# Diagonalization

## Geometric Algorithms
## Lecture 19

CAS CS 132

# Introduction

# Recap Problem

$$A = \begin{bmatrix} -1 & h & 2 \\ 0 & 2 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

Hint: dim(Nul(B))
= (# non pivots of B)
= 3 − (# pivots of B)
(by rank-nullity)

For what values of $h$ is $\dim(H) = 2$, where $H$ is the eigenspace of $A$ for the eigenvalue $-1$?

Hint: eigenspace of A for −1 is Nul(A − (−1)I)

# **Answer:** $h = 3$

$\dim(\text{Nul}(A - (-1)I)) = 2 = \#$ non-pivot columns.

$$A + I = \begin{bmatrix} -1+1 & h & 2 \\ 0 & 2+1 & 2 \\ 0 & 0 & -1+1 \end{bmatrix} = \begin{bmatrix} 0 & h & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 3 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 2/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{h = 3}$$

# Objectives

1. Finish our discussion on the characteristic polynomial.
2. Motivate diagonalization via linear dynamical systems and changes of coordinate systems.
3. Describe how to diagonalize a matrix.

# Keywords

multiplicity

similar matrices

diagonalizable matrices

change of basis

eigenbasis

# Recap

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**polynomial in $\lambda$**

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**In Reality.** We'll use

*numpy.linalg.eig(A)*

# Example

$$A = \begin{bmatrix} 1 & -1 \\ 7 & -3 \end{bmatrix}$$

ad − bc

polynomial

$$\det \begin{bmatrix} 1-\lambda & -1 \\ 7 & -3-\lambda \end{bmatrix} = (\lambda - 1)(\lambda + 3) + 7 = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$$

factor

`The only eigenvalues of` $A$ `is` $2$.

# Last Remarks on the Characteristic Polynomial

# Example: Triangular matrix

$$\begin{bmatrix} 1 & -3 & 0 & 6 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

The characteristic polynomial of a triangular matrix comes pre-factored:

$$(1-\lambda)(-\lambda)(1-\lambda)(4-\lambda) =$$

$$-\lambda(1-\lambda)^2(4-\lambda) = \boxed{\lambda(\lambda-1)^2(\lambda-4)}$$

$\lambda = 0, 1, 4$ are eigenvalues.

# An Observation: Multiplicity

$$\lambda^{1}(\lambda - 1)^{2}(\lambda - 4)^{1}$$

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This is called the **(algebraic) multiplicity** of the root.

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This is called the **(algebraic) multiplicity** of the root.

**Is the multiplicity meaningful in this context?**

# Multiplicity and Dimension

**Theorem.** The dimension of the eigenspace of $A$ for the eigenvalue $\lambda$ is at most the multiplicity of $\lambda$ in $\det(A - \lambda I)$ (and at least $1$).

**The multiplicity is an upper bound on "how large" the eigenspace is.**

# Example

*Let* $A$ *be a* $5 \times 5$ *matrix with characteristic polynomial* $(x-1)^3(x-3)(x+5)$.

- *What is* $\text{rank}(A)$? 5 dim ( Nul (A) ) = dim ( Nul (A - 0I) ) = 0

- ~~*What is the minimum possible rank of* $A - I$?~~

# Motivating Diagonalization via Linear Dynamical Systems

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We will be almost exclusively interested of eigenbases of $\mathbb{R}^n$ when $A \in \mathbb{R}^{n \times n}$.

**<u>The Question.</u> When can we describe any vector in $\mathbb{R}^n$ as a unique linear combination of eigenvectors of $A$?**

# Recall: Linear Dynamical Systems

initial vector

$$\mathbf{v}_1 = A\mathbf{v}_0$$

$$\mathbf{v}_2 = A\mathbf{v}_1 = A^2\mathbf{v}_0$$

$$\mathbf{v}_3 = A\mathbf{v}_2 = A^3\mathbf{v}_0$$

$$\mathbf{v}_4 = A\mathbf{v}_3 = A^4\mathbf{v}_0$$

$$\vdots$$

vectors after time t

A **linear dynamical system** describes a sequence of **state vectors** starting at $\mathbf{v}_0$.

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$$\vdots$$

**multiplying by $A$ changes the state.**

A **linear dynamical system** describes a sequence of **state vectors** starting at $\mathbf{v}_0$.

demo

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Given $\mathbf{v}_k = A\mathbf{v}_{k-1} = A^k\mathbf{v}_0$, if

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then

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closed-form solution

Verify: $A^k(\alpha_1\vec{b}_1 + \alpha_2\vec{b}_2) = \alpha_1 A^k\vec{b}_1 + \alpha_2 A^k\vec{b}_2$

$= \alpha_1\lambda_1^k\vec{b}_1 + \alpha_2\lambda_2^k\vec{b}_2$

# Application: Eigenbases and Limiting Behavior

**Theorem.** If $A$ has an eigenbasis with eigenvalues

$$\lambda_1 \geq \lambda_2 \ldots \geq \lambda_k$$

then $\mathbf{v}_k \sim \lambda_1^k \mathbf{u}$ for some vector $\mathbf{u}$.

**In the long term, the system grows exponentially in $\lambda_1$.**

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$\lambda_2 < \lambda_1$

**In the long term, the system grows exponentially in $\lambda_1$.**

Verify:

$$V_k = \frac{\alpha_1 \lambda_1^k b_1}{\lambda_1^k} + \frac{\alpha_2 \lambda_2^k b_2}{\lambda_1^k}$$

$\varepsilon^k \quad 0 < \varepsilon < 1$

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Given a basis $\mathscr{B}$ for $\mathbb{R}^n$, we only need to know how $A \in \mathbb{R}^n$ behaves on $\mathscr{B}$.

Sometimes, $A$ behaves simply on $\mathscr{B}$, as in the case of eigenbases.

**What we're really doing is changing our coordinate system to expose a behavior of $A$.**

# Recap: Change of Basis

# Recall: Bases define Coordinate Systems

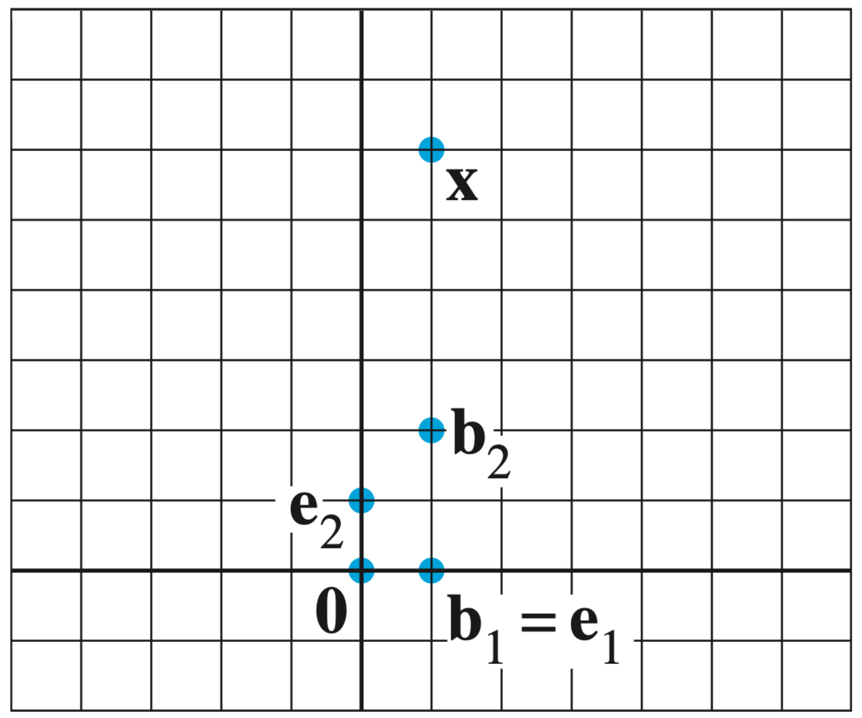



**FIGURE 1** Standard graph paper.

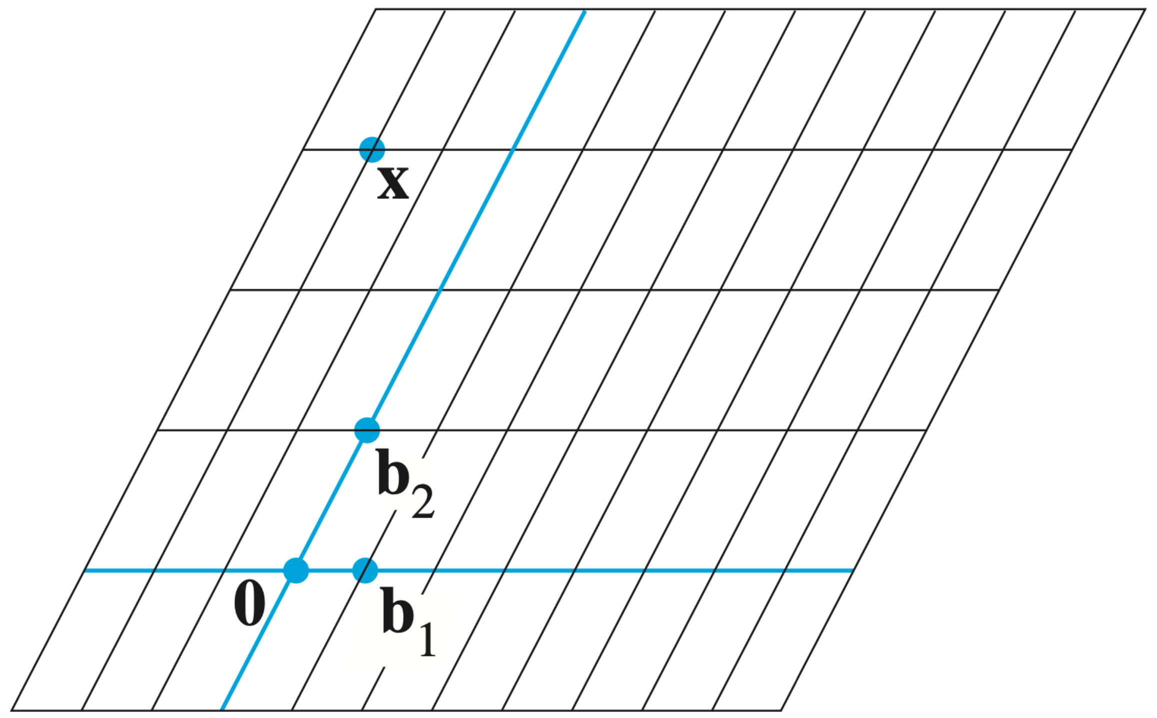
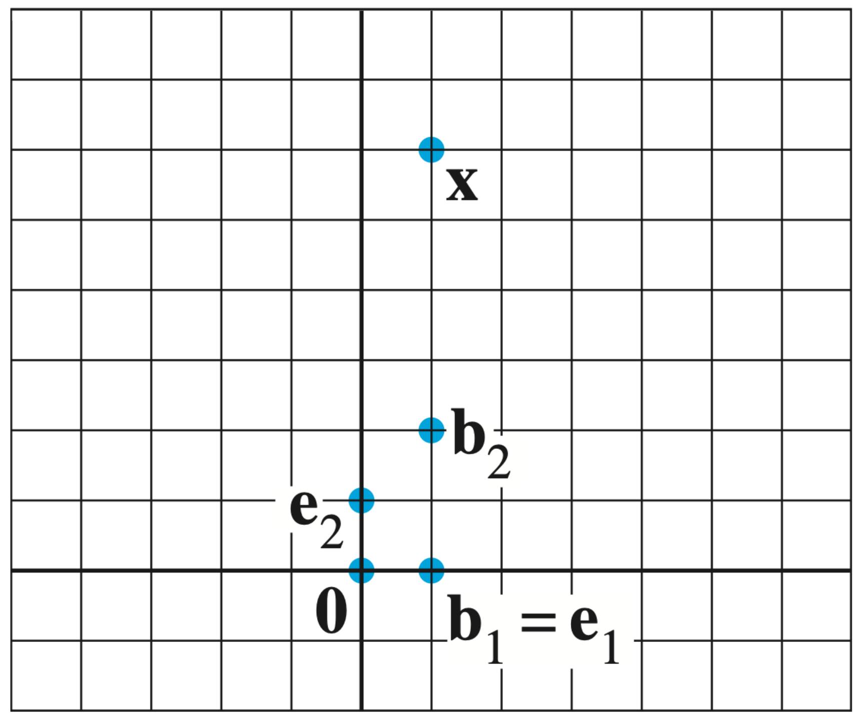



**FIGURE 2** $\mathcal{B}$-graph paper.

Linear Algebra and its Applications (Lay, Lay, McDonald)

# Recall: Bases define Coordinate Systems

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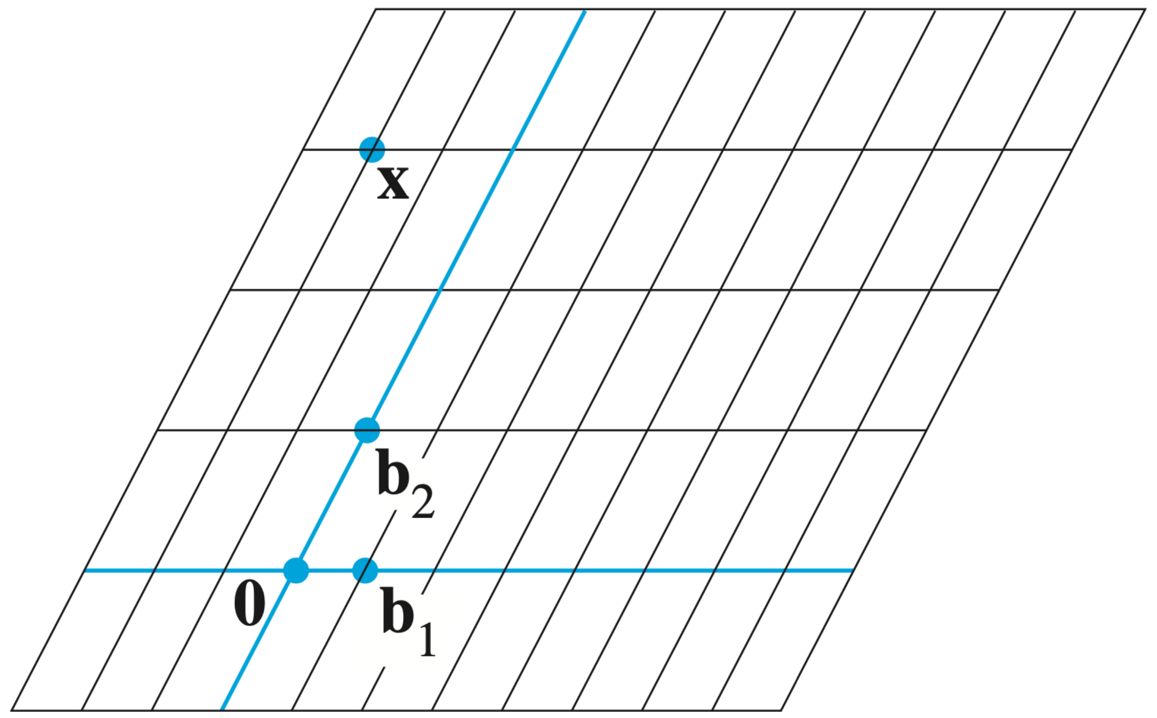


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Given a basis $\mathscr{B}$ of $\mathbb{R}^n$, there is **exactly one way** to write every vector as a linear combination of vectors in $\mathscr{B}$.

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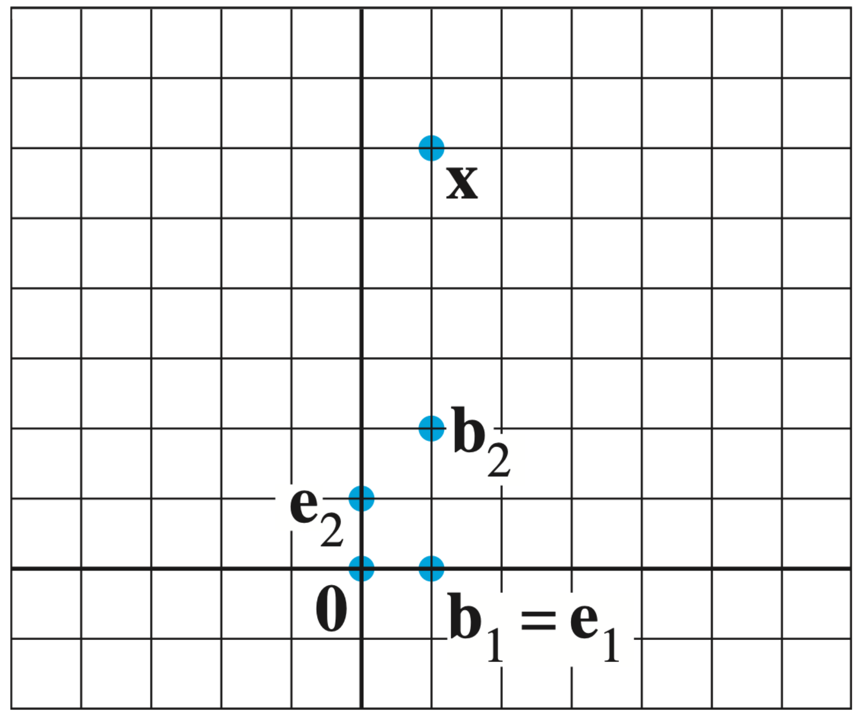

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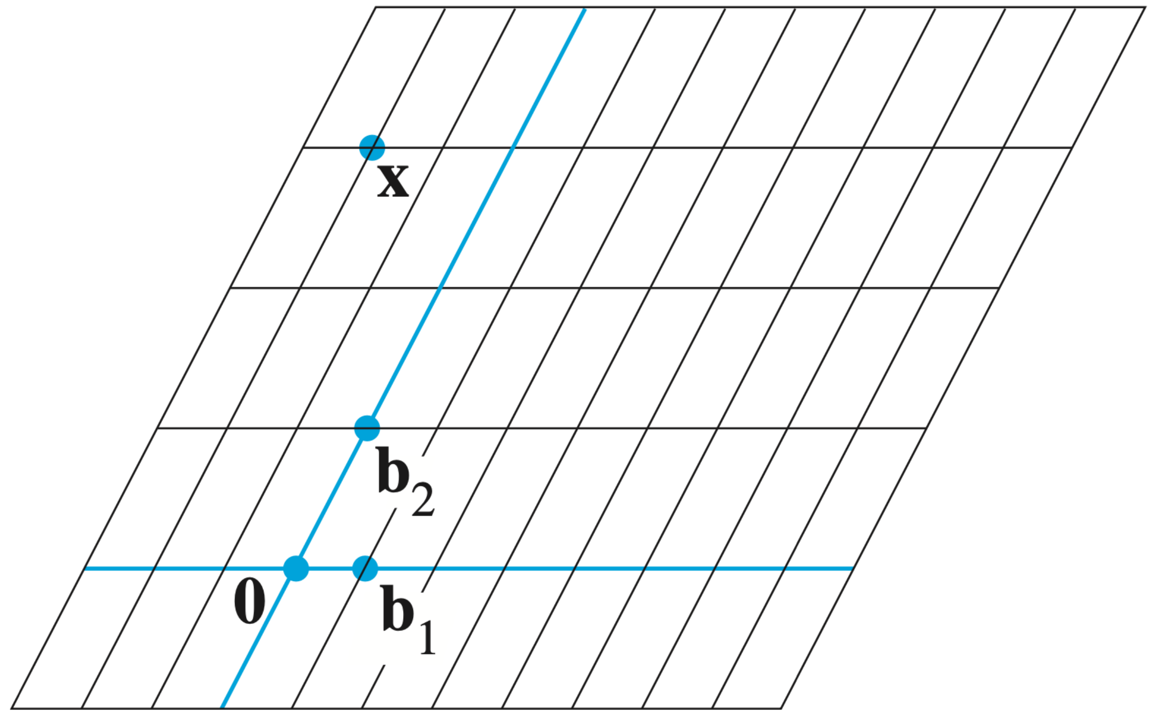

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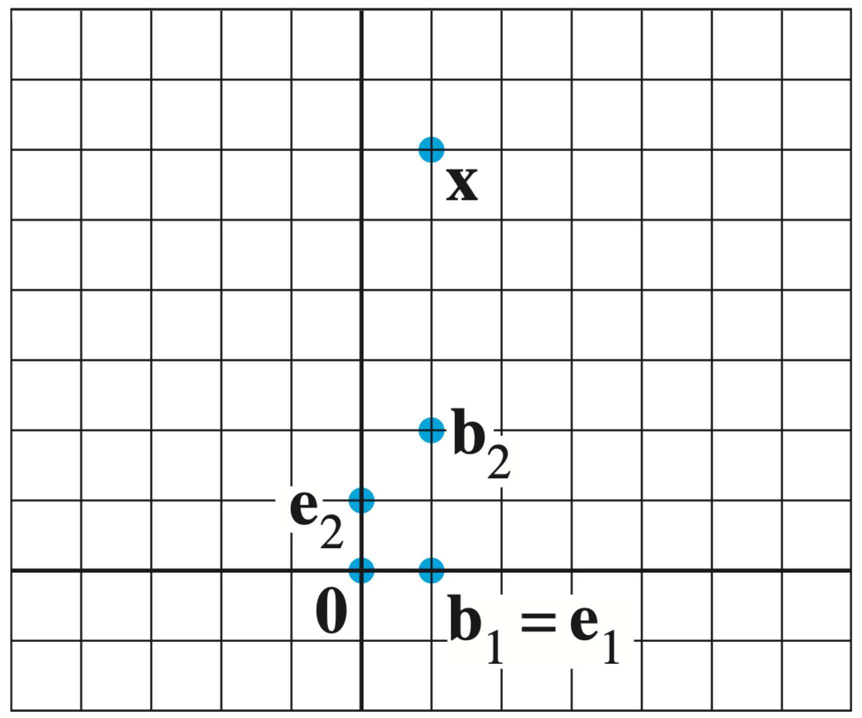


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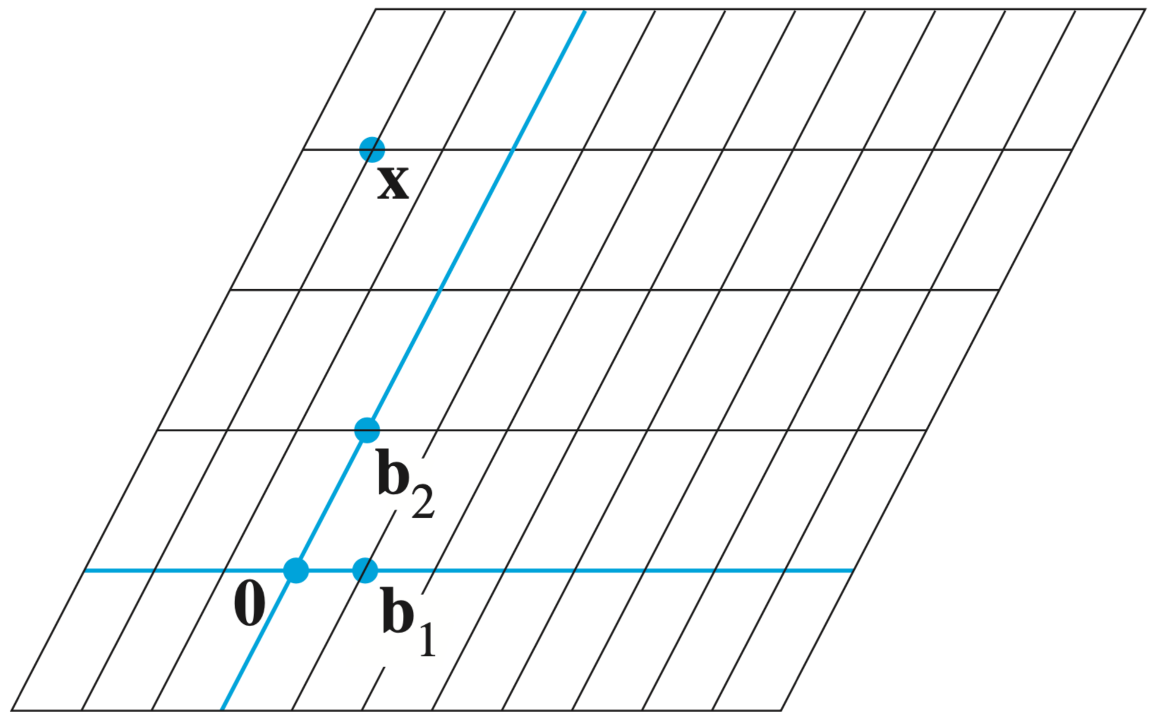


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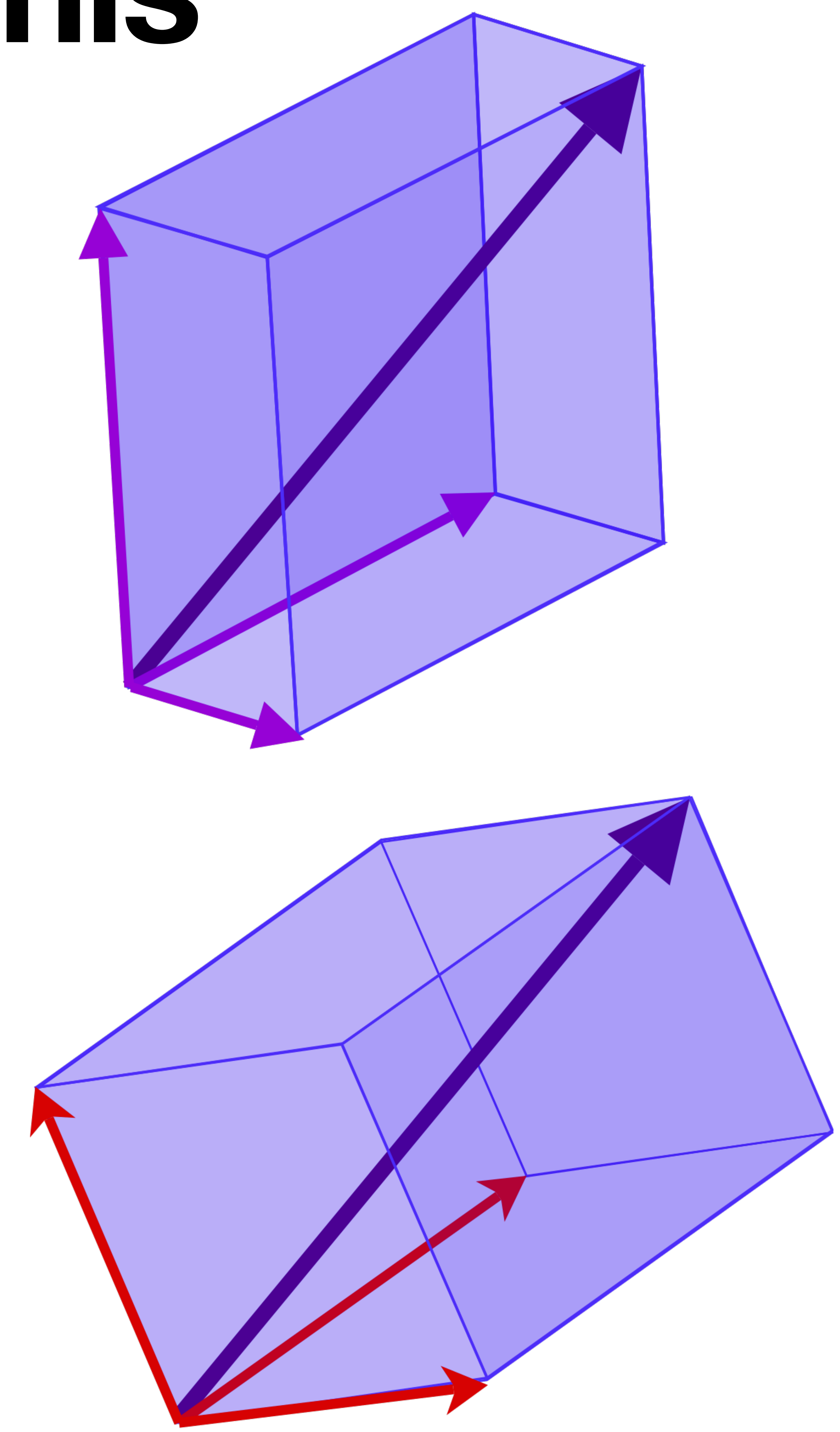
**$\mathscr{B}$ defines a "different grid for our graph paper"**

Linear Algebra and its Applications (Lay, Lay, McDonald)

# Recall: How to think about this

Changing the coordinate system "warps space".

**The Question.** how do we represent a vector **v** in the warped space if we wanted it to "be in the same place"?



https://commons.wikimedia.org/wiki/File:3d_two_bases_same_vector.svg

# Recall: Coordinate Vectors

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Let $\mathbf{v}$ be a vector in a $\mathbb{R}^n$ and let $\mathscr{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ be a basis of $\mathbb{R}^n$ where

unique lin. comb.

$$\mathbf{v} = a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \ldots + a_n\mathbf{b}_n$$

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$$[\mathbf{v}]_{\mathscr{B}} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

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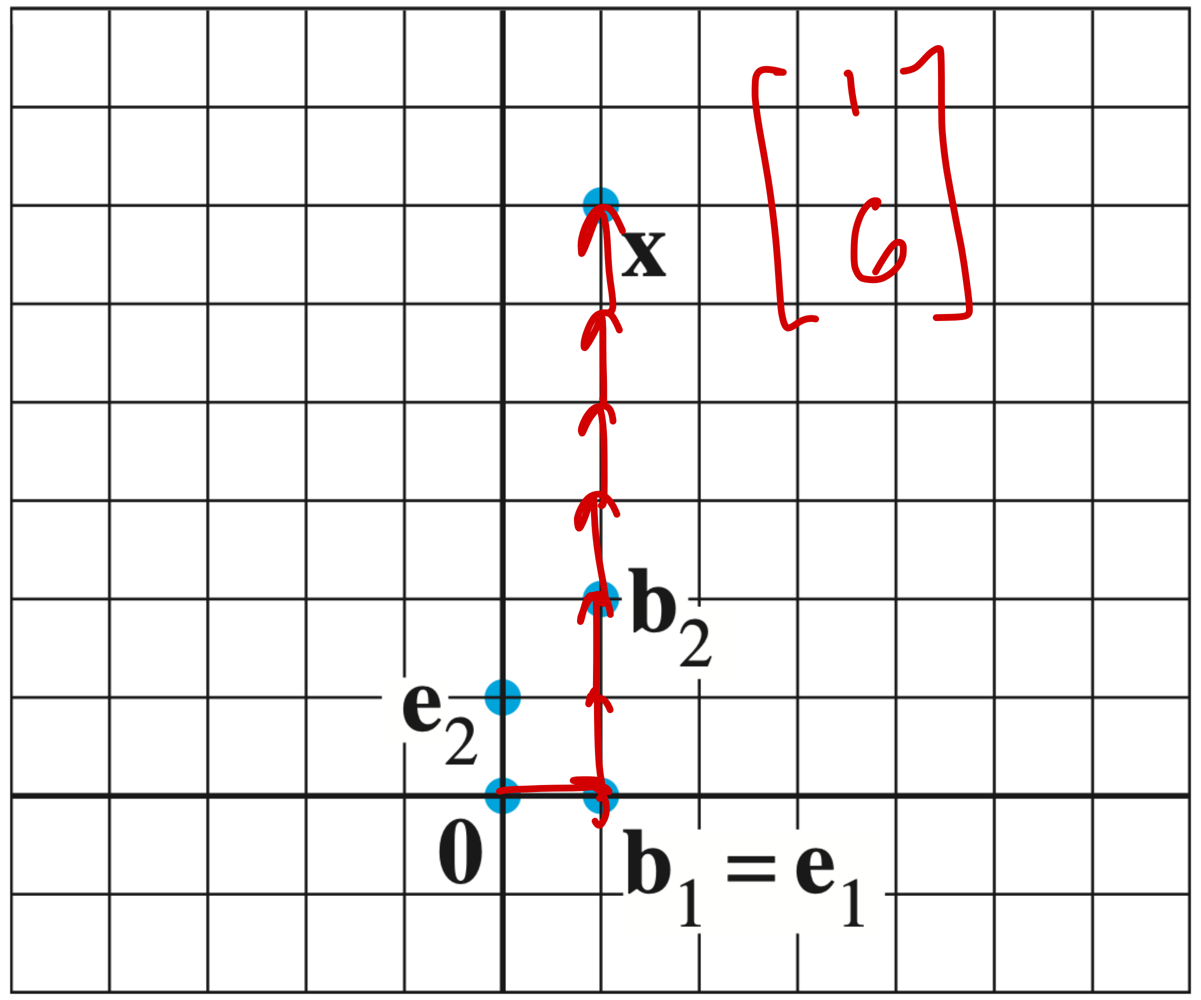


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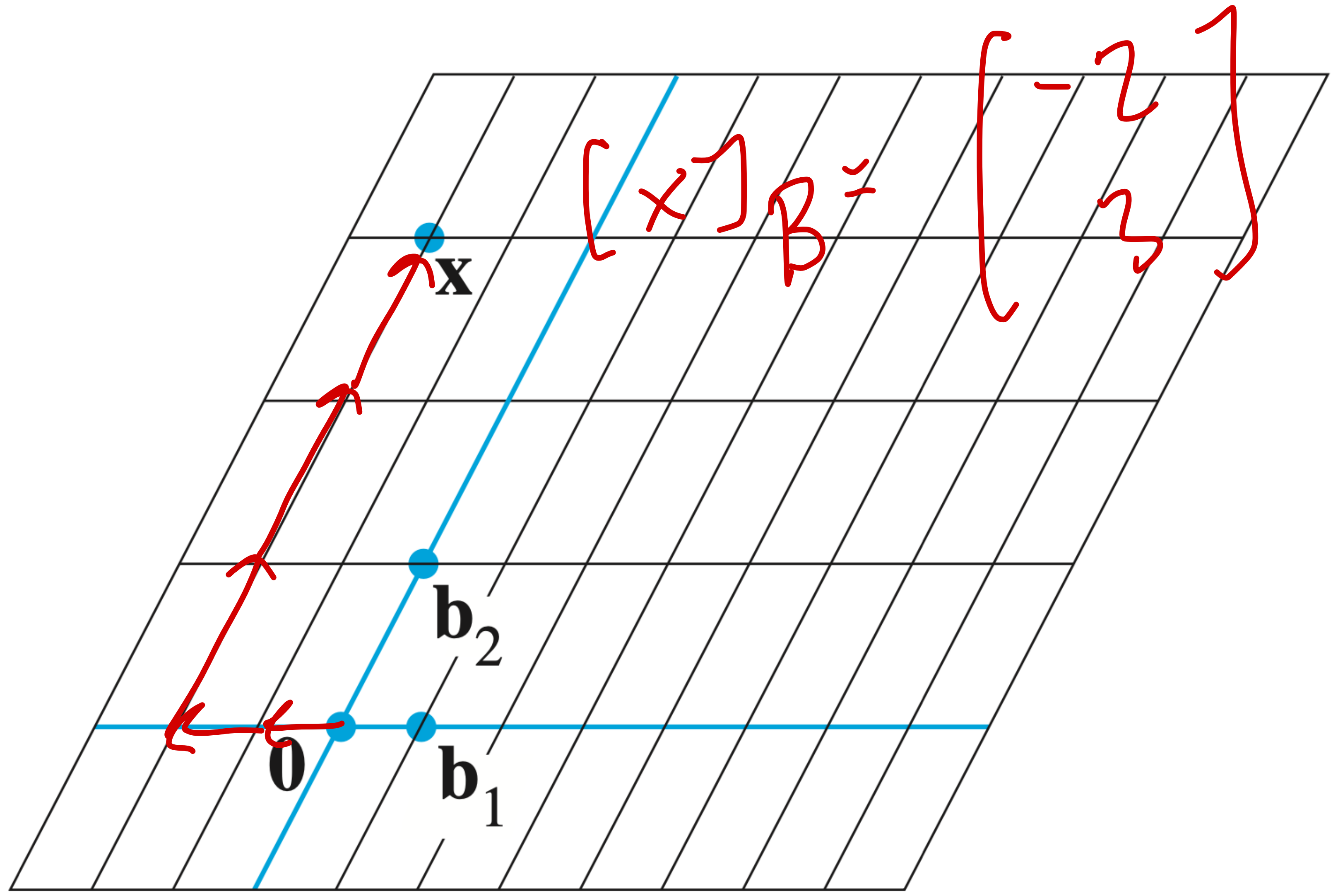


**FIGURE 2** $\mathcal{B}$-graph paper.

# Question (Conceptual)

Hint: $[\vec{b}_1\ \vec{b}_2 \dots \vec{b}_n]\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = c_1\vec{b}_1 + \dots c_n\vec{b}_n$

We know that if a $n \times n$ matrix $B = [\mathbf{b}_1\ \mathbf{b}_2\ \dots\ \mathbf{b}_n]$ is invertible, then the columns of $B$ form a basis $\mathscr{B}$ of $\mathbb{R}^n$.

*What is the matrix that implements the transformation*

$$\mathbf{x} \mapsto [\mathbf{x}]_{\mathscr{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

$B^{-1}$

*where* $\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \dots + c_n\mathbf{b}_n$*?*

# Change of Basis Matrix

**Theorem.** If $\mathscr{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ form a basis of $\mathbb{R}^n$, then

$$[\mathbf{x}]_{\mathscr{B}} = [\mathbf{b}_1 \ \mathbf{b}_2 \ \ \ldots \ \ \mathbf{b}_n]^{-1}\mathbf{x}$$

**Matrix inverses perform changes of bases.**

# How To: Change of Basis

**Question.** Given a basis $\mathscr{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ of $\mathbb{R}^n$, find the matrix which implements $\mathbf{x} \mapsto [\mathbf{x}]_{\mathscr{B}}$.

**Solution.** Construct the matrix $[\mathbf{b}_1 \ \mathbf{b}_2 \ \ldots \ \mathbf{b}_n]^{-1}$.

# Diagonalization

# Diagonal Matrices

ex. $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -0.4 & 0 & 0 \\ 0 & 0 & 22 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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**Definition.** A $n \times n$ matrix $A$ is **diagonal** if

$i \neq j$ if and only if $A_{ij} = 0$.

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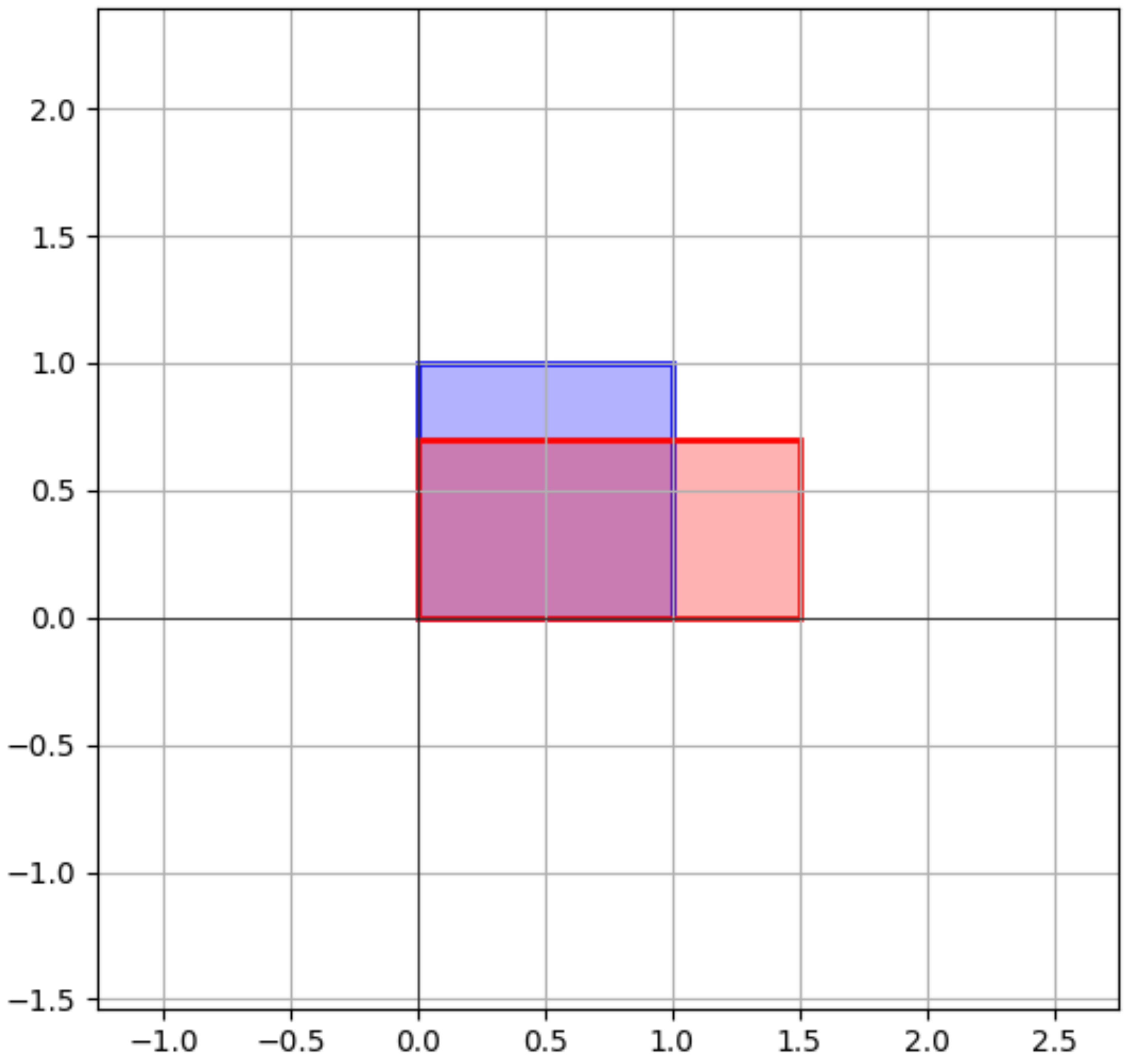
**Diagonal matrices are scaling matrices.**

# Recall: Unequal Scaling

`The scaling matrix` *`affects each component of a vector in a simple way.`*

**`The diagonal entries`** **<u>`scale`</u>** **`each corresponding entry.`**

$$\begin{bmatrix} 1.5 & 0 \\ 0 & 0.7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1.5x \\ 0.7y \end{bmatrix}$$




$$\begin{bmatrix} 1.5 & 0 \\ 0 & 0.7 \end{bmatrix}$$

**High level question:**
When do matrices "behave" like scaling matrices "up to" change of basis?

# Scaling and Eigenvectors

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$$\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} (x\mathbf{e}_1 + y\mathbf{e}_2) = x2\mathbf{e}_1 + y(-3)\mathbf{e}_2$$

$$A \begin{bmatrix} x \\ y \end{bmatrix}_{\mathscr{B}} = A(x\mathbf{b}_1 + y\mathbf{b}_2) = x\lambda_1\mathbf{b}_1 + y\lambda_2\mathbf{b}_2$$

# Scaling and Eigenvectors

**The idea.** Matrices behave like scaling matrices on eigenvectors.

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very similar

**The fundamental question:**

Can we expose this behavior in terms of a *matrix factorization*?

# Recall: Matrix Factorization

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A **factorization** of a matrix $A$ is an equation which expresses $A$ as a product of one or more matrices, e.g.,

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Factorizations can:

- make working with $A$ easier
- expose important information about $A$

# Similar Matrices

change of basis

$$A = PBP^{-1}$$

**Definition.** A matrix $A$ is **similar** to a matrix $B$ if there is some invertible matrix $P$ such that $A = PBP^{-1}$.

**$A$ and $B$ are the same up to a change of basis.**

# Similar Matrices and Eigenvalues

**Theorem.** Similar matrices have the <u>same eigenvalues</u>.

Verify: $\det(A - \lambda I) = \det(PBP^{-1} - \lambda I)$

$= \det(PBP^{-1} - P(\lambda I)P^{-1})$

$= \det(P(B - \lambda I)P^{-1})$

$= \det(P)\det(B - \lambda I)\det(P^{-1}) = \det(B - \lambda I)$

$\det(P^{-1}) = 1/\det(P)$

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# Diagonalizable Matrices

**Definition.** A matrix $A$ is **diagonalizable** if it is similar to a diagonal matrix.

*There is an invertible matrix $P$ and diagonal matrix $D$ such that $A = PDP^{-1}$.*

**Diagonalizable matrices are the same as scaling matrices up to a change of basis.**

# Important: Not all Matrices are Diagonalizable

**This is very different from the LU factorization.**

We will need to figure out which matrices are diagonalizable.

*<u>Question</u>. Is the zero matrix diagonalizable?*

Yes

$$\begin{bmatrix} 0 & & & 0 \\ & 0 & & \\ & & 0 & \\ 0 & & & \ddots & \\ & & & & 0 \end{bmatrix}$$ is diagonal

# Application: Matrix Powers

only take the power of $B$

**Theorem.** If $A = PBP^{-1}$, then $A^k = PB^kP^{-1}$.

It may be easier to take the power of $B$ (as in the case of diagonal matrices).

Verify: $AAA = (PBP^{-1})(PBP^{-1})(PBP^{-1})$

$PB^3P^{-1}$

# How To: Matrix Powers

**Question.** Given $A$ is diagonalizable, determine $A^k$.

**Solution.** Find it's diagonalization $PDP^{-1}$ and then compute $PD^kP^{-1}$.

*Remember that*

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}^k = \begin{bmatrix} a^k & 0 & 0 \\ 0 & b^k & 0 \\ 0 & 0 & c^k \end{bmatrix}$$

But how do we find the diagonalization...

# Diagonalization and Eigenvectors

Suppose we have a diagonalization

$$A = PDP^{-1}$$

What do we know about it?

# Columns of $P$ are eigenvectors

$$A = [\mathbf{p}_1 \ \mathbf{p}_2 \ \mathbf{p}_3] \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} [\mathbf{p}_1 \ \mathbf{p}_2 \ \mathbf{p}_3]^{-1}$$

Verify:

$$A\vec{p}_2 = PD\left([\vec{p}_1 \ \vec{p}_2 \ \vec{p}_3]^{-1}\vec{p}_2\right) \quad \vec{e}_2$$

$$= PD\vec{e}_2$$

$$= P\lambda_2\vec{e}_2 = \lambda_2 P\vec{e}_2 = \lambda_2\vec{p}_2$$

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In fact, the columns of $P$ form an **eigenbasis** of $\mathbb{R}^n$ for $A$.

And the entries of $D$ are the **eigenvalues** associated to each eigenvector.

**A diagonalization exposes a lot of information about $A$.**

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We can use the same recipe to go in the other direction, given an eigenbasis, we can **build a diagonalization.**

# Diagonalizing a Matrix

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The diagonal of $D$ are the eigenvalues for each column of $P$.

**The matrix $P^{-1}$ is a change of basis to this eigenbasis of $A$.**

# Step 1: Eigenvalues

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & 3 \\ 3 & 3 & 1 \end{bmatrix}$$

Find all the eigenvalues of $A$.

*Find the roots of* $\det(A - \lambda I)$.

*e.g.*

$$\det(A - \lambda I) = -(\lambda - 1)(\lambda + 2)^2$$

# Step 2: Eigenvectors

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & 3 \\ 3 & 3 & 1 \end{bmatrix}$$

$$\lambda_1 = 1$$

$$\lambda_2 = -2$$

Find **bases** of the corresponding eigenspaces.

*e.g.*

$$\text{Nul}(A - I) = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$$\text{Nul}(A + 2I) = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

# Step 3: Construct P

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & 3 \\ 3 & 3 & 1 \end{bmatrix}$$

$$\lambda_1 = 1$$

$$\lambda_2 = -2$$

$$\text{Nul}(A - I) = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

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we could fail

If there are $n$ eigenvectors from the previous step they form an **eigenbasis.**

Build the matrix with these vectors as the columns

*e.g.*

$$P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

# Step 5: Construct D

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & 3 \\ 3 & 3 & 1 \end{bmatrix}$$

$$\lambda_1 = 1$$

$$\lambda_2 = -2$$

$$P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Build the matrix with eigenvalues as diagonal entries.

**Note the order.** It should be the same as the order of columns of $P$.

e.g.

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

# Step 6: Invert P

Find the inverse of $P$ (we know how to do this).

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & 3 \\ 3 & 3 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

# Putting it Together

$$\underset{A}{\begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & 3 \\ 3 & 3 & 1 \end{bmatrix}} = \underset{P}{\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}} \underset{D}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}} \underset{P^{-1}}{\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1}}$$

eigenvalues

eigen basis

"change of basis"

# How to: Diagonalizing a Matrix

**Question.** Find a diagonalization of $A \in \mathbb{R}^n$, or determine that $A$ is not diagonalizable.

**Solution.**

1. Find the eigenvalues of $A$, and bases for their eigenspaces. If these eigenvectors don't form a basis of $\mathbb{R}^n$, then $A$ is **not diagonalizable.**

2. Otherwise, build a matrix $P$ whose columns are the eigenvectors of $A$.

3. Then build a diagonal matrix $D$ whose entries are the eigenvalues of $A$ *in the same order.*

4. Invert $P$.

5. The diagonalization of $A$ is $PDP^{-1}$.

We know how to do every step, its a matter of putting it all together.

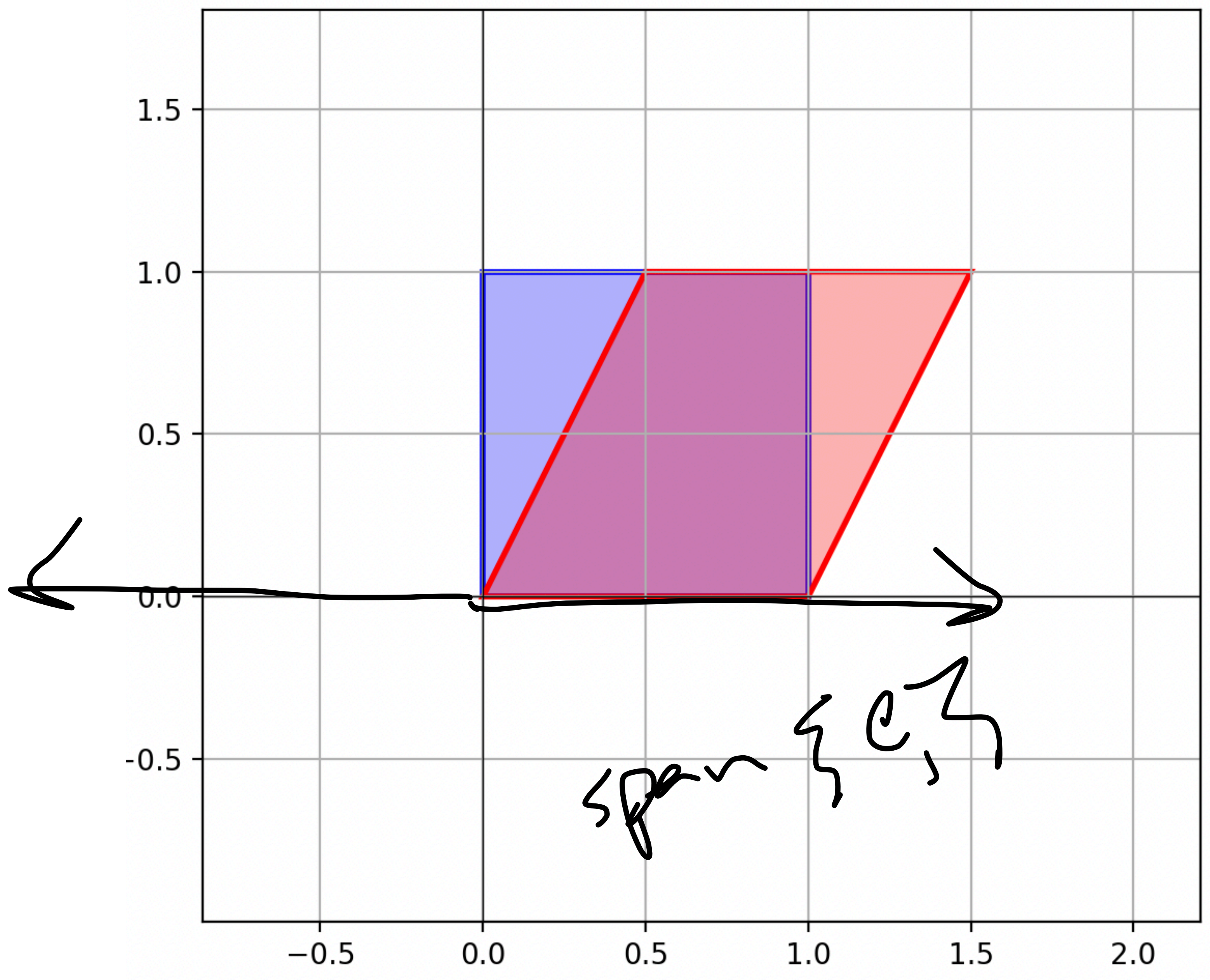
# Example of Failure: Shearing

$$A = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}$$

The shearing matrix has a single eigenvalue with an eigenspace of dimension 1.

**We can't build an eigenbasis of $\mathbb{R}^2$ for $A$.**

In other words, $A$ is not diagonalizable.



# Important case: Distinct Eigenvalues

ex. $\begin{bmatrix} 1 & -3 & 4 & 2 \\ 0 & -2 & 3 & -1 \\ 0 & 0 & 10 & 5 \\ 0 & 0 & 0 & 6 \end{bmatrix}$

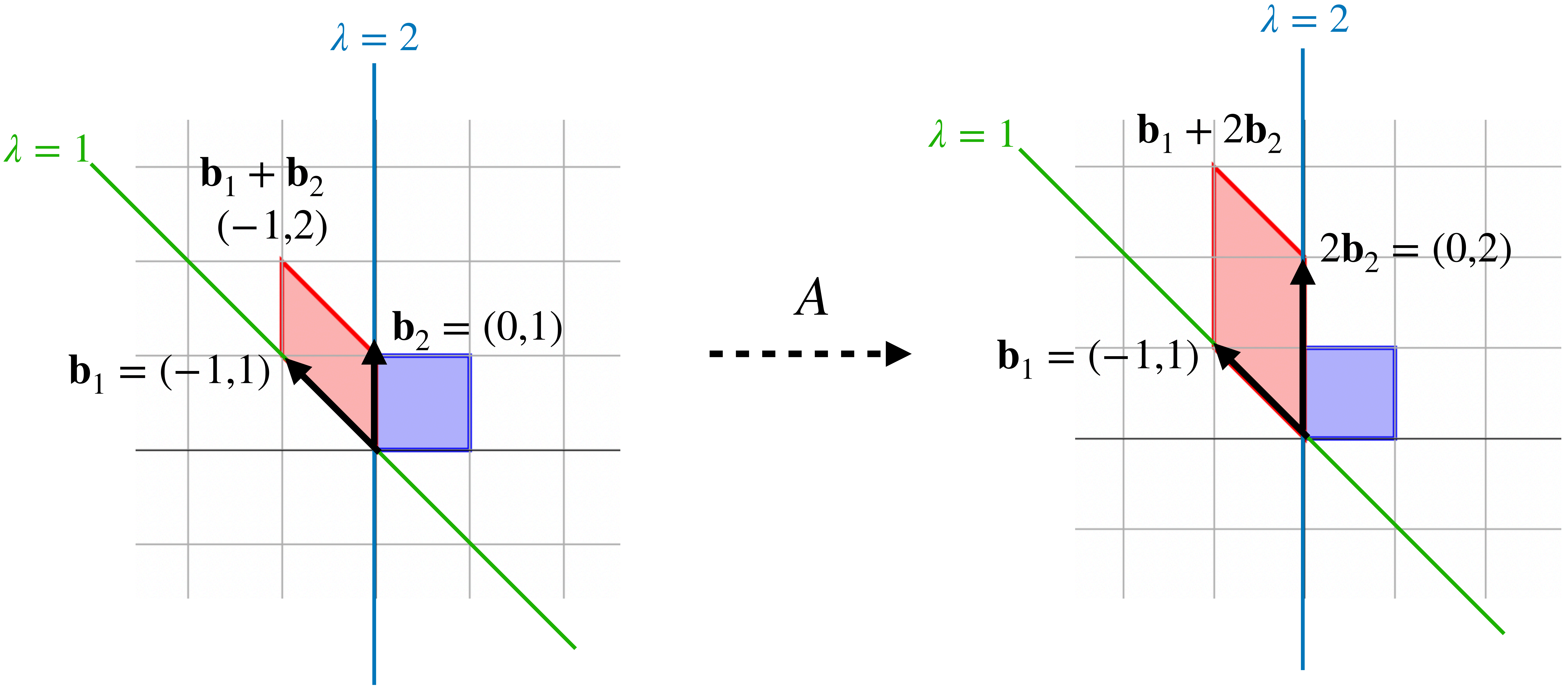
**Theorem.** If an $n \times n$ matrix has has $n$ distinct eigenvalues, then it is diagonalizable.

This is because eigenvectors with distinct eigenvalues are *linearly independent.*

# The Picture

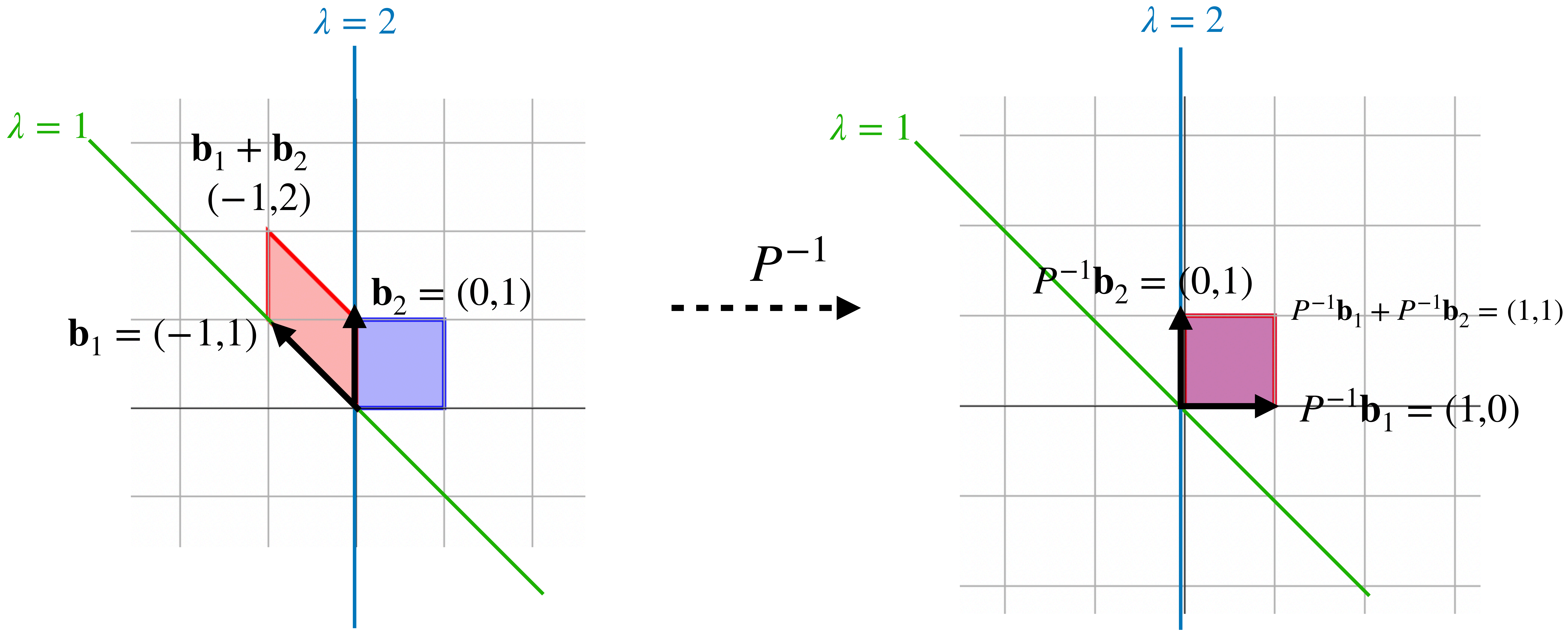
# Example (Geometric)

$$A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$$



# Example (Geometric)

$$P^{-1} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}^{-1}$$

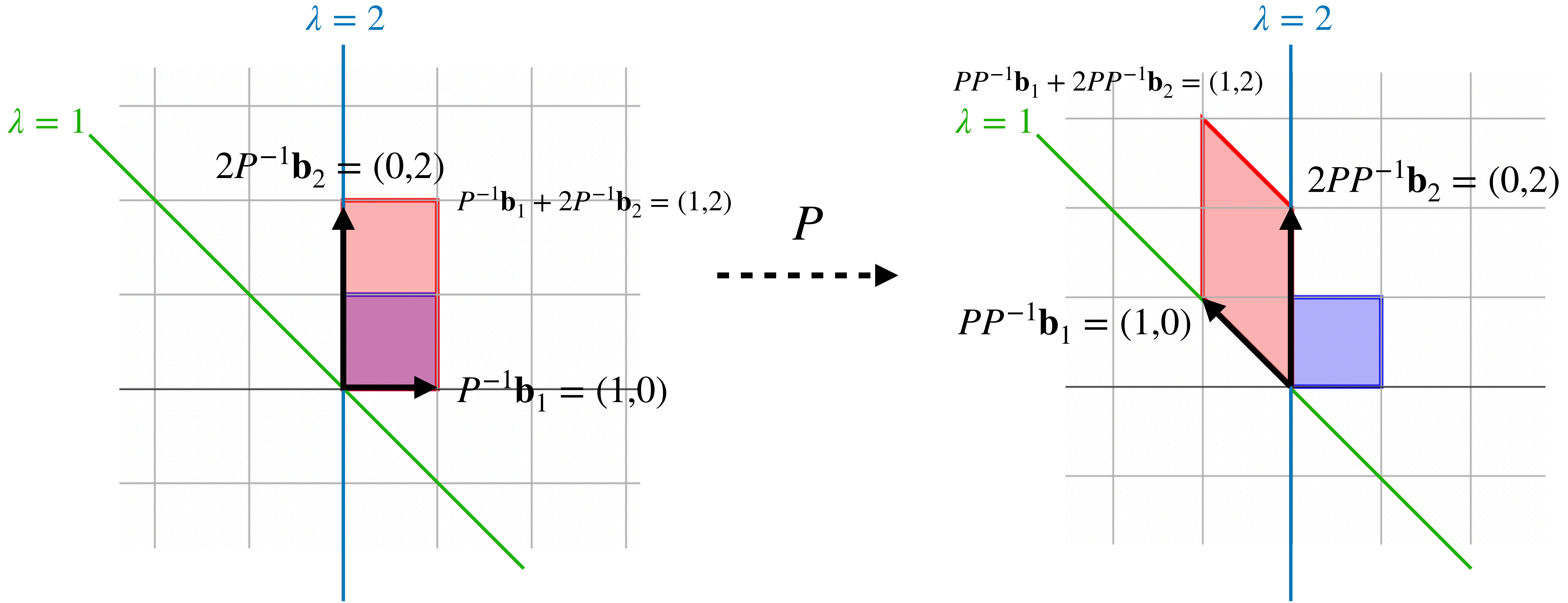


# Example (Geometric)

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

# Example (Geometric)

$$P = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$$



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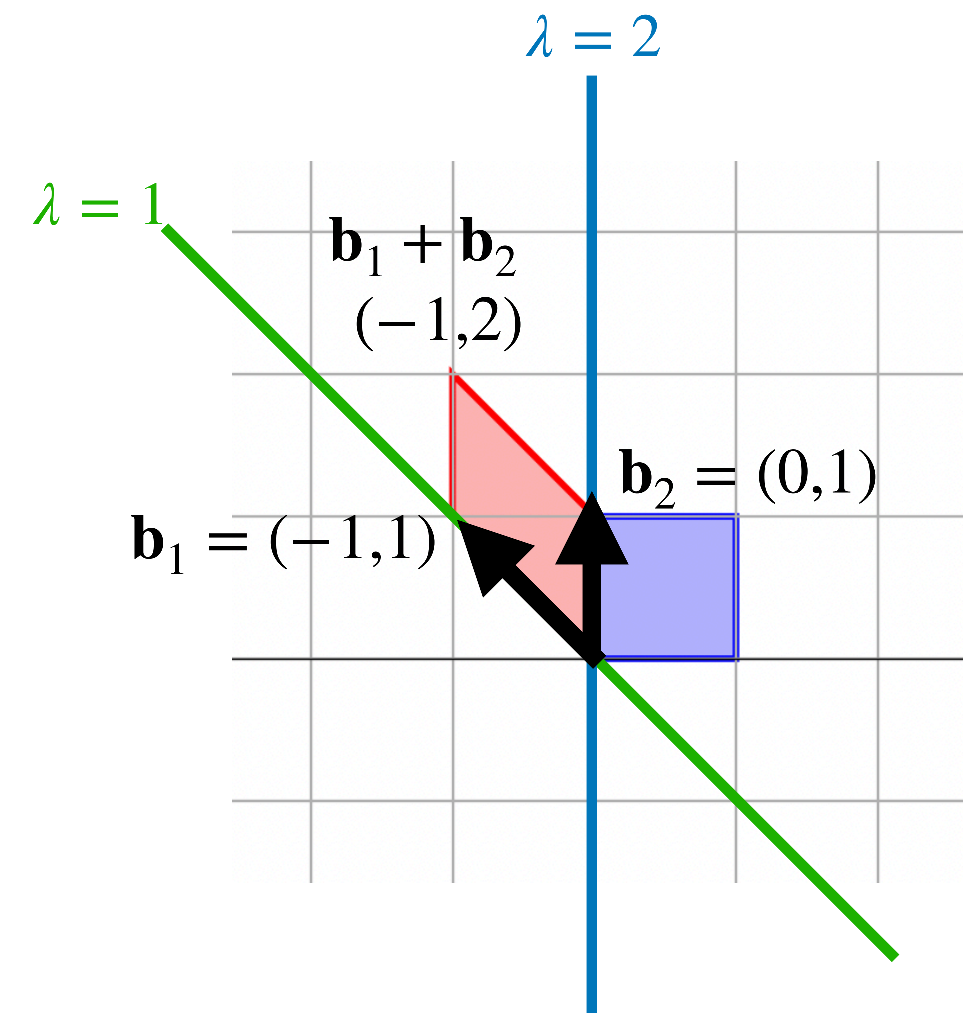


$$A = PDP^{-1}$$

$$\begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}^{-1}$$