Diagonalization

Geometric Algorithms

Lecture 19

CAS CS 132

Introduction

Recap Problem

$$A = \begin{bmatrix} -1 & h & 2 \\ 0 & 2 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

For what values of h is $\dim(H) = 2$, where H is the eigenspace of A for the eigenvalue -1?

Hint: eigenspace of A for -1 $= \text{Nul}(A + I)$

Hint: $\dim(\text{Nul}(B))$ = # of non-pivot columns of B

= 3 - (# of pivots) by rank-nullity

Answer: $h = 3$

$$\dim\left(\text{Nul}\left(\overset{A+I}{A - (-1)I}\right)\right)$$

$$\begin{bmatrix} -1 & h & 2 \\ 0 & 2 & 2 \\ 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & h & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$h = 3$

$$\dim(\text{Nul}(A+I)) =$$

$$\dim\{\vec{v} : (A+I)\vec{v} = 0\}$$

= # free variables in general form solution to $(A+I)\vec{x} = \vec{0}$

Objectives

1. Finish our discussion on the characteristic polynomial.
2. Motivate diagonalization via linear dynamical systems and changes of coordinate systems.
3. Describe how to diagonalize a matrix.

Keywords

multiplicity

similar matrices

diagonalizable matrices

change of basis

eigenbasis

Recap

Recall: Determinants and Invertibility

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$\det(A - \lambda I) = 0$ $\equiv$ $(A - \lambda I)\mathbf{x} = \mathbf{0}$ **has nontrivial solutions**

$\equiv$ λ is an eigenvalue of A

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polynomial in λ

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How To: Finding Eigenvalues

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Question. Determine the eigenvalues of A.

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Solution. Find the *roots* of the characteristic polynomial of A, which is

$$\det(A - \lambda I)$$

viewed as a *polynomial* in λ.

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In Reality. We'll use

numpy.linalg.eig(A)

Example

$$A = \begin{bmatrix}1 & -1\\ 7 & -3\end{bmatrix}$$

$$\det\begin{bmatrix}1-\lambda & -1\\ 7 & -3-\lambda\end{bmatrix} = \underbrace{\boxed{(\lambda-1)(\lambda+3)+7}}_{\text{polynomial}} = \lambda^2-4\lambda+4 = \underbrace{\boxed{(\lambda-2)^2}}_{\text{Factor}}$$

The only eigenvalues of A is 2.

Last Remarks on the Characteristic Polynomial

Example: Triangular matrix

$$\begin{bmatrix} 1 & -3 & 0 & 6 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

The characteristic polynomial of a triangular matrix comes <u>pre-factored</u>:

$$(1-\lambda)(-\lambda)(1-\lambda)(4-\lambda)$$

$$-\lambda(\lambda-1)^2(4-\lambda)$$

$$= \lambda(\lambda-1)^2(\lambda-4)$$

An Observation: Multiplicity

$$\lambda^{\boxed{1}}(\lambda-1)^{\boxed{2}}(\lambda-4)^{\boxed{1}}$$

multiplicities

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In the examples so far, we've seen a number appear as a root <u>multiple times</u>.

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This is called the **(algebraic) multiplicity** of the root.

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In the examples so far, we've seen a number appear as a root multiple times.

This is called the **(algebraic) multiplicity** of the root.

Is the multiplicity meaningful in this context?

Multiplicity and Dimension

$p(x)(1-\lambda)^2$

Theorem. The dimension of the eigenspace of A for the eigenvalue λ is at most the multiplicity of λ in $\det(A - \lambda I)$ (and at least 1).

The multiplicity is an upper bound on "how large" the eigenspace is.

Example

Let A *be a* 5×5 *matrix with characteristic polynomial* $(x-1)^3(x-3)(x+5)$.

- *What is* $\text{rank}(A)$*?* 5   0 is <u>not</u> an eignvalue

  $A\vec{x} = \vec{0}$ has no nontrivial solutions

- *What is the minimum possible rank of* $A - I$*?*

$$\dim(\text{Nul}(A - I)) \leq 3$$

$$\text{rank}(A) \geq 5 - 3 = 2$$

Motivating Diagonalization via Linear Dynamical Systems

Recall: Eigenbasis

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We will be almost exclusively interested of eigenbases of $\mathbb{R}^n$ when $A \in \mathbb{R}^{n \times n}$.

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Definition. An eigenbasis of H for the matrix A is a basis of H made up of eigenvectors of A.

We will be almost exclusively interested of eigenbases of $\mathbb{R}^n$ when $A \in \mathbb{R}^{n \times n}$.

The Question. **When can we describe any vector in $\mathbb{R}^n$ as a unique linear combination of eigenvectors of A?**

Recall: Linear Dynamical Systems

$$\mathbf{v}_1 = A\mathbf{v}_0$$

$$\mathbf{v}_2 = A\mathbf{v}_1 = A^2\mathbf{v}_0$$

$$\mathbf{v}_3 = A\mathbf{v}_2 = A^3\mathbf{v}_0$$

$$\mathbf{v}_4 = A\mathbf{v}_3 = A^4\mathbf{v}_0$$

$$\vdots$$

A **linear dynamical system** describes a sequence of **state vectors** starting at $\mathbf{v}_0$.

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multiplying by A changes the state.

A **linear dynamical system** describes a sequence of **state vectors** starting at $\mathbf{v}_0$.

demo

Eigenbases and Closed-Form solutions

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`Given` $\mathbf{v}_k = A\mathbf{v}_{k-1} = A^k\mathbf{v}_0$`, if`

$$\mathbf{v}_0 = \alpha_1\mathbf{b}_1 + \alpha_2\mathbf{b}_2 + \alpha_3\mathbf{b}_3$$

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then

$$A^k\mathbf{v}_0 = \alpha_1\lambda_1^k\mathbf{b}_1 + \alpha_2\lambda_2^k\mathbf{b}_2 + \alpha_3\lambda_3^k\mathbf{b}_3$$

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$$\boxed{\vec{v}_k = A\,\vec{v}_{k-1}}$$ recursive

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$$A^k\mathbf{v}_0 = \alpha_1\lambda_1^k\mathbf{b}_1 + \alpha_2\lambda_2^k\mathbf{b}_2 + \alpha_3\lambda_3^k\mathbf{b}_3$$

closed-form solution

Verify: $A^k(\alpha_1\vec{b}_1 + \alpha_2\vec{b}_2) = \alpha_1 A^k\vec{b}_1 + \alpha_2 A^k\vec{b}_2$

$$= \alpha_1\lambda_1^k\vec{b}_1 + \alpha_2\lambda_2^k\vec{b}_2$$

Application: Eigenbases and Limiting Behavior

Theorem. If A has an eigenbasis with eigenvalues

$$\lambda_1 \geq \lambda_2 \ldots \geq \lambda_k$$

then $\mathbf{v}_k \sim \lambda_1^k \mathbf{u}$ for some vector $\mathbf{u}$.

In the long term, the system grows <u>exponentially in λ_1</u>.

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Verify:

$$v_k = \frac{\alpha_1 \overset{1}{\cancel{\lambda_1^k}} \vec{b}_1}{\cancel{\lambda_1^k}} + \frac{a_2 \overset{\varepsilon^k}{\cancel{\lambda_2^k}} b_2}{\cancel{\lambda_1^k}}$$

$0 < \varepsilon < 1$

The Takeaway

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Given a basis $\mathscr{B}$ for $\mathbb{R}^n$, we only need to know how $A \in \mathbb{R}^n$ behaves on $\mathscr{B}$.

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The Takeaway

Given a basis $\mathscr{B}$ for $\mathbb{R}^n$, we only need to know how $A \in \mathbb{R}^n$ behaves on $\mathscr{B}$.

Sometimes, A behaves simply on $\mathscr{B}$, as in the case of eigenbases.

What we're really doing is changing our coordinate system to expose a behavior of A.

Recap: Change of Basis

Recall: Bases define Coordinate Systems

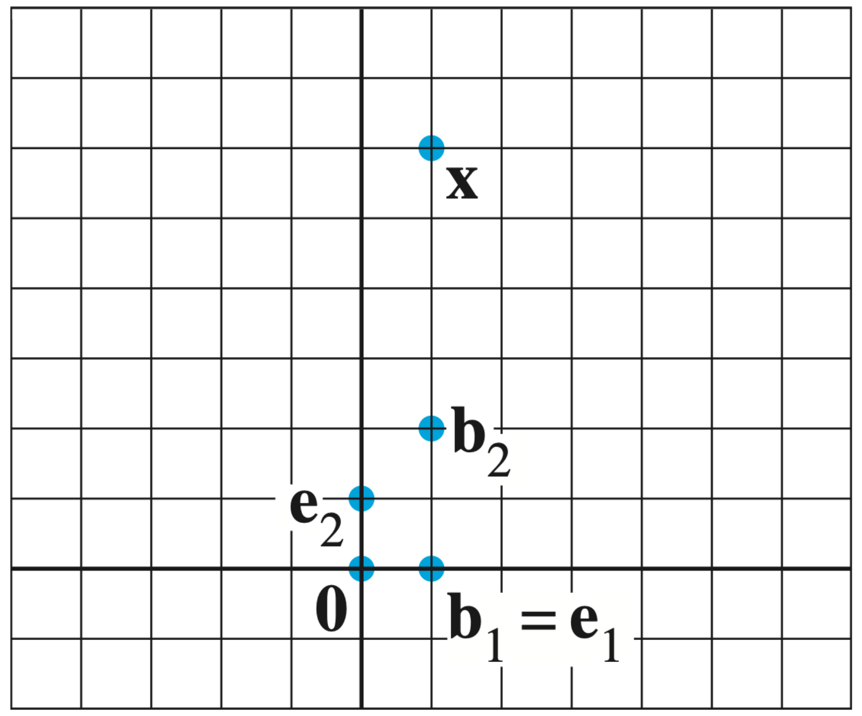

FIGURE 1 Standard graph paper.

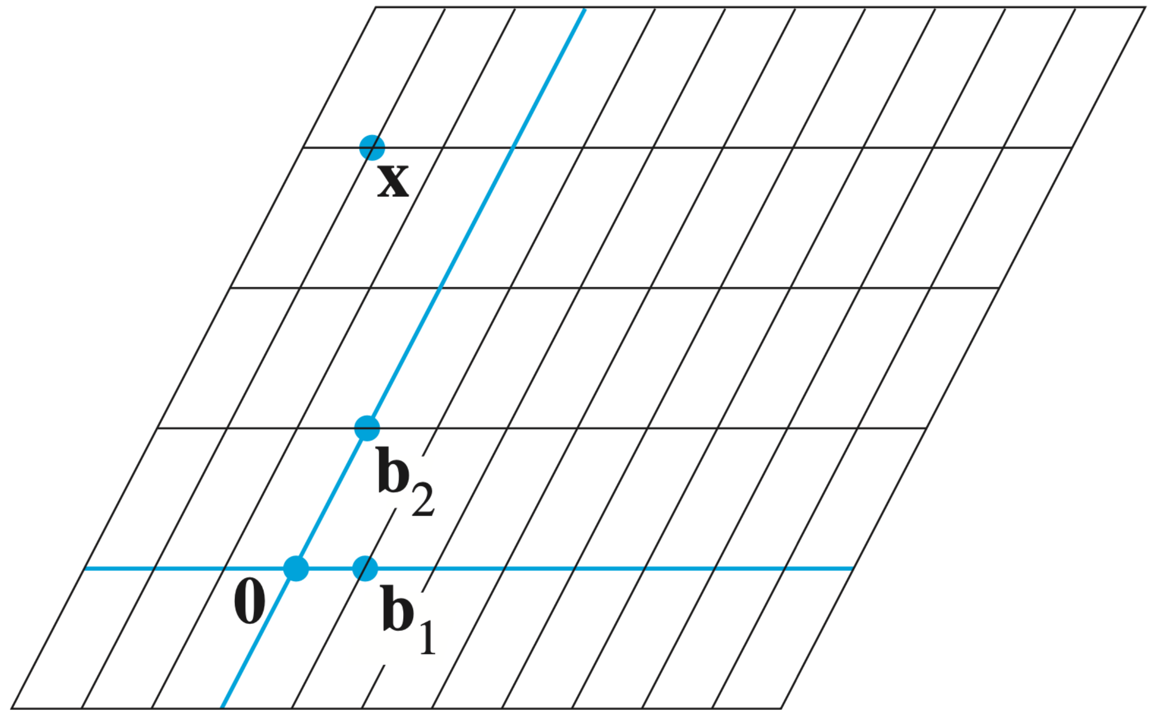

FIGURE 2 $\mathcal{B}$-graph paper.

Linear Algebra and its Applications (Lay, Lay, McDonald)

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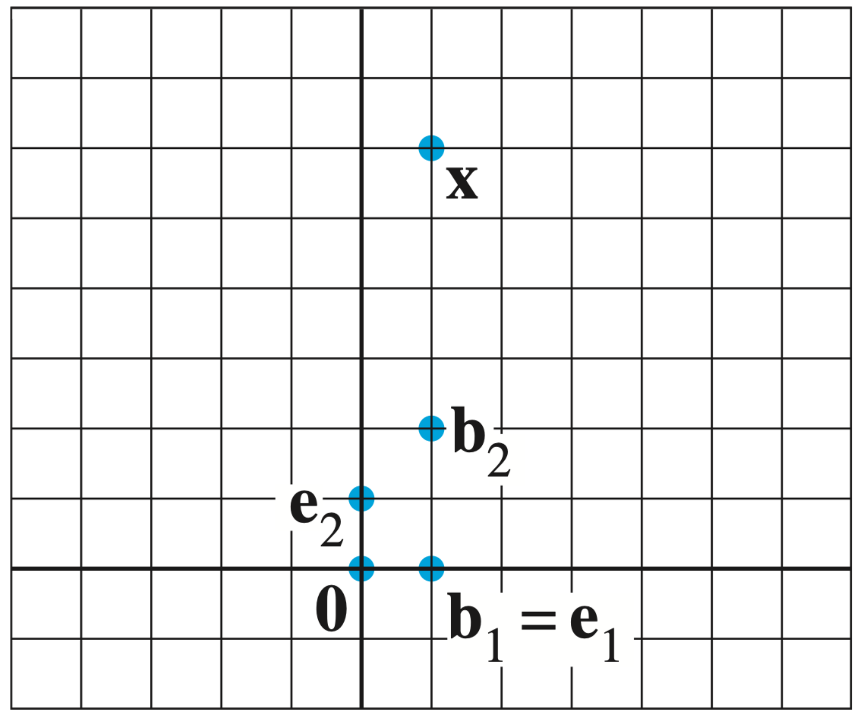

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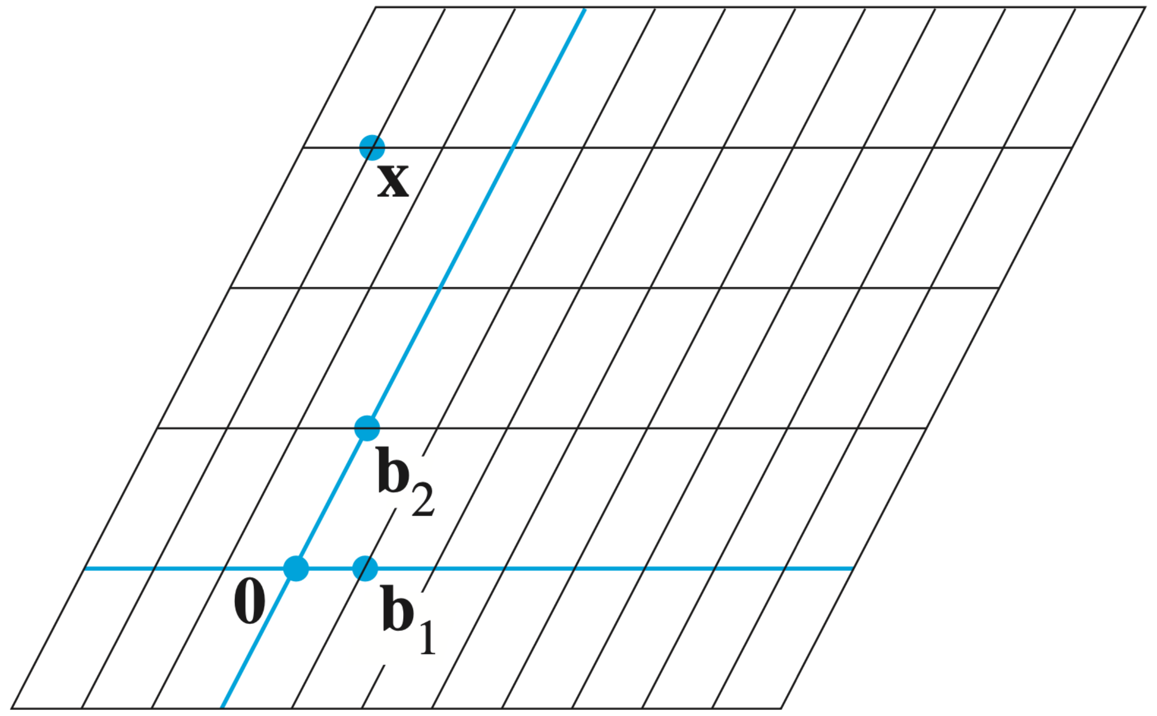

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Given a basis $\mathscr{B}$ of $\mathbb{R}^n$, there is **exactly one way** to write every vector as a linear combination of vectors in $\mathscr{B}$.

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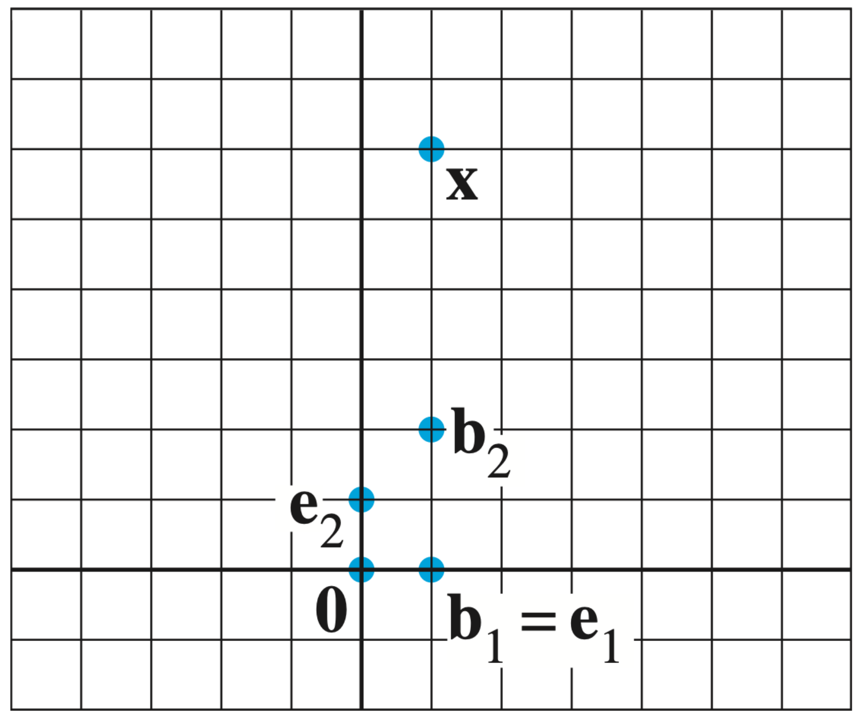

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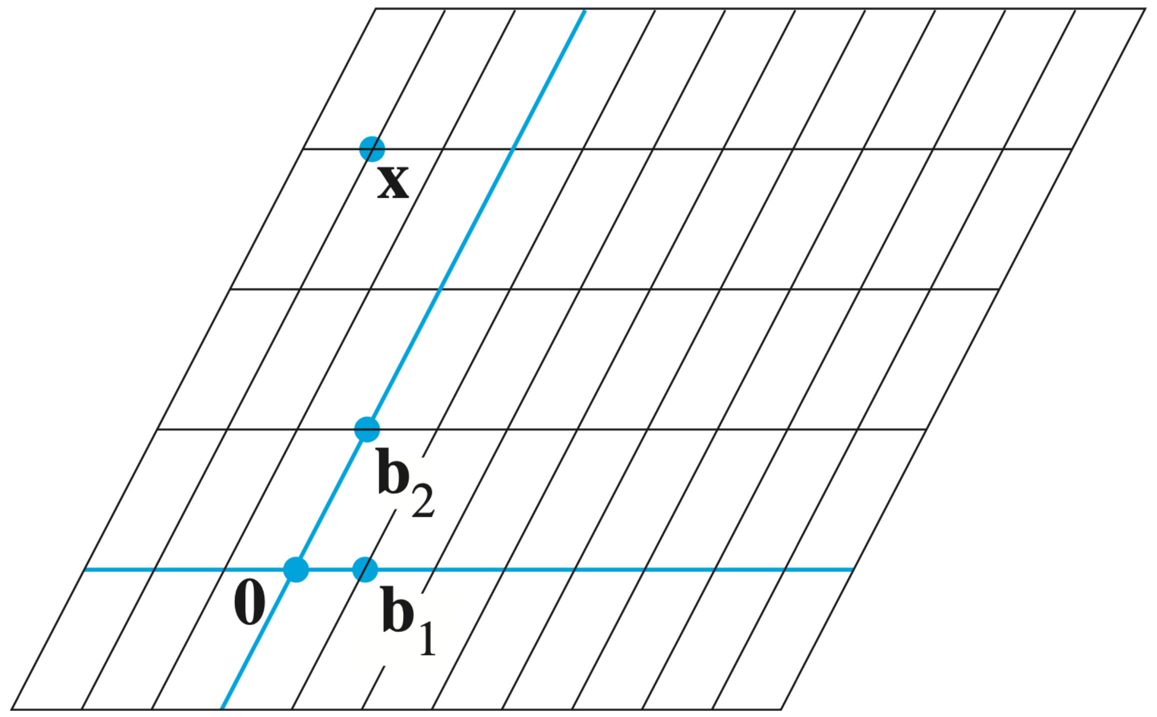

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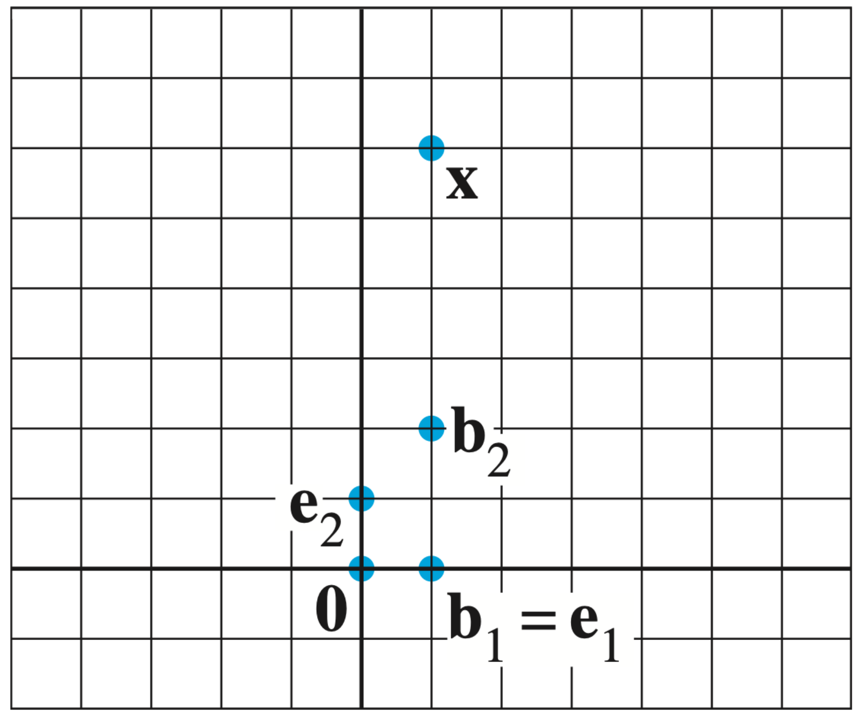

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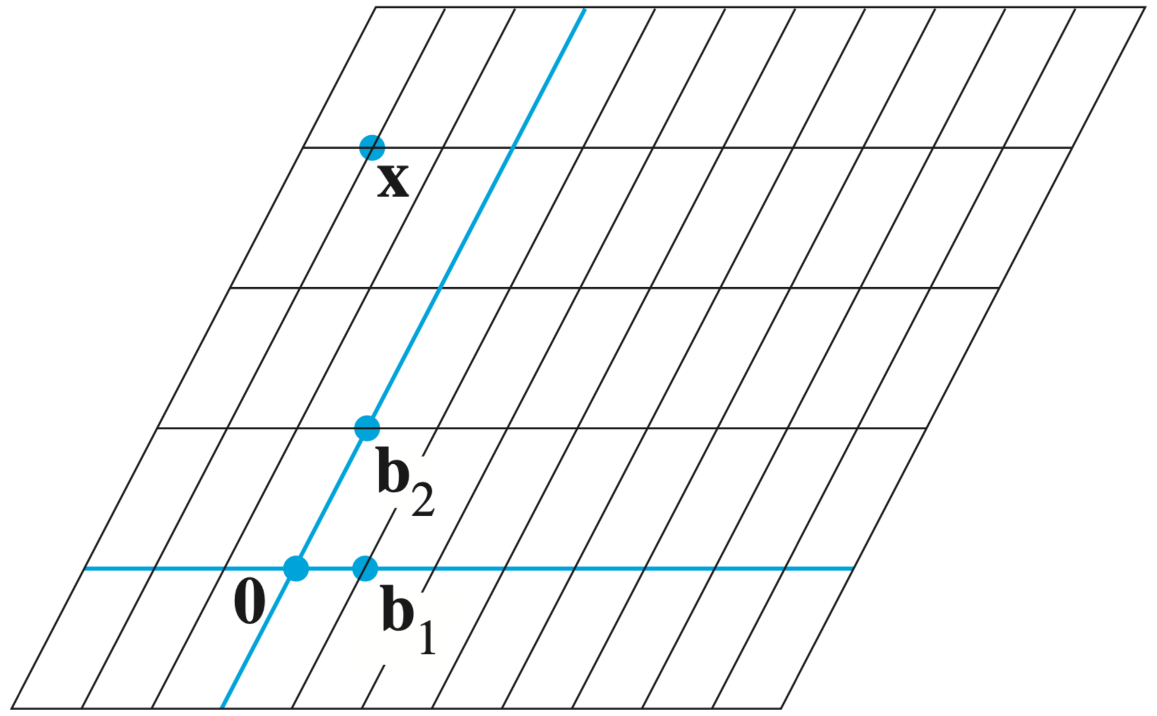

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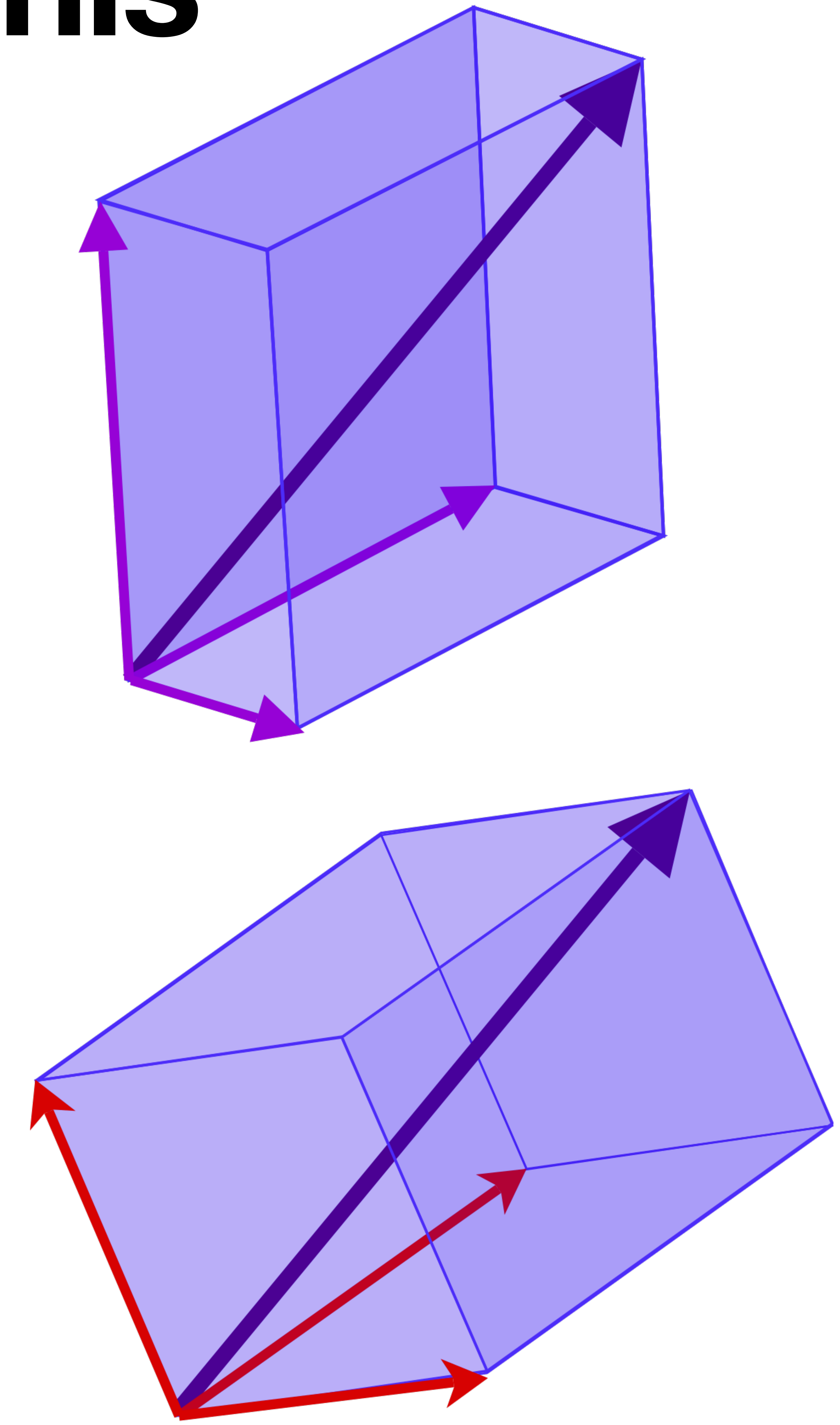
$\mathscr{B}$ defines a "different grid for our graph paper"

Linear Algebra and its Applications (Lay, Lay, McDonald)

Recall: How to think about this



Changing the coordinate system "warps space".

The Question. how do we represent a vector $\mathbf{v}$ in the warped space if we wanted it to "be in the same place"?

https://commons.wikimedia.org/wiki/File:3d_two_bases_same_vector.svg

Recall: Coordinate Vectors

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Let $\mathbf{v}$ be a vector in a $\mathbb{R}^n$ and let $\mathscr{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ be a basis of $\mathbb{R}^n$ where

$$\mathbf{v} = a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \ldots + a_n\mathbf{b}_n$$

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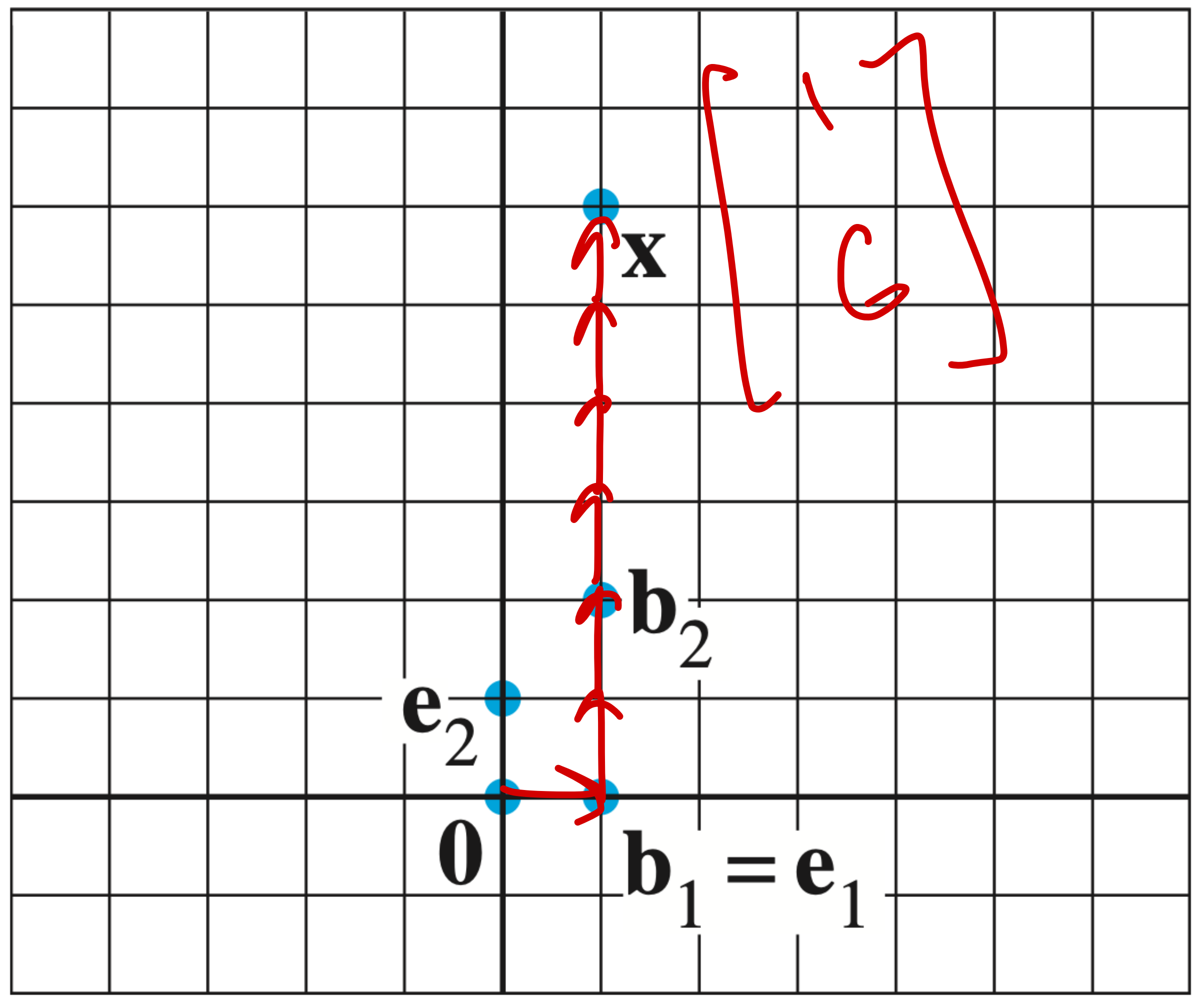
$$[\mathbf{v}]_{\mathscr{B}} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n$$

Recall: Coordinate Vectors



$$\begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

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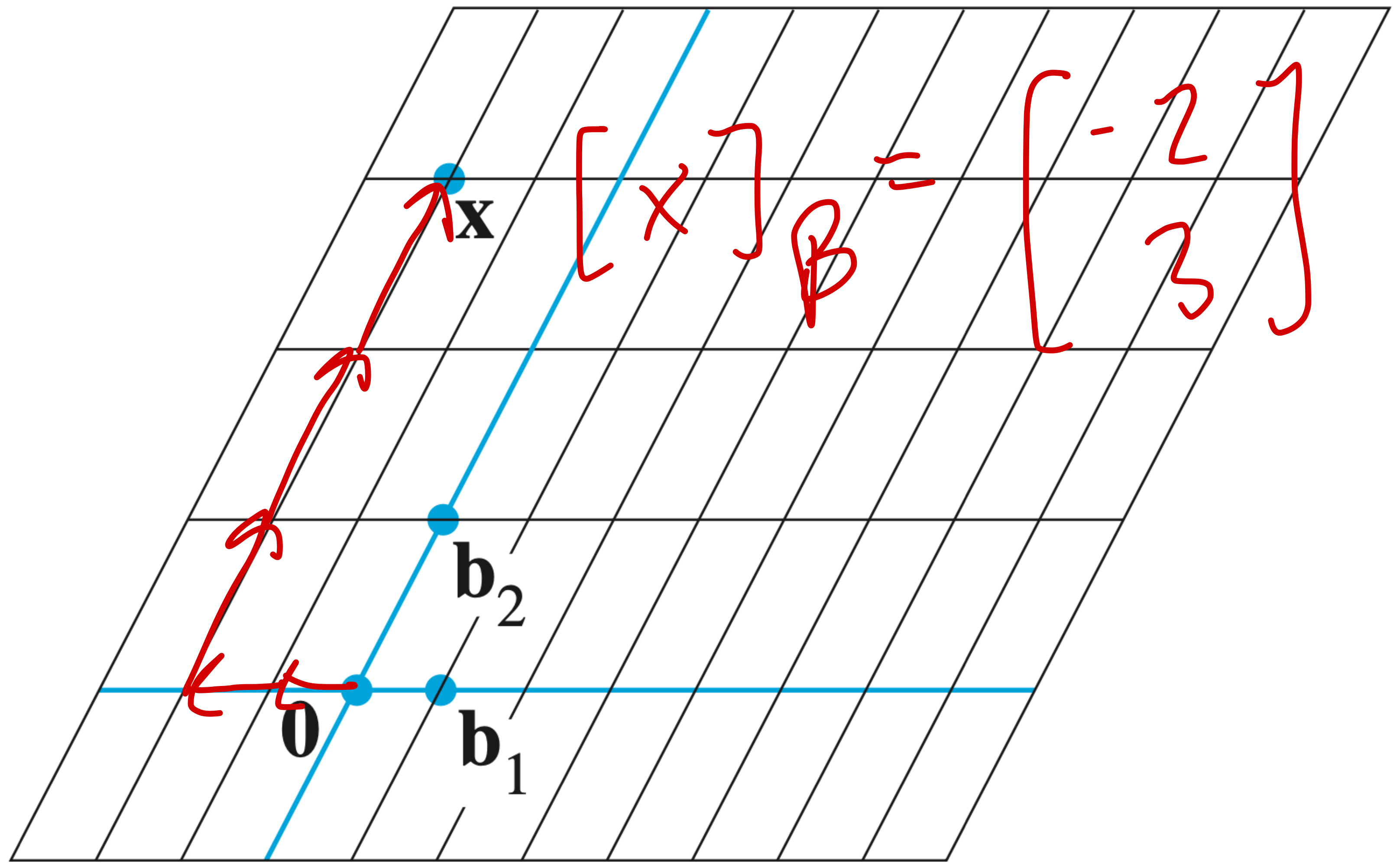

$$[x]_B = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

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Linear Algebra and its Applications (Lay, Lay, McDonald)

Question (Conceptual)

Hint: $\begin{bmatrix} b_1 & b_2 & \dots & b_n \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = c_1 \vec{b}_1 + c_2 \vec{b}_2 + \dots + c_n \vec{b}_n$

We know that if a $n \times n$ matrix $B = [\mathbf{b}_1\ \mathbf{b}_2\ \dots\ \mathbf{b}_n]$ is invertible, then the columns of B form a basis $\mathscr{B}$ of $\mathbb{R}^n$.

What is the matrix that implements the transformation

$$\mathbf{x} \mapsto [\mathbf{x}]_{\mathscr{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

where $\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \dots + c_n\mathbf{b}_n$*?*

Change of Basis Matrix

Theorem. If $\mathscr{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ form a basis of $\mathbb{R}^n$, then

$$[\mathbf{x}]_{\mathscr{B}} = [\mathbf{b}_1\ \mathbf{b}_2\ \ldots\ \mathbf{b}_n]^{-1}\mathbf{x}$$

Matrix inverses perform changes of bases.

B^{-1} $\qquad$ $B^{-1}x$ $\underline{\text{is}}$ $[x]_B$

How To: Change of Basis

Question. Given a basis $\mathscr{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ of $\mathbb{R}^n$, find the matrix which implements $\mathbf{x} \mapsto [\mathbf{x}]_{\mathscr{B}}$.

Solution. Construct the matrix $[\mathbf{b}_1\ \mathbf{b}_2\ \ldots\ \mathbf{b}_n]^{-1}$.

Diagonalization

Diagonal Matrices

ex. $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -0.4 & 0 & 0 \\ 0 & 0 & 22 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -0.4 & 0 & 0 \\ 0 & 0 & 22 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Definition. A $n \times n$ matrix A is **diagonal** if

$i \neq j$ if and only if $A_{ij} = 0$.

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Only the diagonal entries can be nonzero.

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$$i \neq j \quad \text{if and only if} \quad A_{ij} = 0.$$

Only the diagonal entries can be nonzero.

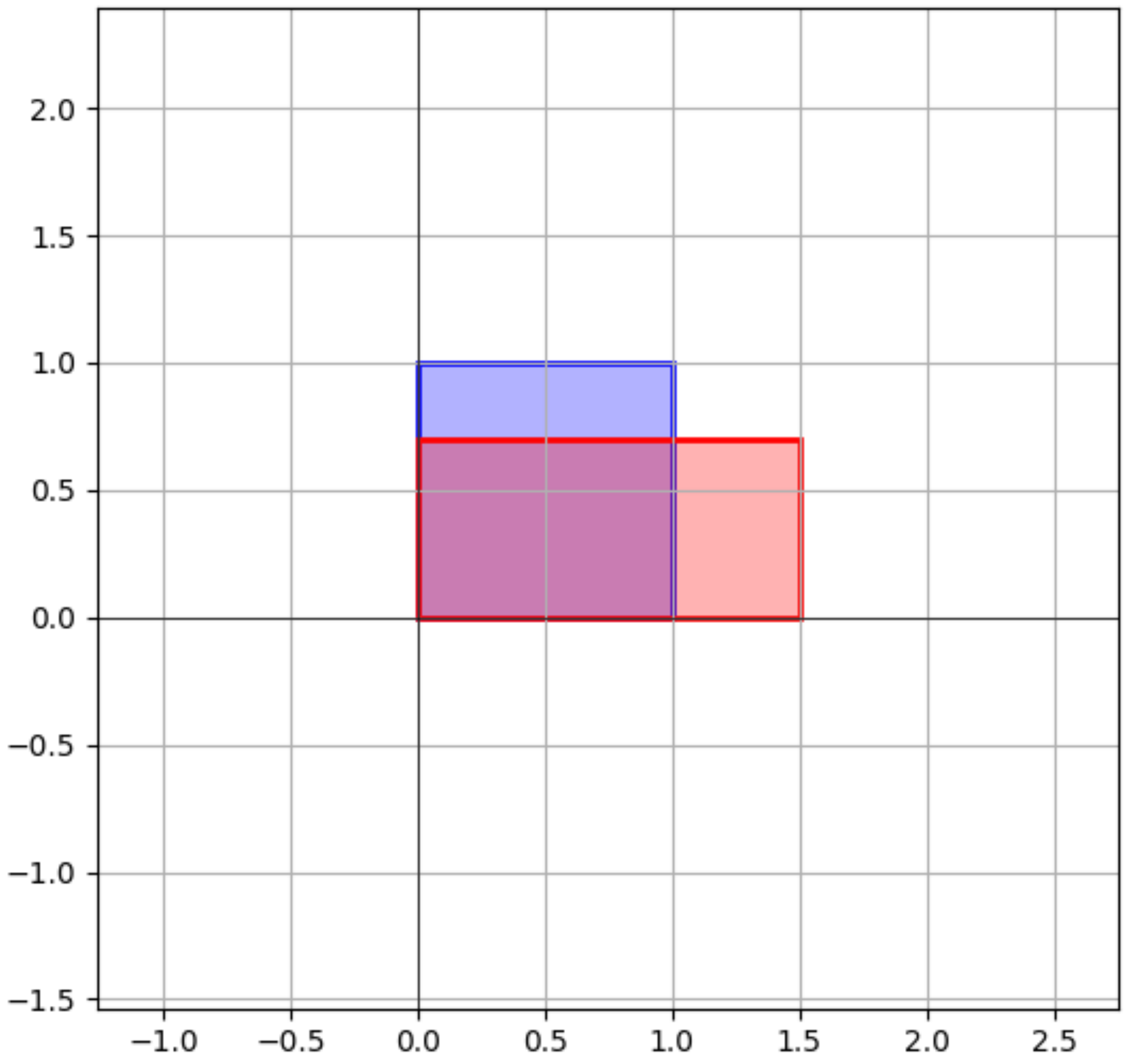
Diagonal matrices are scaling matrices.

Recall: Unequal Scaling

`The scaling matrix` *`affects each component of a vector in a simple way.`*

`The diagonal entries` **<u>`scale`</u>** **`each corresponding entry.`**

$$\begin{bmatrix} 1.5 & 0 \\ 0 & 0.7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1.5x \\ 0.7y \end{bmatrix}$$



$$\begin{bmatrix} 1.5 & 0 \\ 0 & 0.7 \end{bmatrix}$$

High level question:

When do matrices "behave" like scaling matrices "up to" change of basis?

Scaling and Eigenvectors

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The idea. Matrices behave like scaling matrices on eigenvectors.

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$$\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} (x\mathbf{e}_1 + y\mathbf{e}_2) = x2\mathbf{e}_1 + y(-3)\mathbf{e}_2$$

$$A \begin{bmatrix} x \\ y \end{bmatrix}_{\mathscr{B}} = A(x\mathbf{b}_1 + y\mathbf{b}_2) = x\lambda_1\mathbf{b}_1 + y\lambda_2\mathbf{b}_2$$

Scaling and Eigenvectors

The idea. Matrices behave like scaling matrices on eigenvectors.

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The fundamental question:

Can we expose this behavior in terms of a *matrix factorization*?

Recall: Matrix Factorization

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$$A = PBP^{-1}$$

Factorizations can:

- make working with A easier
- expose important information about A

Similar Matrices

$$A = PBP^{-1}$$

change of basis to
the columns of P.

Definition. A matrix A is **similar** to a matrix B if there is some invertible matrix P such that $A = PBP^{-1}$.

A and B are the same up to a change of basis.

Similar Matrices and Eigenvalues

Theorem. Similar matrices have the <u>same eigenvalues</u>.

Verify: $\boxed{\det(\boxed{A - \lambda I})} = \det(PBP^{-1} - \lambda I)$

$$= \det(PBP^{-1} - \lambda PIP^{-1})$$

$$= \det(P(\boxed{B - \lambda I})P^{-1})$$

$$= \cancel{\det(P)}\det(B - \lambda I)\underset{\overset{\shortparallel}{1/\det(P)}}{\cancel{\det(P^{-1})}} \qquad = \boxed{\det(B - \lambda I)}$$

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Diagonalizable Matrices

Definition. A matrix A is **diagonalizable** if it is similar to a diagonal matrix.

There is an invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$.

Diagonalizable matrices are the same as scaling matrices up to a change of basis.

Important: Not all Matrices are Diagonalizable

This is very different from the LU factorization.

We will need to figure out which matrices are diagonalizable.

Question. Is the zero matrix diagonalizable? Yes.

$$\begin{bmatrix} 0 & & 0 \\ & \ddots & \\ 0 & & 0 \end{bmatrix}$$ is diagonal

Application: Matrix Powers

only take the power of B

Theorem. If $A = PBP^{-1}$, then $A^k = PB^kP^{-1}$.

It may be easier to take the power of B (as in the case of diagonal matrices).

Verify:

$$AAA = (PB\cancel{P^{-1}})(\cancel{P}B\cancel{P^{-1}})(\cancel{P}BP^{-1})$$

$$PB^3P^{-1}$$

How To: Matrix Powers

Question. Given A is diagonalizable, determine A^k.

Solution. Find it's diagonalization PDP^{-1} and then compute PD^kP^{-1}.

Remember that

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}^k = \begin{bmatrix} a^k & 0 & 0 \\ 0 & b^k & 0 \\ 0 & 0 & c^k \end{bmatrix}$$

But how do we find the diagonalization...

Diagonalization and Eigenvectors

Suppose we have a diagonalization

$$A = PDP^{-1}$$

What do we know about it?

Columns of P are eigenvectors

$$A = [\mathbf{p}_1\ \mathbf{p}_2\ \mathbf{p}_3]\begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix}[\mathbf{p}_1\ \mathbf{p}_2\ \mathbf{p}_3]^{-1}$$

`Verify:`

$$\boxed{A\vec{p}_2} = PD\overset{\vec{e}_2}{\boxed{[p_1\ \ p_2\ \ p_3]^{-1}\vec{p}_2}}$$

$$P(\lambda_2\vec{e}_2) = \lambda P\vec{e}_2 = \boxed{\lambda_2\vec{p}_2}$$

Columns of P are eigenvectors

$$A = [\mathbf{p}_1 \ \mathbf{p}_2 \ \mathbf{p}_3] \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} [\mathbf{p}_1 \ \mathbf{p}_2 \ \mathbf{p}_3]^{-1}$$

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And the entries of D are the **eigenvalues** associated to each eigenvector.

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In fact, the columns of P form an **eigenbasis** of $\mathbb{R}^n$ for A.

And the entries of D are the **eigenvalues** associated to each eigenvector.

A diagonalization exposes a lot of information about A.

The Diagonalization Theorem

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Theorem. A matrix is diagonalizable if and only if it has an eigenbasis.

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(we just did the hard part, if a matrix is diagonalizable then it has an **eigenbasis**)

We can use the same recipe to go in the other direction, given an eigenbasis, we can **build a diagonalization.**

Diagonalizing a Matrix

High Level

$$A = PDP^{-1}$$

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The diagonal of D are the eigenvalues for each column of P.

The matrix P^{-1} is a change of basis to this eigenbasis of A.

Step 1: Eigenvalues

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & 3 \\ 3 & 3 & 1 \end{bmatrix}$$

Find all the eigenvalues of A.

Find the roots of $\det(A - \lambda I)$.

e.g.

$$\det(A - \lambda I) = -(\lambda - 1)(\lambda + 2)^2$$

Step 2: Eigenvectors

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & 3 \\ 3 & 3 & 1 \end{bmatrix}$$

$$\lambda_1 = 1$$

$$\lambda_2 = -2$$

`Find` **`bases`** `of the corresponding eigenspaces.`

`e.g.`

$$\text{Nul}(A - I) = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$$\text{Nul}(A + 2I) = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Step 3: Construct P

we could fail at this step.

If there are n eigenvectors from the previous step they form an **eigenbasis.**

Build the matrix with these vectors as the columns

e.g.

$$P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & 3 \\ 3 & 3 & 1 \end{bmatrix}$$

$$\lambda_1 = 1$$

$$\lambda_2 = -2$$

$$\text{Nul}(A - I) = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$$\text{Nul}(A + 2I) = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Step 5: Construct D

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & 3 \\ 3 & 3 & 1 \end{bmatrix}$$

$$\lambda_1 = 1$$

$$\lambda_2 = -2$$

$$P = \begin{array}{c} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \\ \begin{array}{ccc} \underbrace{\quad}_{\lambda_1} & \underbrace{\quad}_{\lambda_2} & \underbrace{\quad}_{\lambda_2} \end{array} \end{array}$$

Build the matrix with eigenvalues as diagonal entries.

Note the order. It should be the same as the order of columns of P.

e.g.

$$D = \begin{bmatrix} \overset{\lambda_1}{1} & 0 & 0 \\ 0 & -2^{\lambda_2} & 0 \\ 0 & 0 & -2^{\lambda_2} \end{bmatrix}$$

Step 6: Invert P

Find the inverse of P (we know how to do this).

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & 3 \\ 3 & 3 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Putting it Together

eigenvalue

eigenvalues

$$\overset{A}{\begin{bmatrix}1&3&3\\-3&-5&3\\3&3&1\end{bmatrix}} = \underbrace{\overset{P}{\begin{array}{ccc}\boxed{-1}&\boxed{-2}&\boxed{-2}\\ \hline \end{array}\begin{bmatrix}1&-1&-1\\-1&1&0\\1&0&1\end{bmatrix}}}_{\text{eigenbasis}} \overset{D}{\begin{bmatrix}\boxed{1}&0&0\\0&\boxed{-2}&0\\0&0&\boxed{-2}\end{bmatrix}} \underbrace{\overset{P^{-1}}{\begin{bmatrix}1&-1&-1\\-1&1&0\\1&0&1\end{bmatrix}^{-1}}}_{\text{Change of basis}}$$

How to: Diagonalizing a Matrix

Question. Find a diagonalization of $A \in \mathbb{R}^n$, or determine that A is not diagonalizable.

Solution.

1. Find the eigenvalues of A, and bases for their eigenspaces. If these eigenvectors don't form a basis of $\mathbb{R}^n$, then A is **not diagonalizable.**
2. Otherwise, build a matrix P whose columns are the eigenvectors of A.
3. Then build a diagonal matrix D whose entries are the eigenvalues of A *in the same order.*
4. Invert P.
5. The diagonalization of A is PDP^{-1}.

We know how to do every step, its a matter of putting it all together.

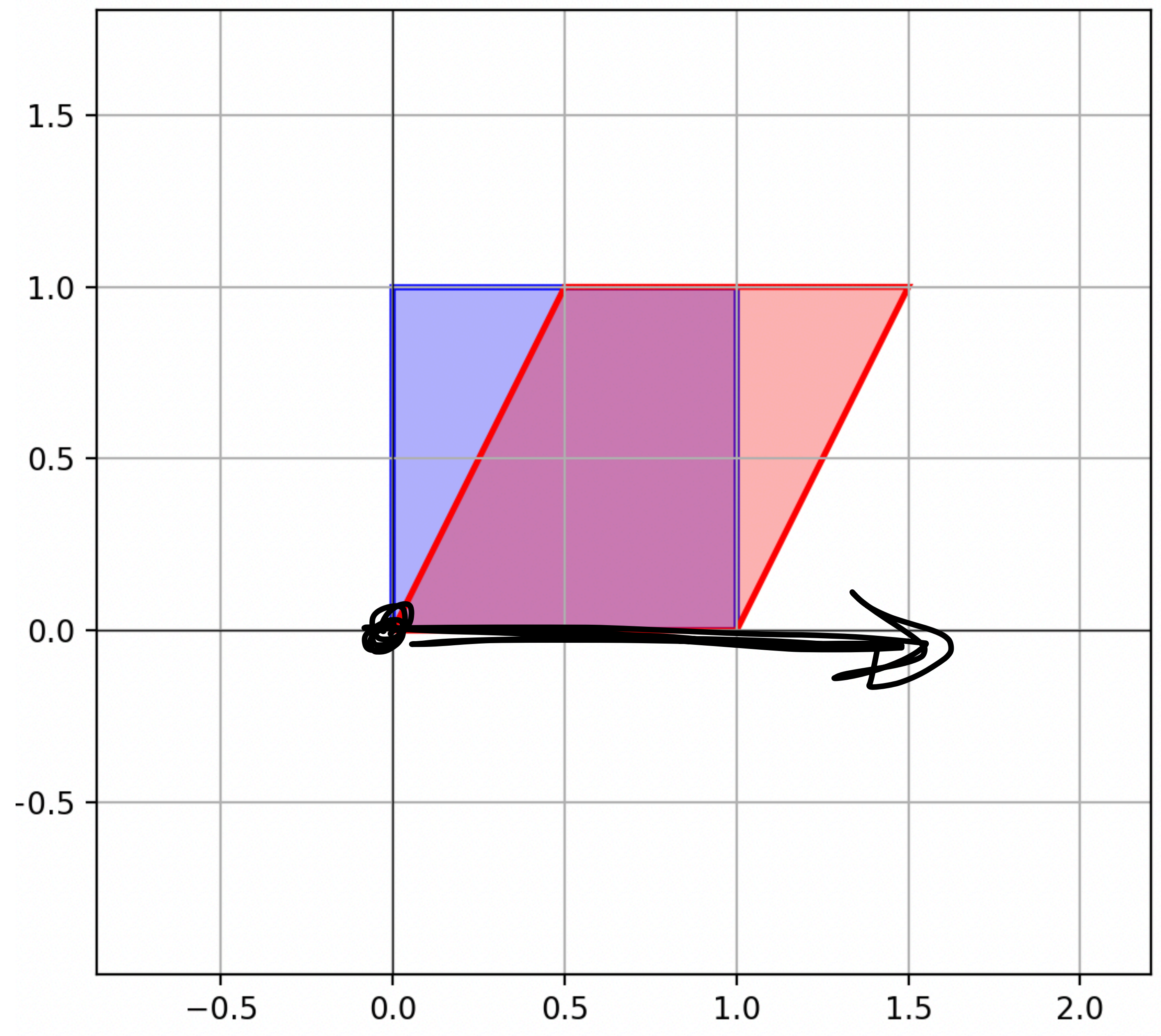
Example of Failure: Shearing

$$A = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}$$

The shearing matrix has a single eigenvalue with an eigenspace of dimension 1.

We can't build an eigenbasis of $\mathbb{R}^2$ for A.

In other words, A is not diagonalizable.



Important case: Distinct Eigenvalues

ex. $$\begin{bmatrix} 1 & -3 & 4 & 2 \\ 0 & -2 & 3 & -1 \\ 0 & 0 & 10 & 5 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

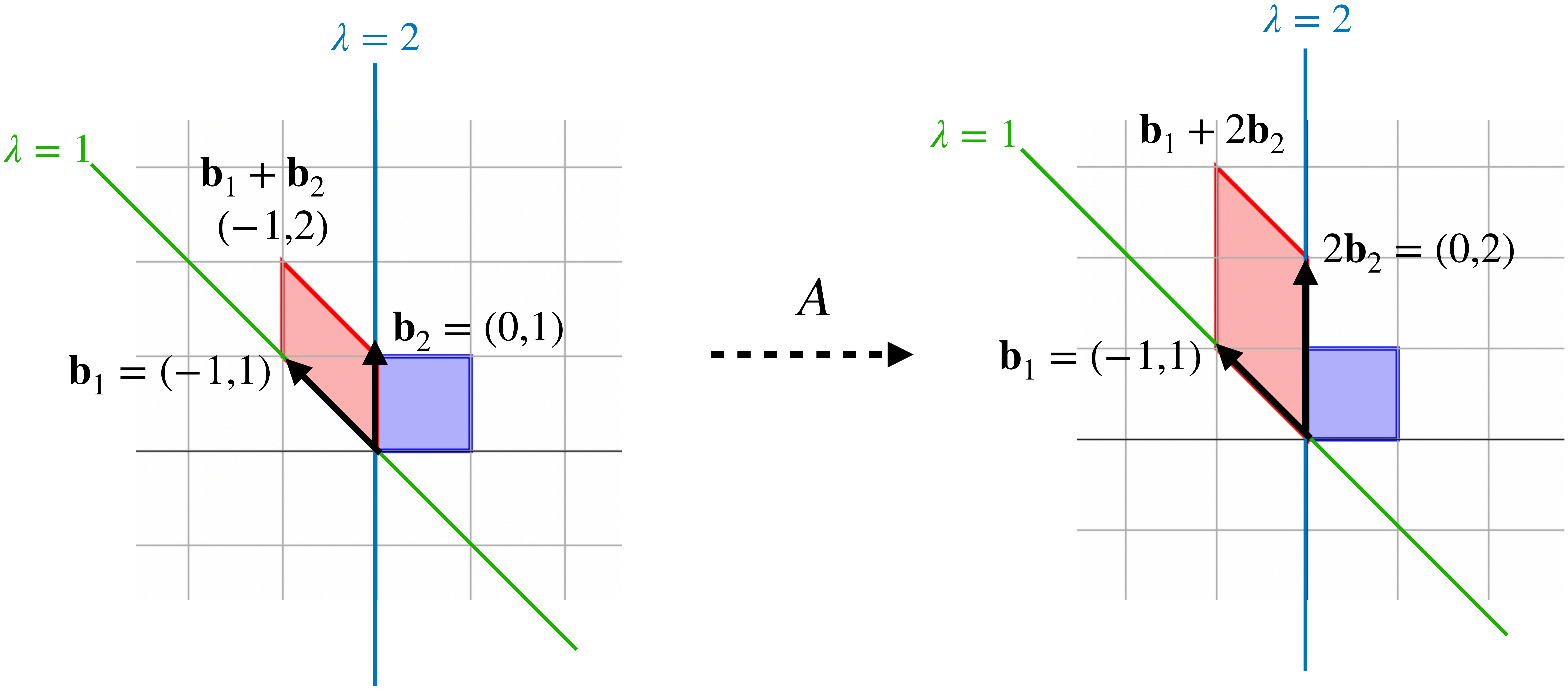
Theorem. If an $n \times n$ matrix has has n distinct eigenvalues, then it is diagonalizable.

This is because eigenvectors with distinct eigenvalues are *linearly independent.*

The Picture

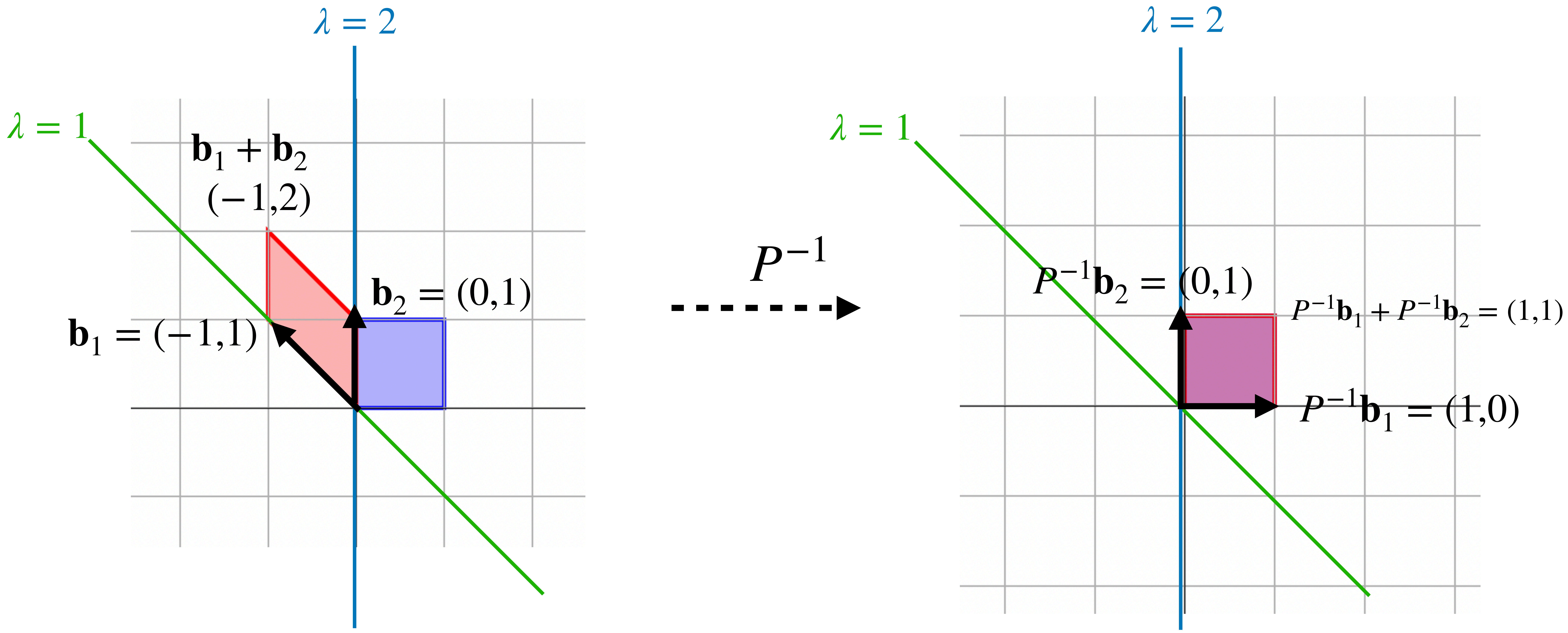
Example (Geometric)

$$A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$$



Example (Geometric)

$$P^{-1} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}^{-1}$$

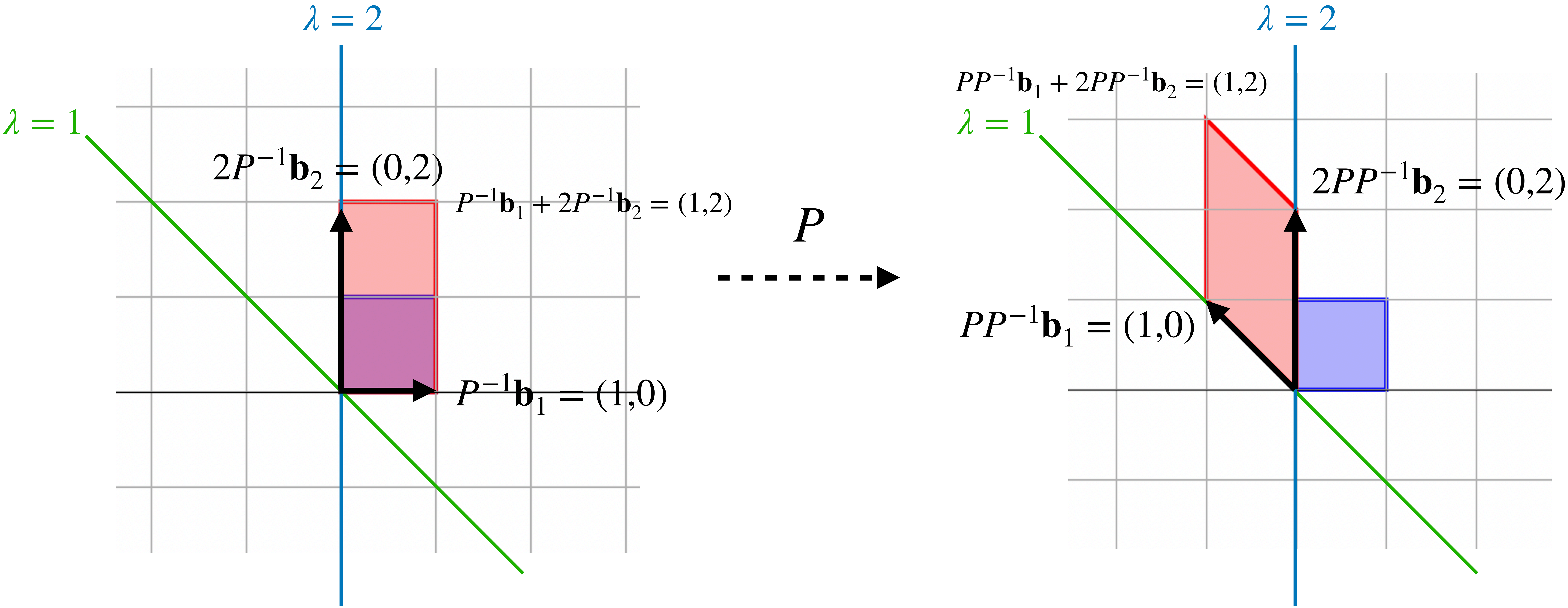


Example (Geometric)

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Example (Geometric)

$$P = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$$



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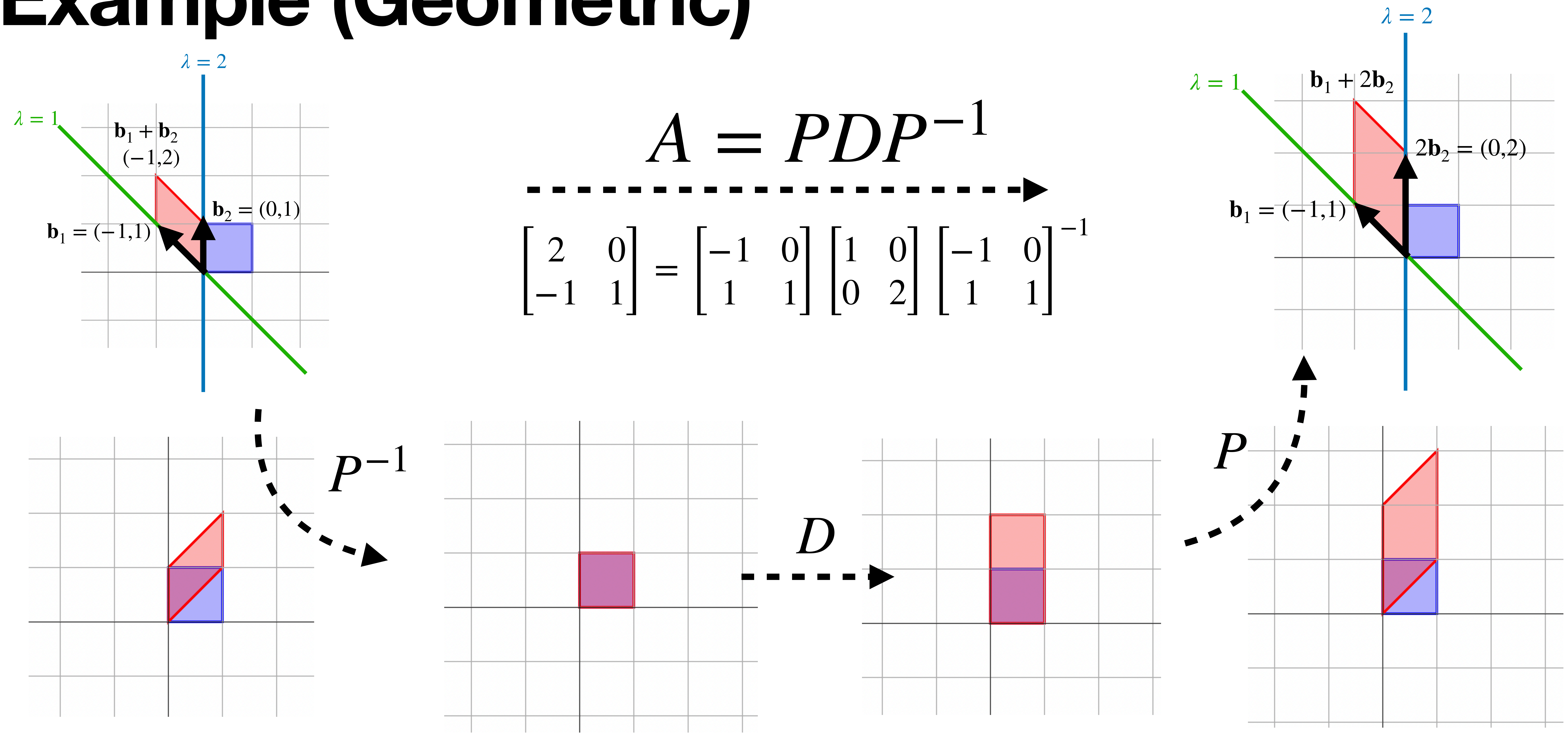

$$A = PDP^{-1}$$

$$\begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}^{-1}$$