

Diagonalization

Geometric Algorithms

Lecture 19

Introduction

Recap Problem

$$A = \begin{bmatrix} -1 & h & 2 \\ 0 & 2 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

For what values of h is $\dim(H) = 2$, where H is the eigenspace of A for the eigenvalue -1 ?

Answer: $h = 3$

Objectives

1. Finish our discussion on the characteristic polynomial.
2. Motivate diagonalization via linear dynamical systems and changes of coordinate systems.
3. Describe how to diagonalize a matrix.

Keywords

multiplicity

similar matrices

diagonalizable matrices

change of basis

eigenbasis

Recap

Recall: Determinants and Invertibility

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$$\begin{aligned} \det(A - \lambda I) = 0 &\equiv (A - \lambda I)\mathbf{x} = \mathbf{0} \text{ has nontrivial solutions} \\ &\equiv \lambda \text{ is an eigenvalue of } A \end{aligned}$$

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In Reality. We'll use

`numpy.linalg.eig(A)`

Example

$$A = \begin{bmatrix} 1 & -1 \\ 7 & -3 \end{bmatrix}$$

$$\det \begin{bmatrix} 1 - \lambda & -1 \\ 7 & -3 - \lambda \end{bmatrix} = (\lambda - 1)(\lambda + 3) + 7 = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$$

The only eigenvalues of A is 2.

Last Remarks on the Characteristic Polynomial

Example: Triangular matrix

$$\begin{bmatrix} 1 & -3 & 0 & 6 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

The characteristic polynomial of a triangular matrix comes pre-factored:

An Observation: Multiplicity

$$\lambda^1 (\lambda - 1)^2 (\lambda - 4)^1 \text{ multiplicities}$$

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This is called the **(algebraic) multiplicity** of the root.

Is the multiplicity meaningful in this context?

Multiplicity and Dimension

Theorem. The dimension of the eigenspace of A for the eigenvalue λ is at most the multiplicity of λ in $\det(A - \lambda I)$ (and at least 1).

The multiplicity is an upper bound on "how large" the eigenspace is.

Example

Let A be a 5×5 matrix with characteristic polynomial $(x - 1)^3(x - 3)(x + 5)$.

» What is $\text{rank}(A)$?

» What is the minimum possible rank of $A - I$?

Motivating Diagonalization via Linear Dynamical Systems

Recall: Eigenbasis

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We will be almost exclusively interested of **eigenbases of \mathbb{R}^n** when $A \in \mathbb{R}^{n \times n}$.

The Question. When can we describe any vector in \mathbb{R}^n as a unique linear combination of eigenvectors of A ?

Recall: Linear Dynamical Systems

$$\mathbf{v}_1 = A\mathbf{v}_0$$

$$\mathbf{v}_2 = A\mathbf{v}_1 = A^2\mathbf{v}_0$$

$$\mathbf{v}_3 = A\mathbf{v}_2 = A^3\mathbf{v}_0$$

$$\mathbf{v}_4 = A\mathbf{v}_3 = A^4\mathbf{v}_0$$

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A **linear dynamical system** describes a sequence of **state vectors** starting at \mathbf{v}_0 .

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multiplying by
 A changes the
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demo

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eigenvalues of A

closed-form solution

Verify:

Application: Eigenbases and Limiting Behavior

Theorem. If A has an eigenbasis with eigenvalues

$$\lambda_1 \geq \lambda_2 \dots \geq \lambda_k$$

then $\mathbf{v}_k \sim \lambda_1^k \mathbf{u}$ for some vector \mathbf{u} .

In the long term, the system grows exponentially in λ_1 .

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Given a basis \mathcal{B} for \mathbb{R}^n , we only need to know how $A \in \mathbb{R}^n$ behaves on \mathcal{B} .

Sometimes, A behaves simply on \mathcal{B} , as in the case of eigenbases.

What we're really doing is changing our coordinate system to expose a behavior of A .

Recap: Change of Basis

Recall: Bases define Coordinate Systems

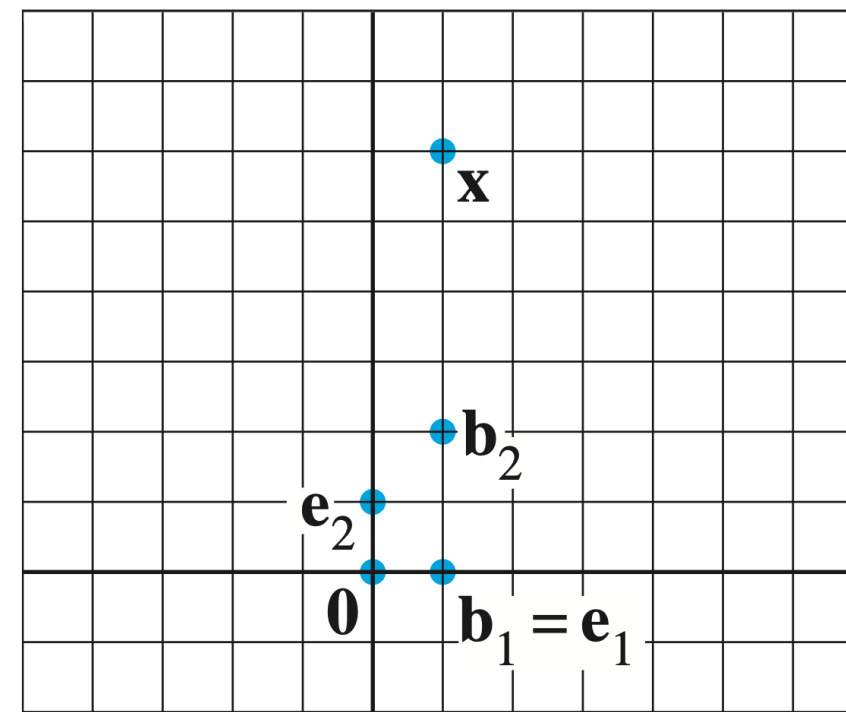


FIGURE 1 Standard graph paper.

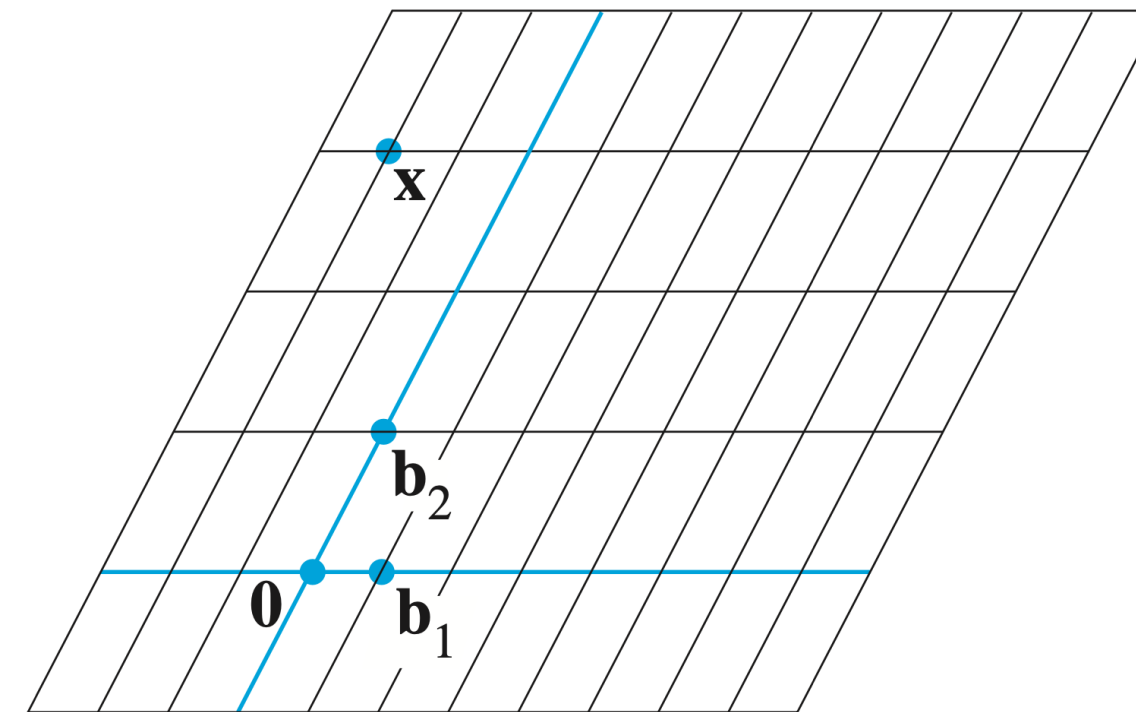


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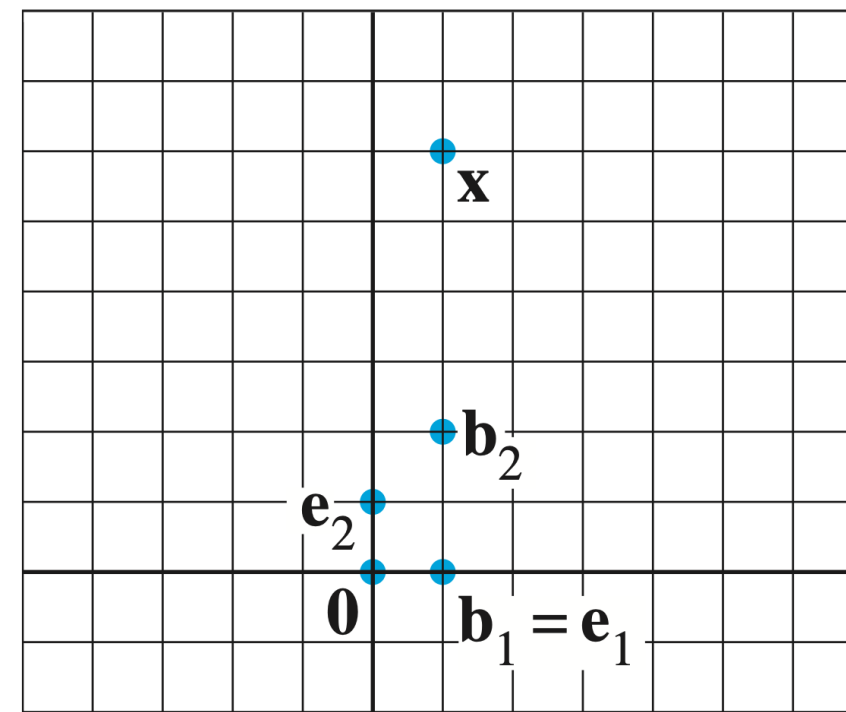


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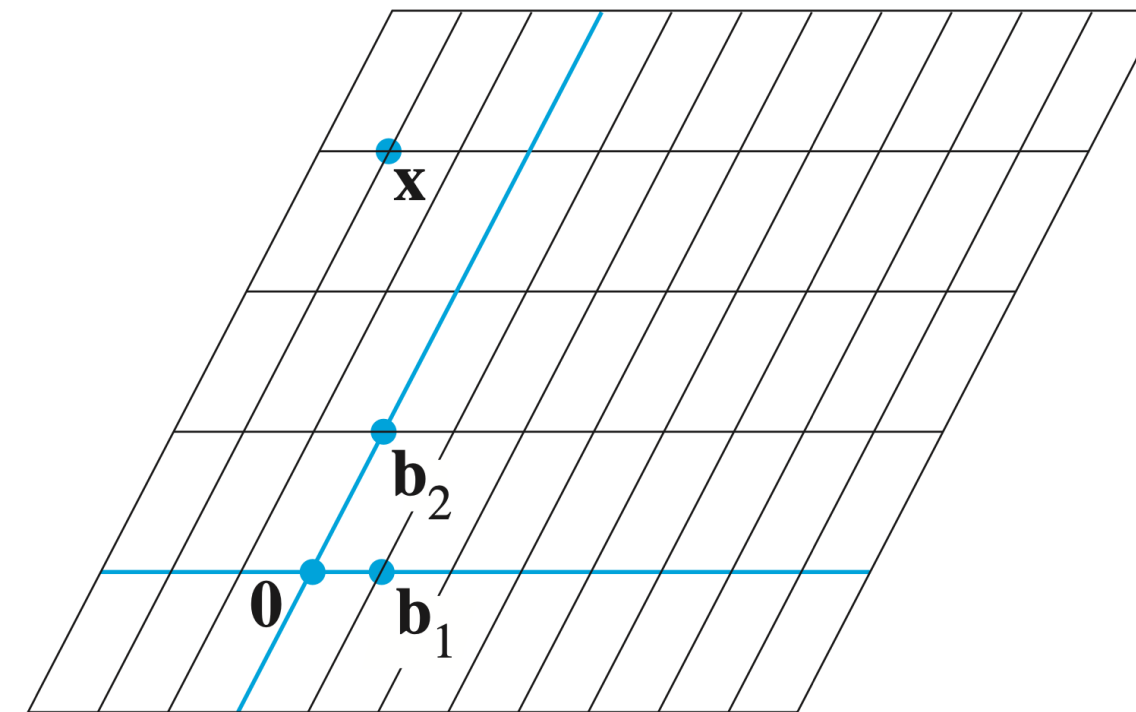


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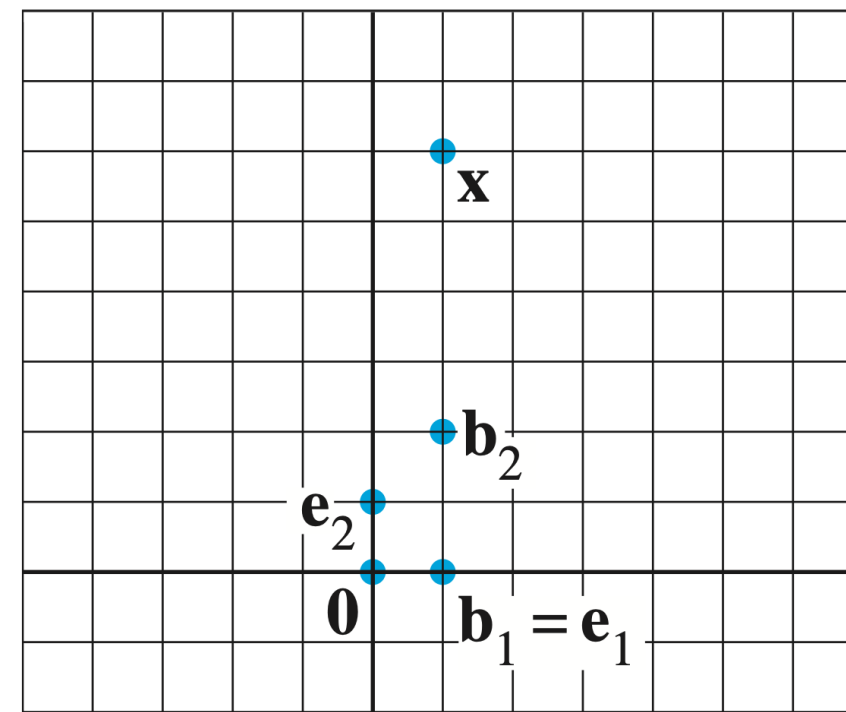


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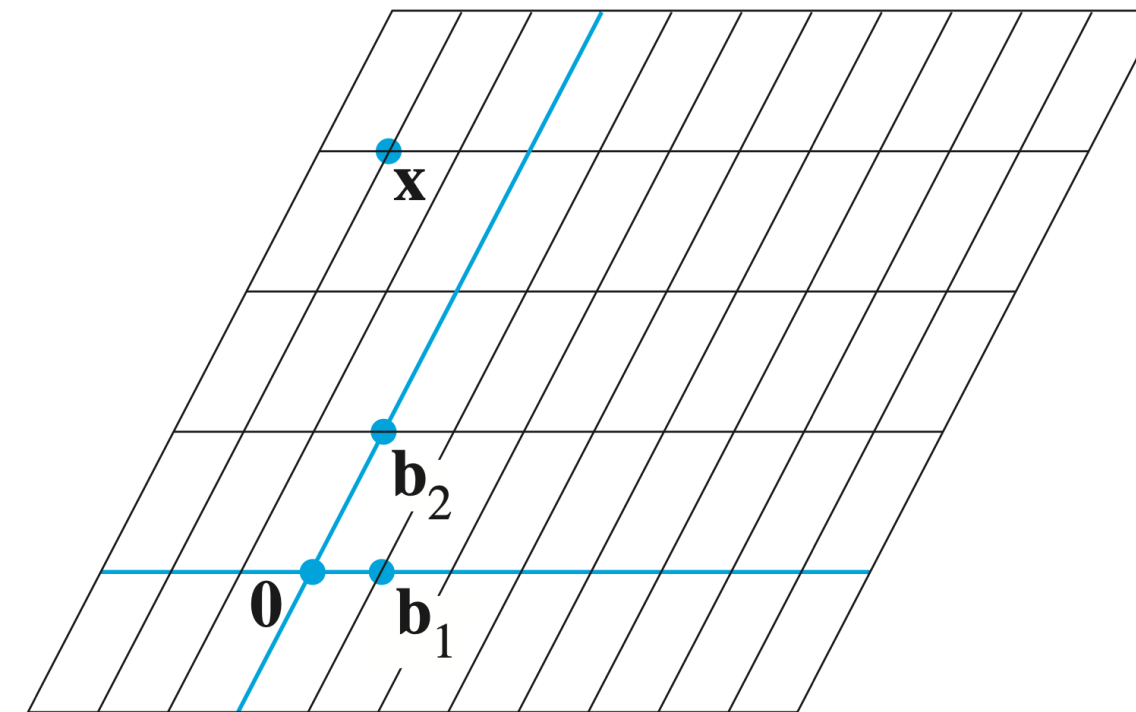


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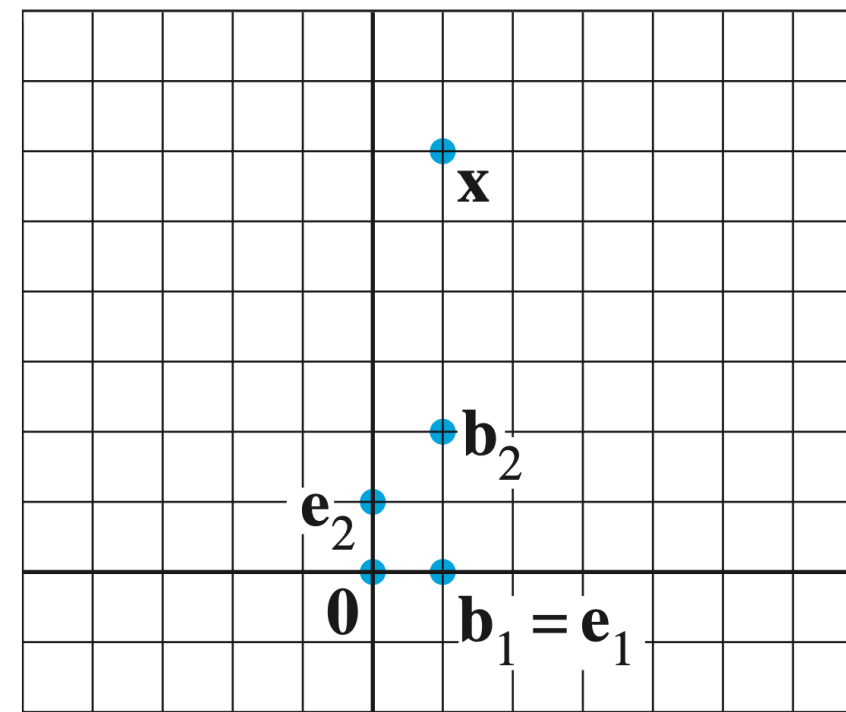


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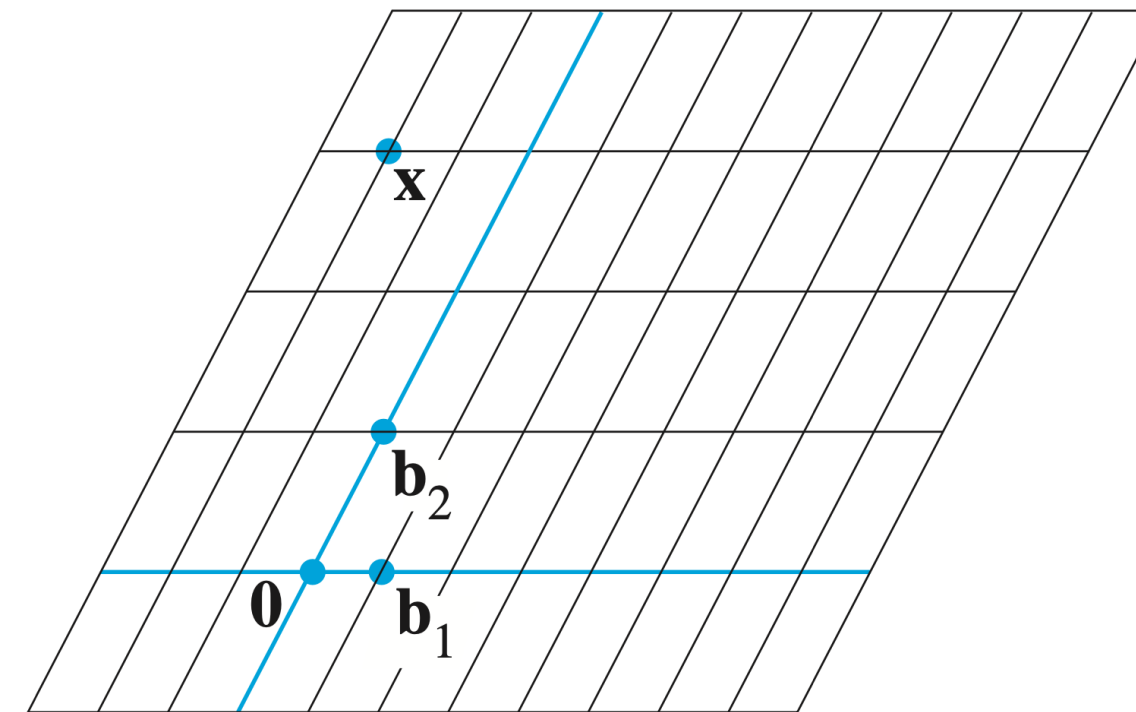


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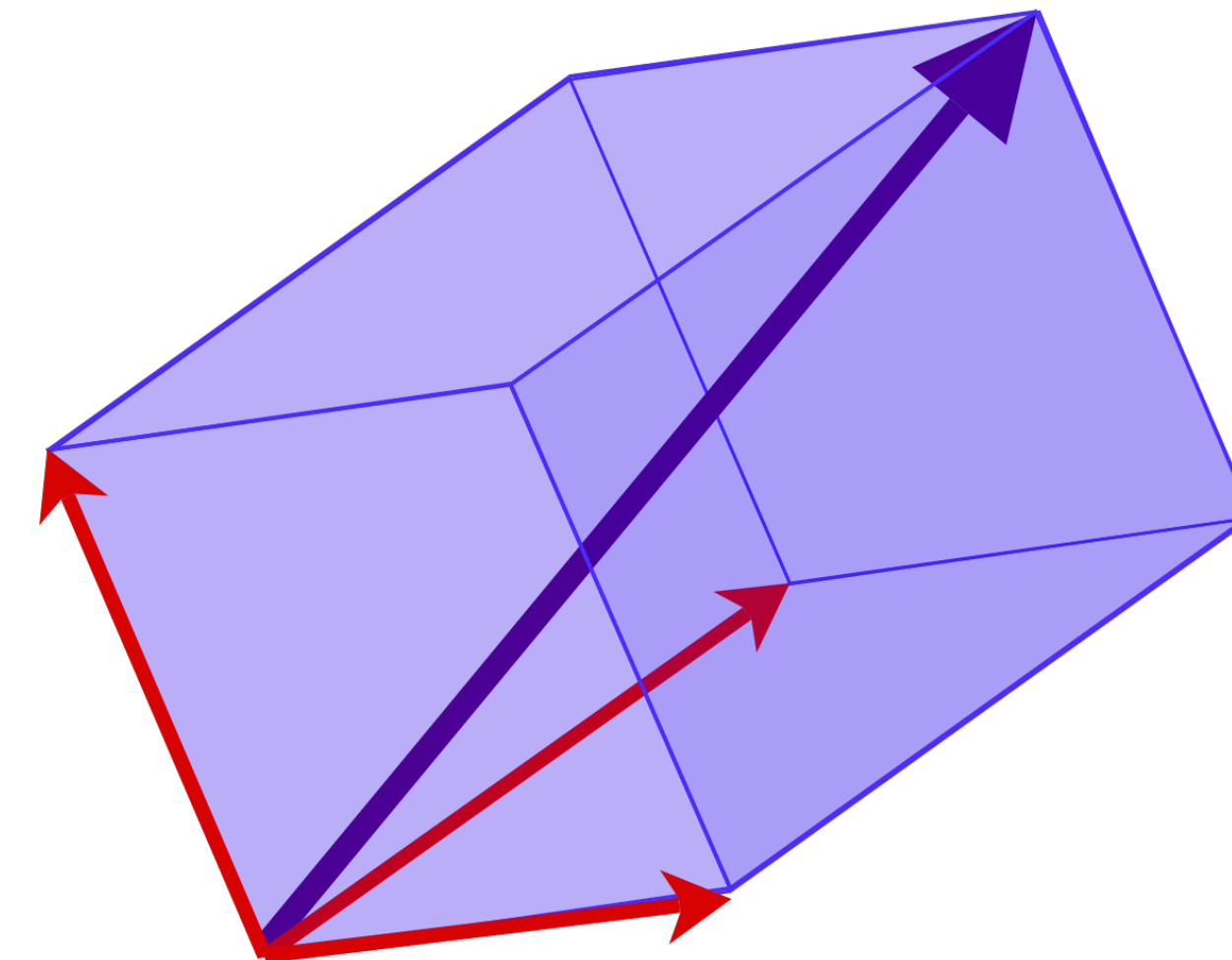
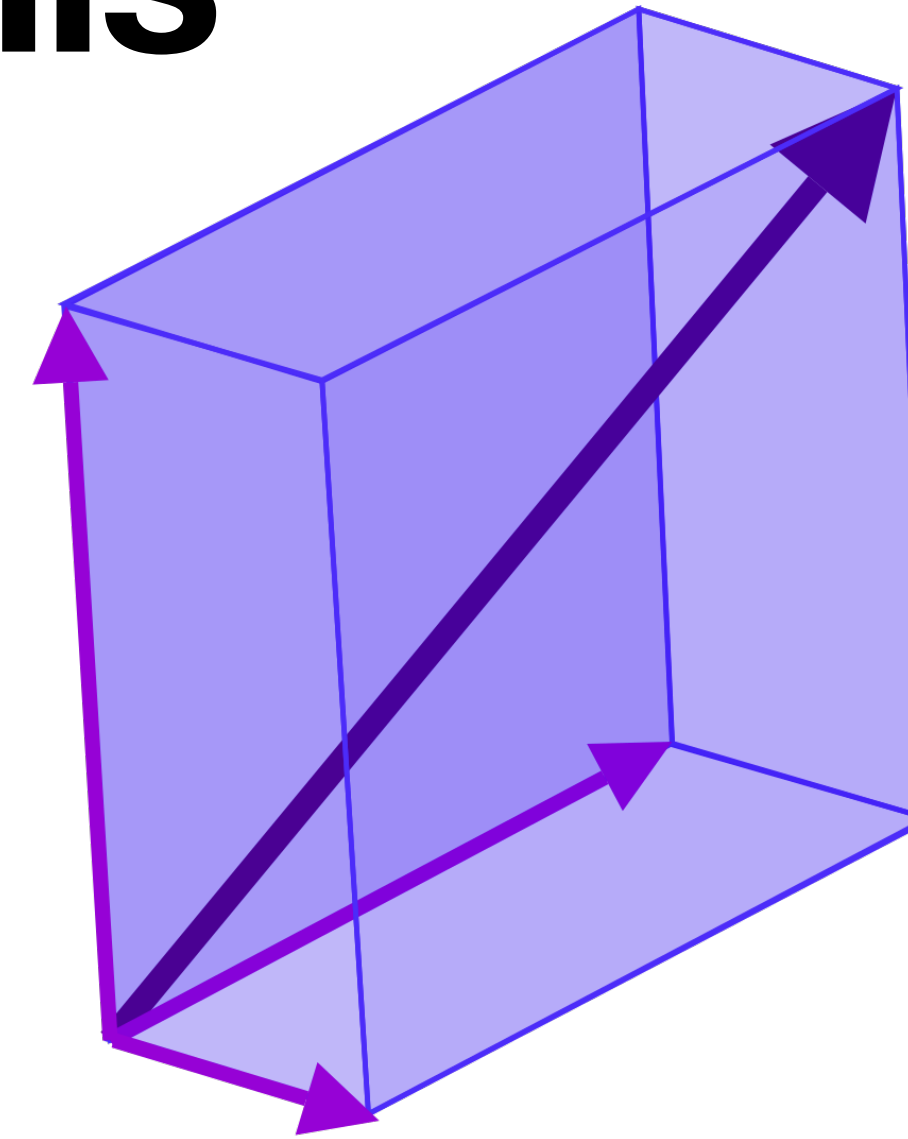
Every basis provides a way to write down *coordinates* of a vector.

\mathcal{B} defines a "different grid for our graph paper"

Recall: How to think about this

Changing the coordinate system "warps space".

The Question. how do we represent a vector \mathbf{v} in the warped space if we wanted it to "be in the same place"?



Recall: Coordinate Vectors

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Let \mathbf{v} be a vector in a \mathbb{R}^n and let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}$ be a basis of \mathbb{R}^n where

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$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

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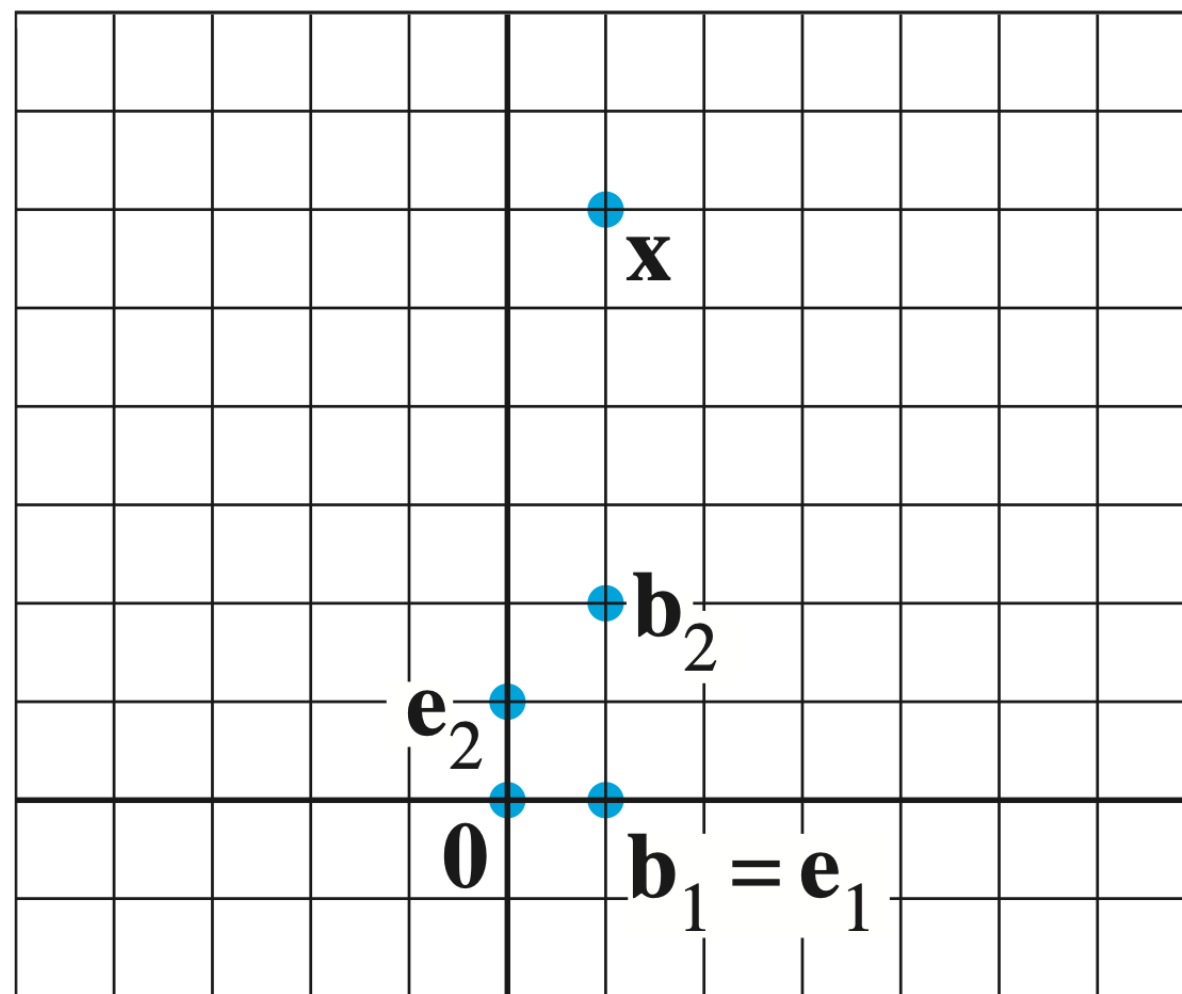


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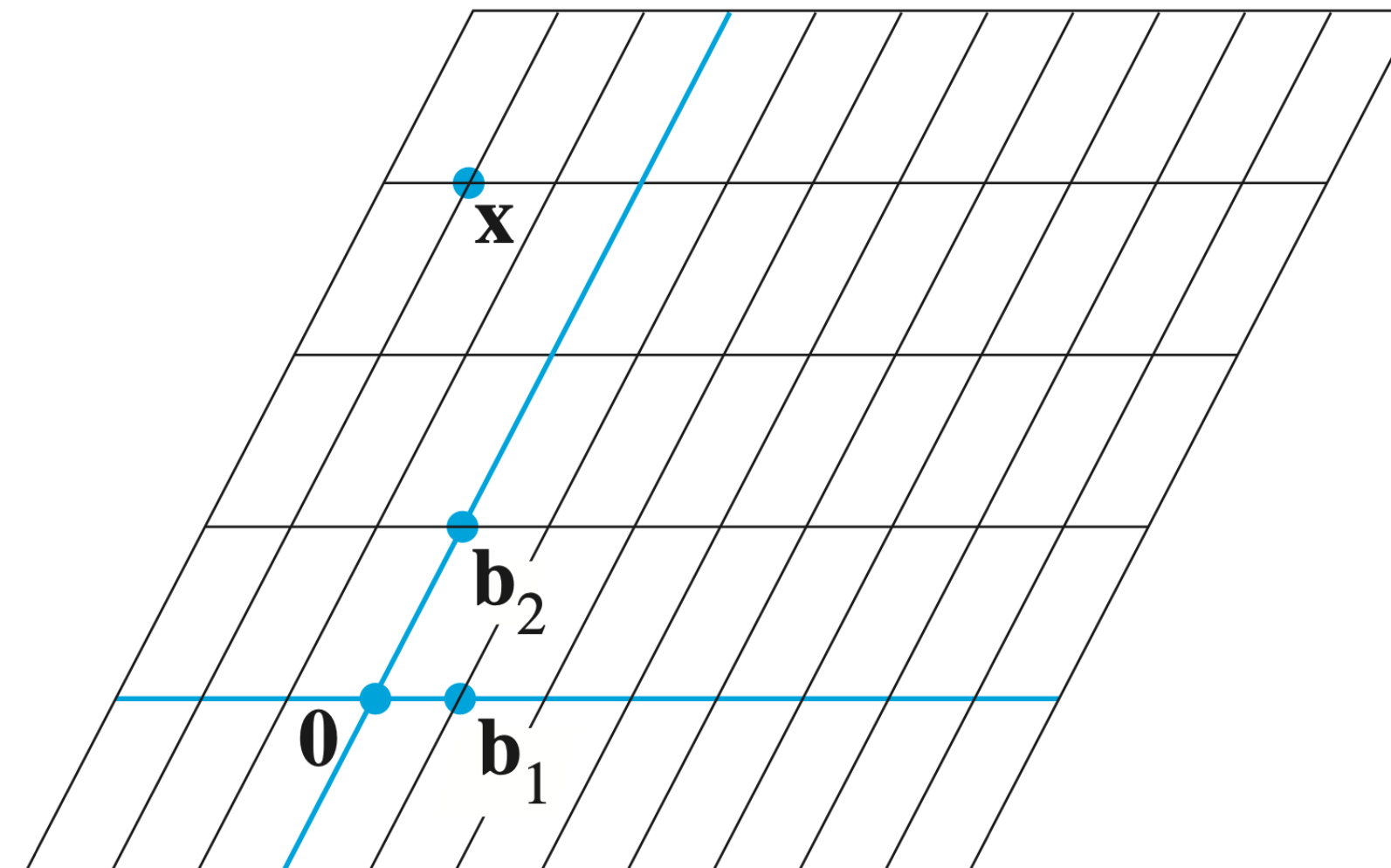


FIGURE 2 B -graph paper.

Question (Conceptual)

We know that if a $n \times n$ matrix $B = [\mathbf{b}_1 \ \mathbf{b}_2 \ \dots \ \mathbf{b}_n]$ is invertible, then the columns of B form a basis \mathcal{B} of \mathbb{R}^n .

What is the matrix that implements the transformation

$$\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

where $\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \dots + c_n\mathbf{b}_n$?

Change of Basis Matrix

Theorem. If $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}$ form a basis of \mathbb{R}^n , then

$$[\mathbf{x}]_{\mathcal{B}} = [\mathbf{b}_1 \ \mathbf{b}_2 \ \dots \ \mathbf{b}_n]^{-1} \mathbf{x}$$

Matrix inverses perform changes of bases.

How To: Change of Basis

Question. Given a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}$ of \mathbb{R}^n , find the matrix which implements $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$.

Solution. Construct the matrix $[\mathbf{b}_1 \ \mathbf{b}_2 \ \dots \ \mathbf{b}_n]^{-1}$.

Diagonalization

Diagonal Matrices

ex.
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -0.4 & 0 & 0 \\ 0 & 0 & 22 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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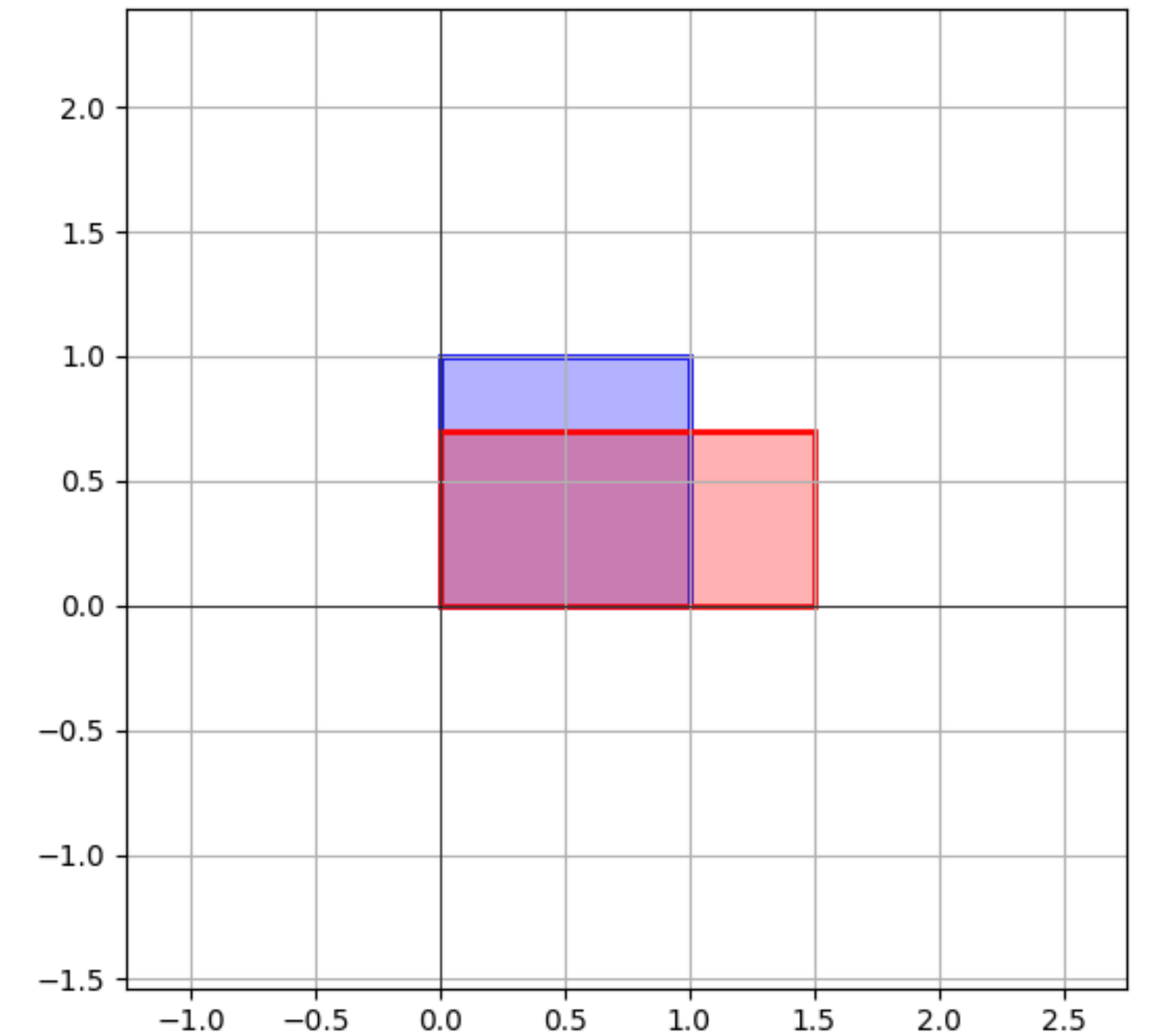
Diagonal matrices are scaling matrices.

Recall: Unequal Scaling

The scaling matrix *affects each component of a vector in a simple way.*

The diagonal entries scale each corresponding entry.

$$\begin{bmatrix} 1.5 & 0 \\ 0 & 0.7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1.5x \\ 0.7y \end{bmatrix}$$



$$\begin{bmatrix} 1.5 & 0 \\ 0 & 0.7 \end{bmatrix}$$

High level question:

When do matrices "behave" like scaling matrices "up to" change of basis?

Scaling and Eigenvectors

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The idea. Matrices behave like scaling matrices on eigenvectors.

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$$\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} (x\mathbf{e}_1 + y\mathbf{e}_2) = x2\mathbf{e}_1 + y(-3)\mathbf{e}_2$$

$$A \begin{bmatrix} x \\ y \end{bmatrix}_{\mathcal{B}} = A(x\mathbf{b}_1 + y\mathbf{b}_2) = x\lambda_1\mathbf{b}_1 + y\lambda_2\mathbf{b}_2$$

Scaling and Eigenvectors

The idea. Matrices behave like scaling matrices on eigenvectors.

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The fundamental question:

Can we expose this behavior in terms of a *matrix factorization*?

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Factorizations can:

- » make working with A easier
- » expose important information about A

Similar Matrices

$$A = PBP^{-1}$$

Definition. A matrix A is **similar** to a matrix B if there is some invertible matrix P such that $A = PBP^{-1}$.

A and B are the same up to a change of basis.

Similar Matrices and Eigenvalues

Theorem. Similar matrices have the same eigenvalues.

Verify:

Diagonalizable Matrices

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Diagonalizable matrices are the same as scaling matrices up to a change of basis.

Important: Not all Matrices are Diagonalizable

This is very different from the LU factorization.

We will need to figure out which matrices are diagonalizable.

Question. *Is the zero matrix diagonalizable?*

Application: Matrix Powers

only take the power of B

Theorem. If $A = PBP^{-1}$, then $A^k = PB^kP^{-1}$.

It may be easier to take the power of B (as in the case of diagonal matrices).

Verify:

How To: Matrix Powers

Question. Given A is diagonalizable, determine A^k .

Solution. Find its diagonalization PDP^{-1} and then compute PD^kP^{-1} .

Remember that

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}^k = \begin{bmatrix} a^k & 0 & 0 \\ 0 & b^k & 0 \\ 0 & 0 & c^k \end{bmatrix}$$

But how do we find the
diagonalization...

Diagonalization and Eigenvectors

Suppose we have a diagonalization

$$A = PDP^{-1}$$

What do we know about it?

Columns of P are eigenvectors

$$A = [\mathbf{p}_1 \ \mathbf{p}_2 \ \mathbf{p}_3] \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} [\mathbf{p}_1 \ \mathbf{p}_2 \ \mathbf{p}_3]^{-1}$$

Verify:

Columns of P are eigenvectors

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And the entries of D are the **eigenvalues** associated to each eigenvector.

Columns of P are eigenvectors

$$A = \overset{\text{eigenbasis}}{[p_1 \ p_2 \ p_3]} \overset{\text{eigenvalues}}{\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}} [p_1 \ p_2 \ p_3]^{-1}$$

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A diagonalization exposes a lot of information about A .

The Diagonalization Theorem

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We can use the same recipe to go in the other direction, given an eigenbasis, we can **build a diagonalization**.

Diagonalizing a Matrix

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The diagonal of D are the eigenvalues for each column of P .

The matrix P^{-1} is a change of basis to this eigenbasis of A .

Step 1: Eigenvalues

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & 3 \\ 3 & 3 & 1 \end{bmatrix}$$

Find all the eigenvalues of A .

Find the roots of $\det(A - \lambda I)$.

e.g.

$$\det(A - \lambda I) = -(\lambda - 1)(\lambda + 2)^2$$

Step 2: Eigenvectors

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & 3 \\ 3 & 3 & 1 \end{bmatrix}$$

$$\lambda_1 = 1$$

Find **bases** of the corresponding eigenspaces. $\lambda_2 = -2$

e.g.

$$\text{Nul}(A - I) = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$$\text{Nul}(A + 2I) = \text{span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Step 3: Construct P

If there are n eigenvectors from the previous step they form an **eigenbasis**.

Build the matrix with these vectors as the columns

e.g.

$$P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & 3 \\ 3 & 3 & 1 \end{bmatrix}$$

$$\lambda_1 = 1$$

$$\lambda_2 = -2$$

$$\text{Nul}(A - I) = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$$\text{Nul}(A + 2I) = \text{span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Step 5: Construct D

Build the matrix with eigenvalues as diagonal entries.

Note the order. It should be the same as the order of columns of P .

e.g.

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & 3 \\ 3 & 3 & 1 \end{bmatrix}$$

$$\lambda_1 = 1$$

$$\lambda_2 = -2$$

$$P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Step 6: Invert P

Find the inverse of P (we know how to do this).

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & 3 \\ 3 & 3 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Putting it Together

$$\begin{matrix} & \mathbf{A} & & \mathbf{P} & & \mathbf{D} & & \mathbf{P}^{-1} \\ \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & 3 \\ 3 & 3 & 1 \end{bmatrix} & = & \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} & \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1} \end{matrix}$$

How to: Diagonalizing a Matrix

Question. Find a diagonalization of $A \in \mathbb{R}^n$, or determine that A is not diagonalizable.

Solution.

1. Find the eigenvalues of A , and bases for their eigenspaces. If these eigenvectors don't form a basis of \mathbb{R}^n , then A is **not diagonalizable**.
2. Otherwise, build a matrix P whose columns are the eigenvectors of A .
3. Then build a diagonal matrix D whose entries are the eigenvalues of A *in the same order*.
4. Invert P .
5. The diagonalization of A is PDP^{-1} .

We know how to do every step, its
a matter of putting it all
together.

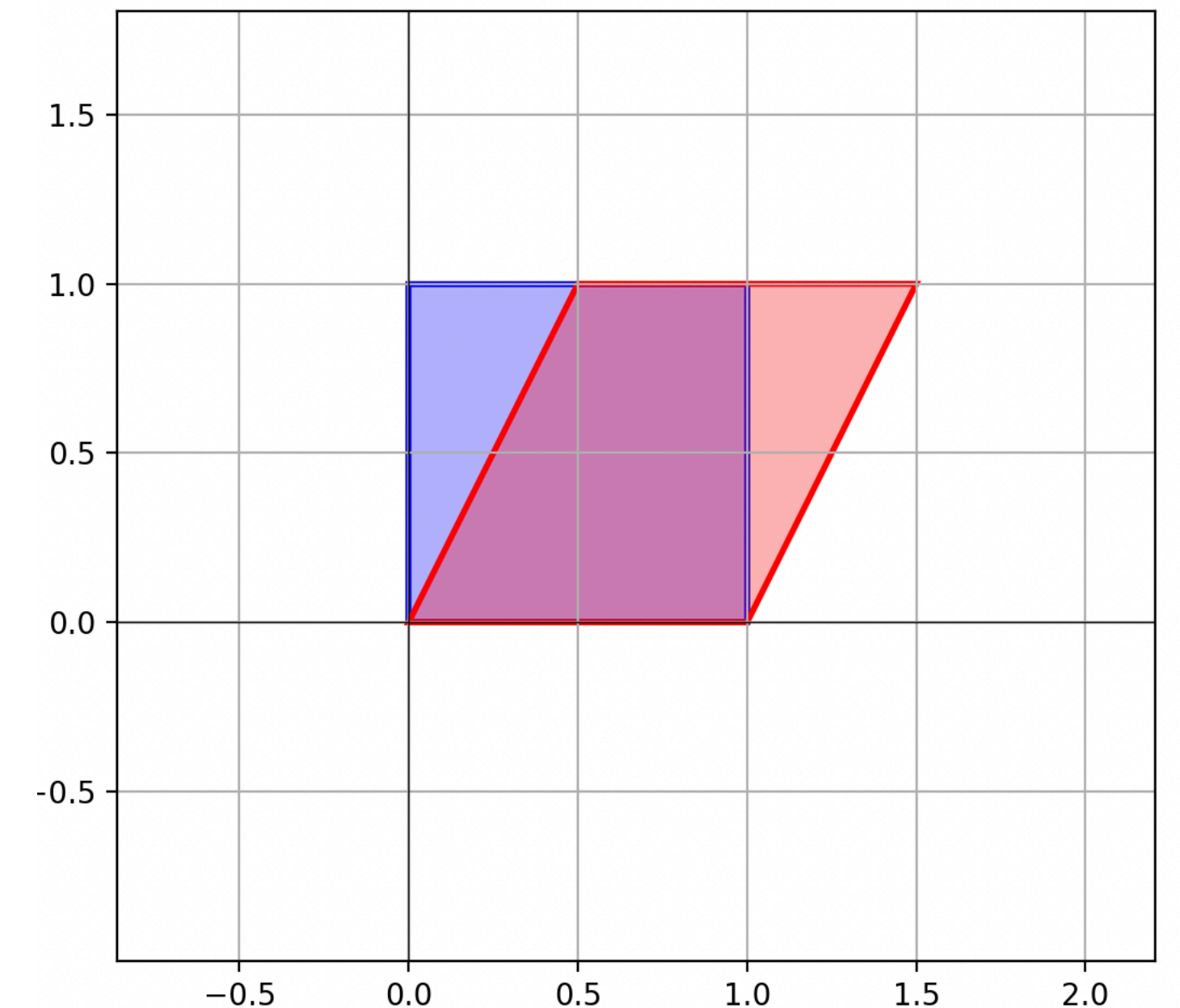
Example of Failure: Shearing

$$A = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}$$

The shearing matrix has a single eigenvalue with an eigenspace of dimension 1.

We can't build an eigenbasis of \mathbb{R}^2 for A .

In other words, A is not diagonalizable.



Important case: Distinct Eigenvalues

ex.
$$\begin{bmatrix} 1 & -3 & 4 & 2 \\ 0 & -2 & 3 & -1 \\ 0 & 0 & 10 & 5 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

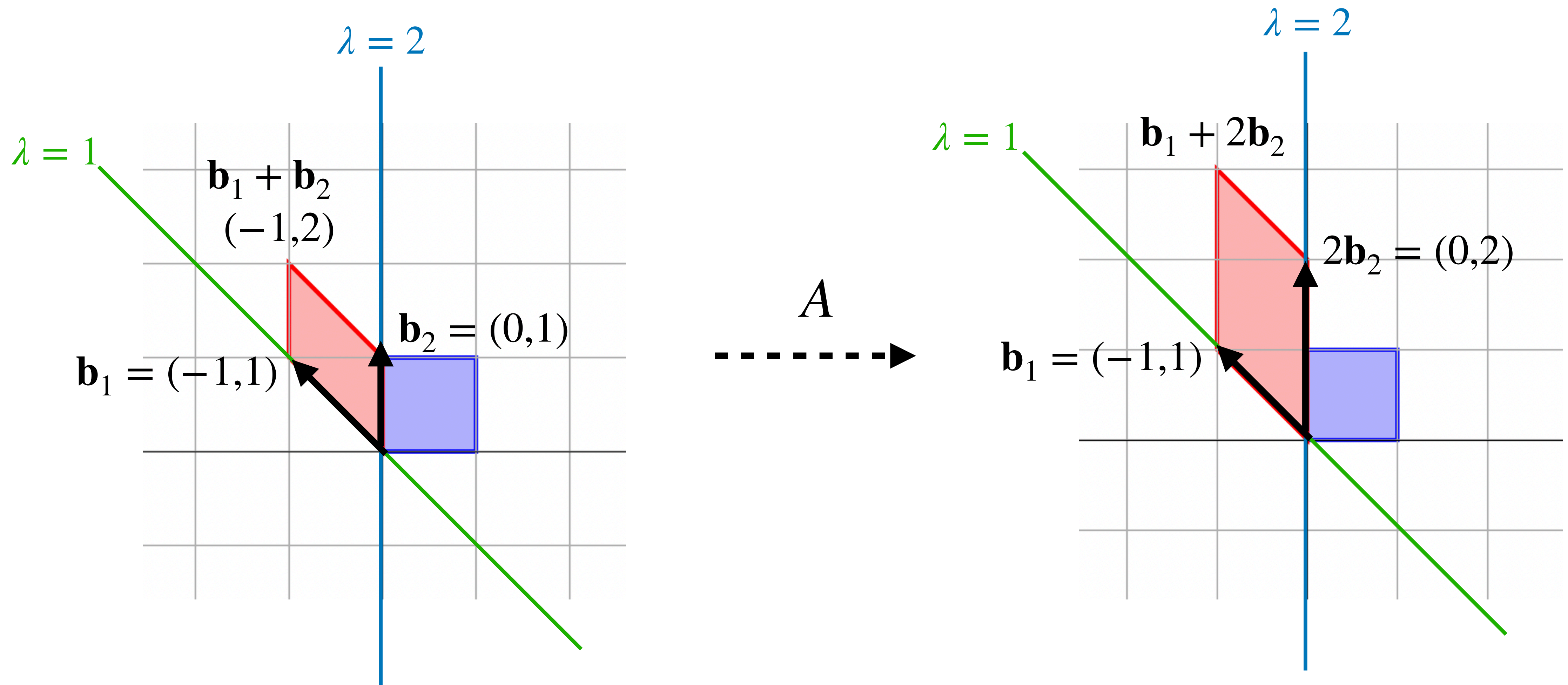
Theorem. If an $n \times n$ matrix has n distinct eigenvalues, then it is diagonalizable.

This is because eigenvectors with distinct eigenvalues are *linearly independent*.

The Picture

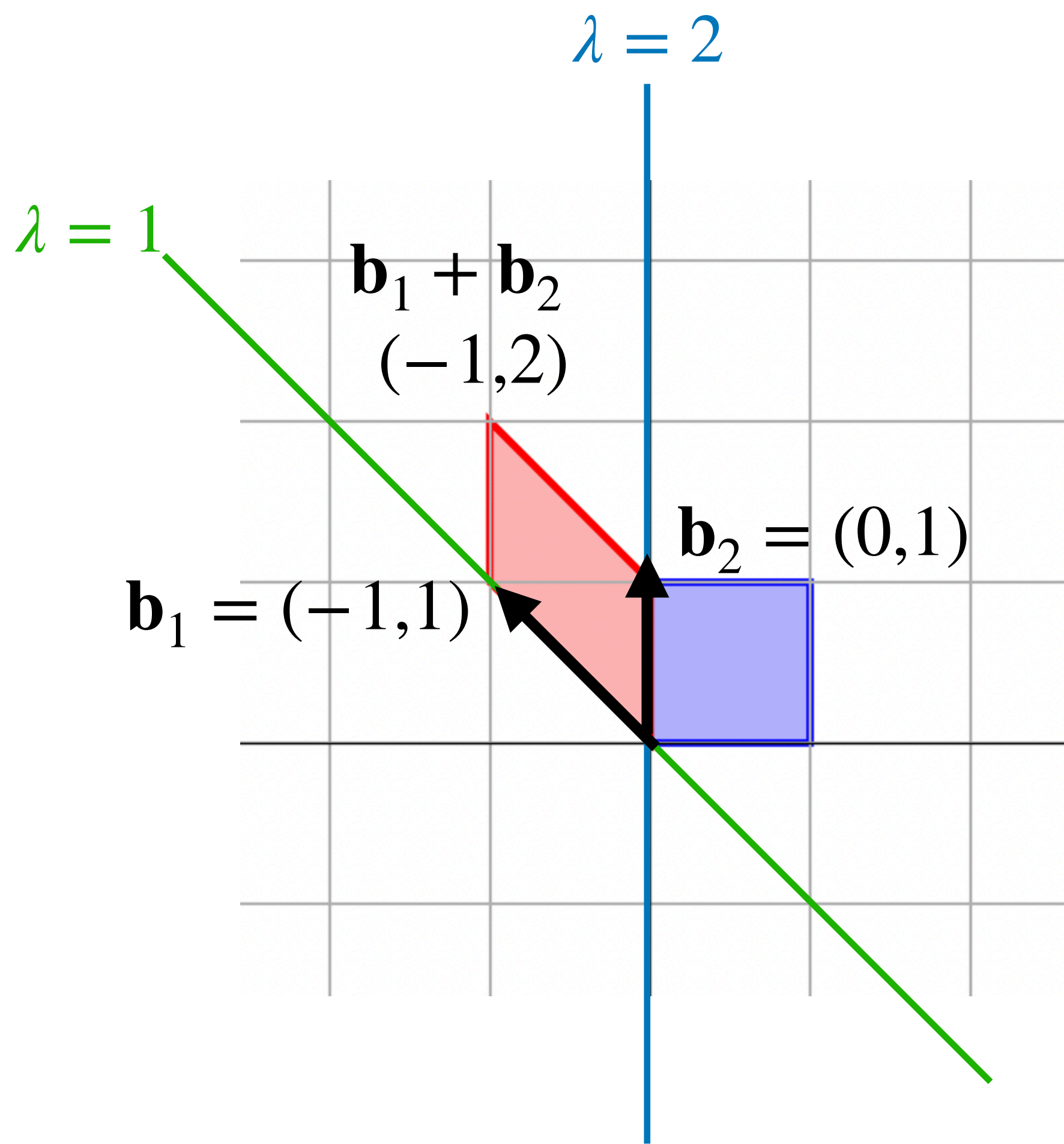
Example (Geometric)

$$A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$$

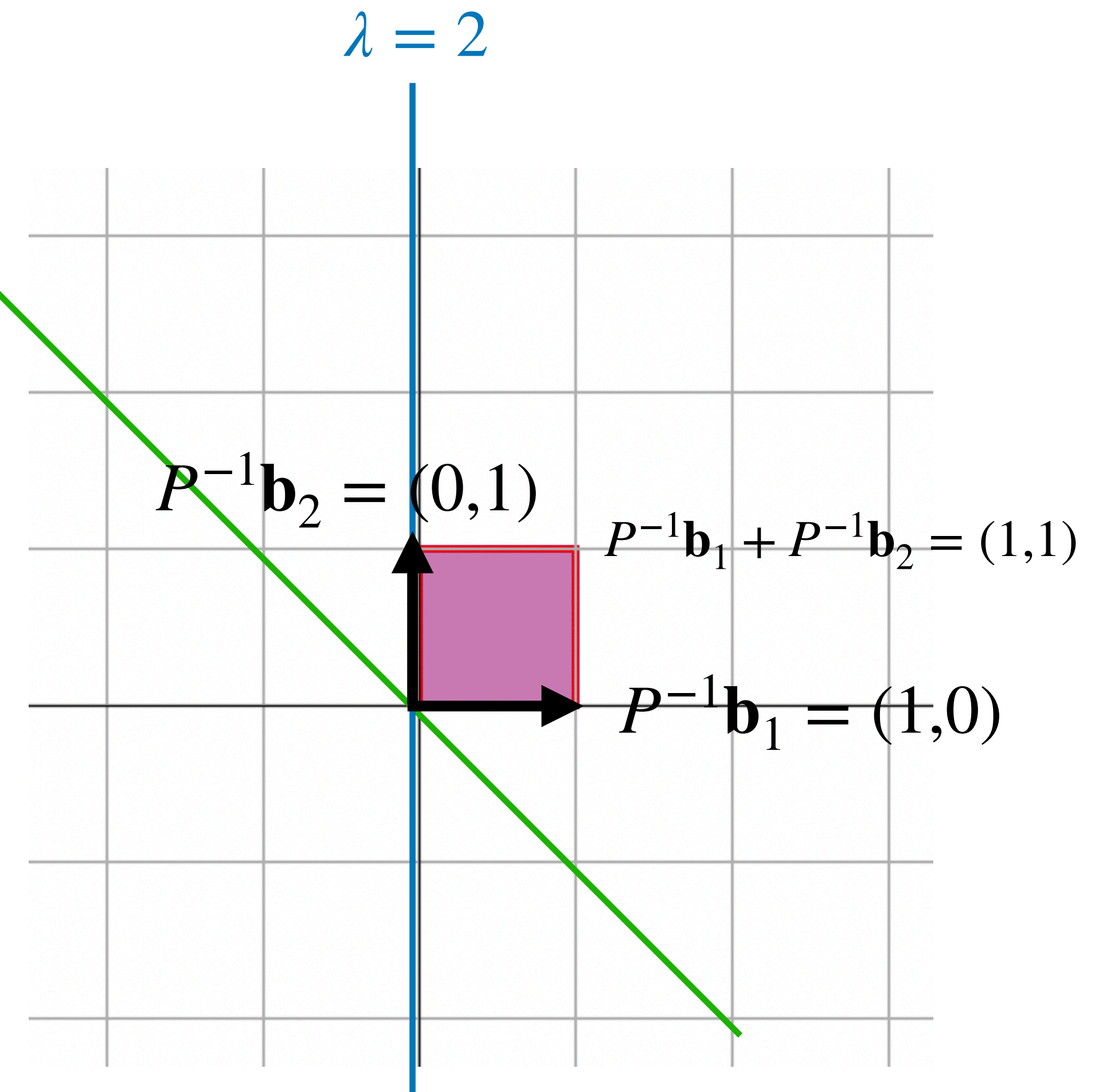


Example (Geometric)

$$P^{-1} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}^{-1}$$

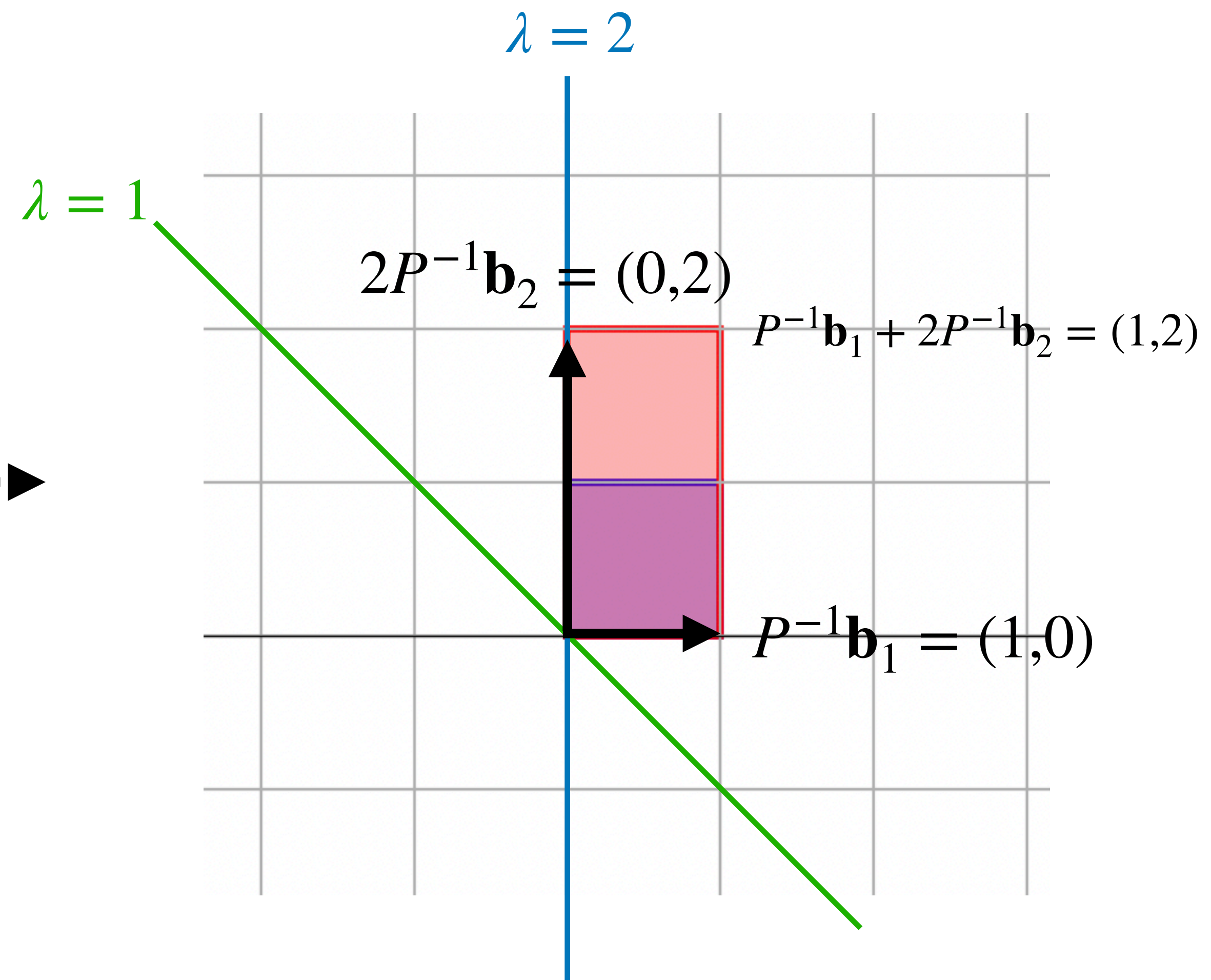
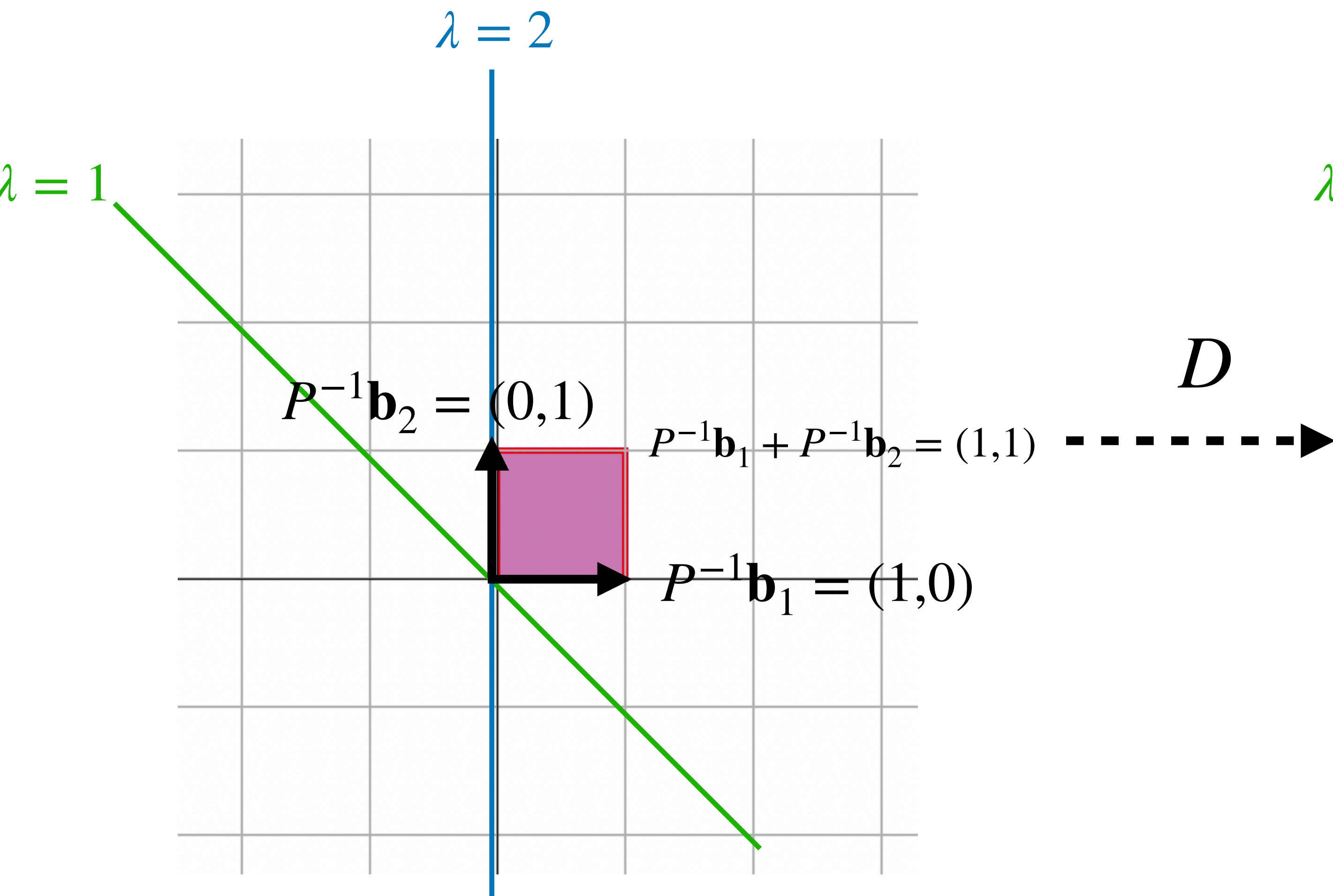


P^{-1}



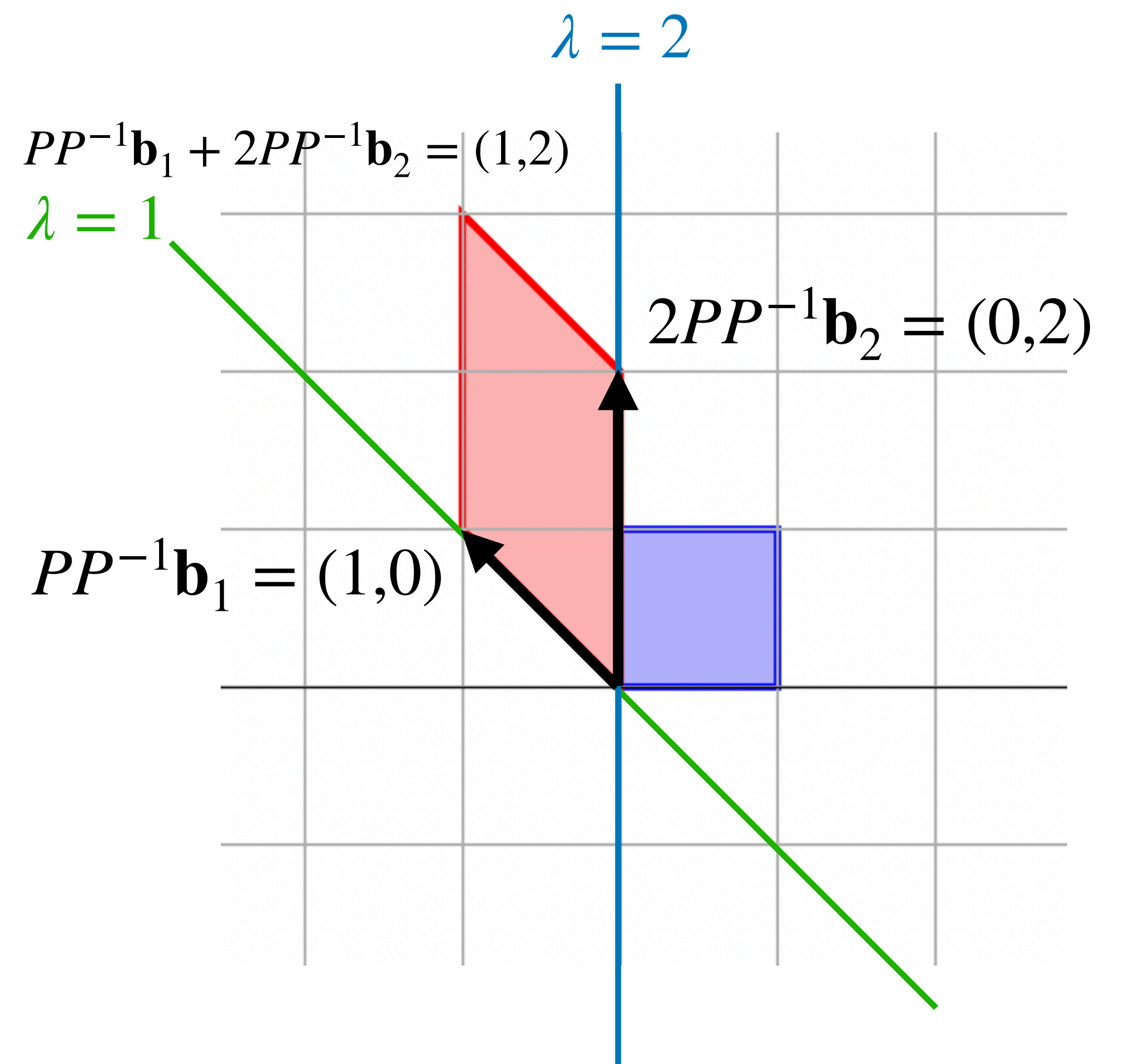
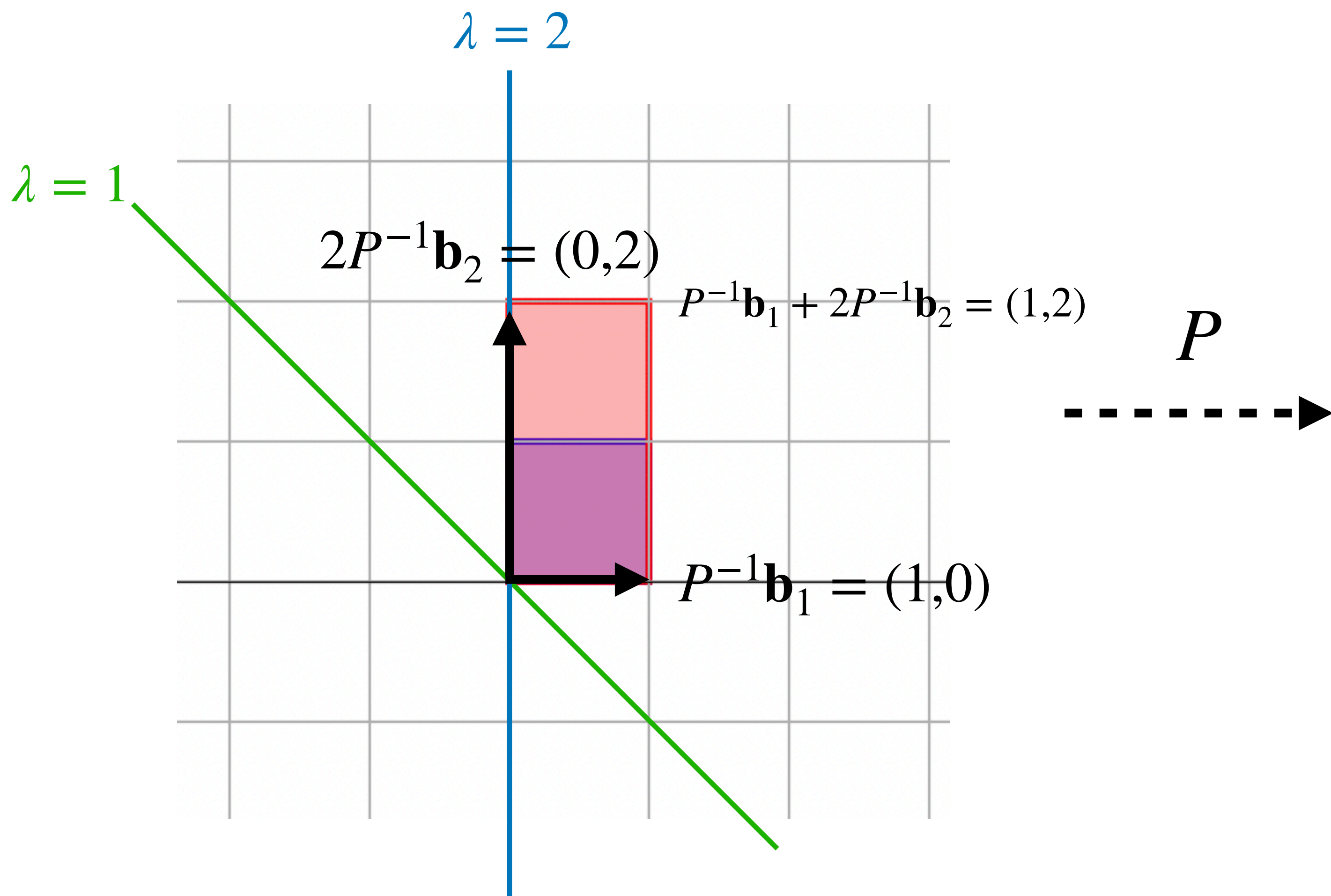
Example (Geometric)

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

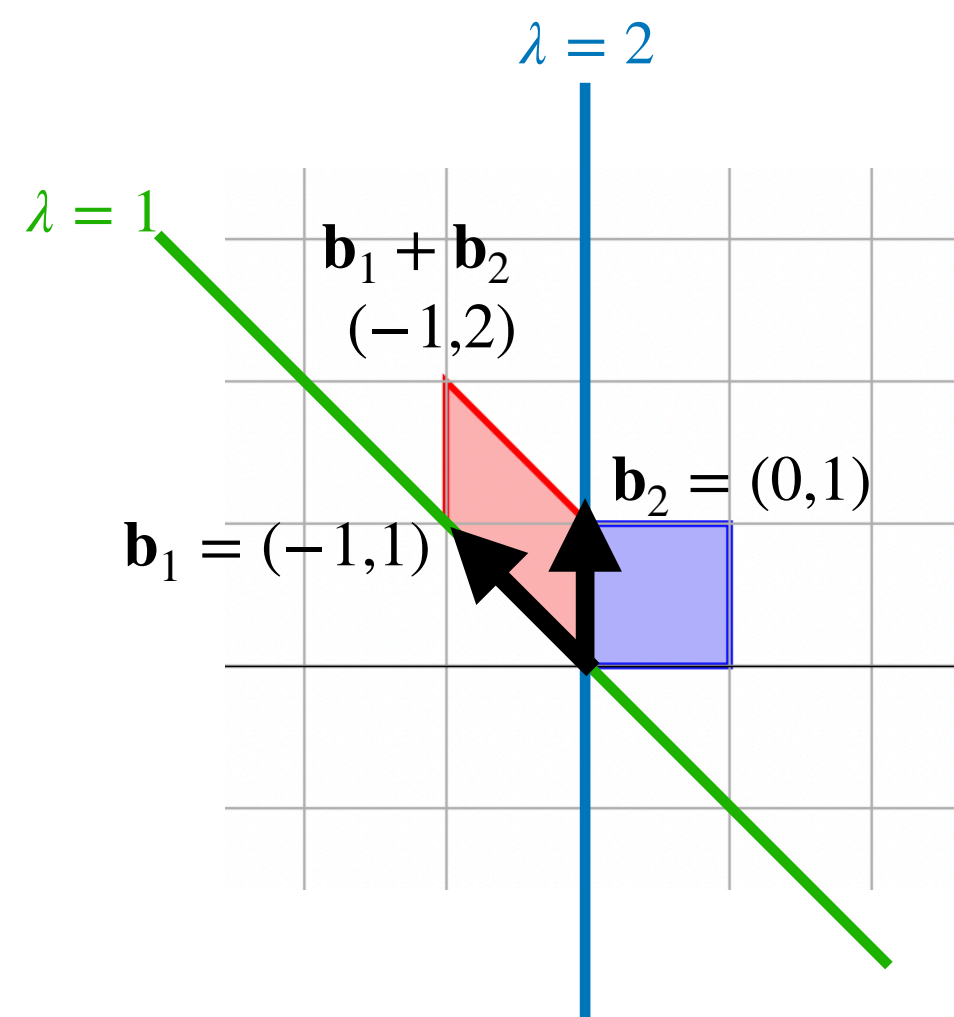


Example (Geometric)

$$P = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$$



Example (Geometric)



$$A = PDP^{-1}$$

$$\begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}^{-1}$$

