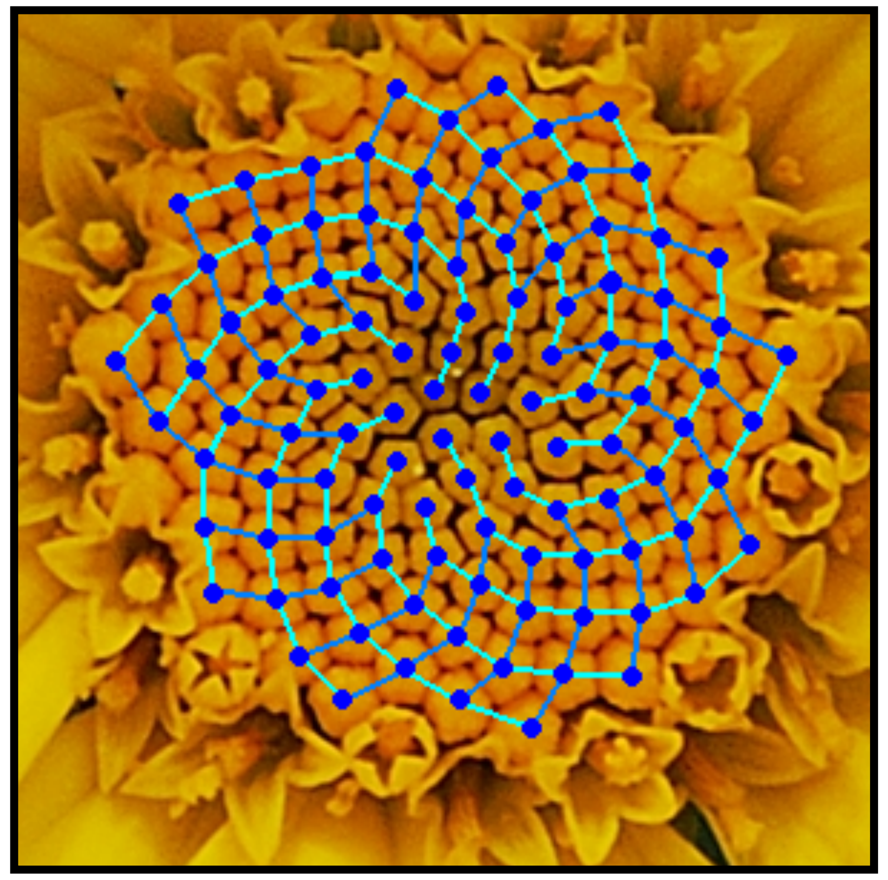
A Special Linear Dynamical System

$$\mathbf{v}_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{v}_k \qquad \mathbf{v}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Consider the system given by the above matrix.

What does this matrix represent?:

Fibonacci Numbers



$$F_0 = 0$$

$$F_1 = 1$$

$$F_k = F_{k-1} + F_{k-2}$$

```
define fib(n):
  curr, next ← 0, 1
  repeat n times:
    curr, next ← next, curr + next
  return curr
```

The Fibonacci numbers are defined in terms of a recurrence relation.

They seem to crop-up in nature.

https://commons.wikimedia.org/wiki/File:FibonacciChamomile.PNG demo

Finding the Eigenvalues (Looking forward a bit)

$$\begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix}$$

Recall: $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if $\det A = 0$

Golden Ratio

$$\varphi = \frac{1+\sqrt{5}}{2} \qquad \frac{F_{k+1}}{F_k} \to \varphi \text{ as } k \to \infty$$

The "long term behavior" is the Fibonacci sequence is defined by the golden ratio.

This is the largest eigenvalue of $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$**.**

Challenge Problem

Find the eigenvalues for these eigenvectors.

Find a closed-form solution for the Fibonacci sequence.

Summary

Eigenvectors of A are "just stretched" by A.

We can easily describe what A does to $\mathbf{v}$ if we can write $\mathbf{v}$ in terms of eigenvectors of A.

Eigenvalues of A give us information about A, like the long term behavior of the dynamical system described by A.