

Analytic Geometry in \mathbb{R}^n

Geometric Algorithms

Lecture 21

Introduction

Recap Problem

Let A be a 4×4 matrix with eigenvalues 3 and -2 where $\dim(\text{Nul}(A + 2I)) = 3$.

True or False: *A must be diagonalizable.*

Answer: True

The set of eigenvectors we get from the diagonalization procedure is of size 4, which means there is an eigenbasis of \mathbb{R}^4 for A .

Objectives

1. Recall what we learned in algebra class.
2. Connect the familiar notions of lengths, distances, and angles to inner products.
3. Begin discussing the fundamental concept of orthogonality.

Keywords

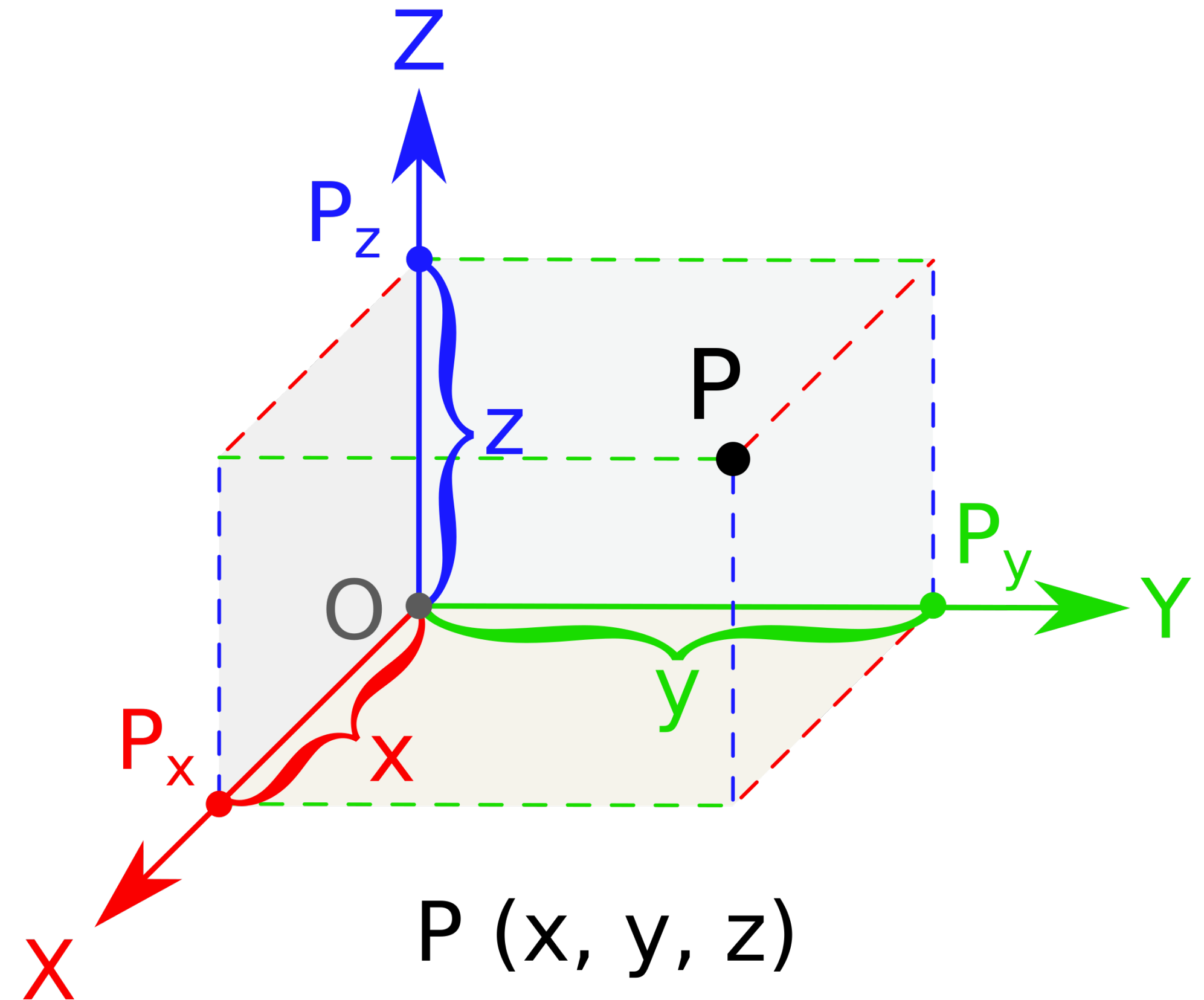
inner product

norm

orthogonal

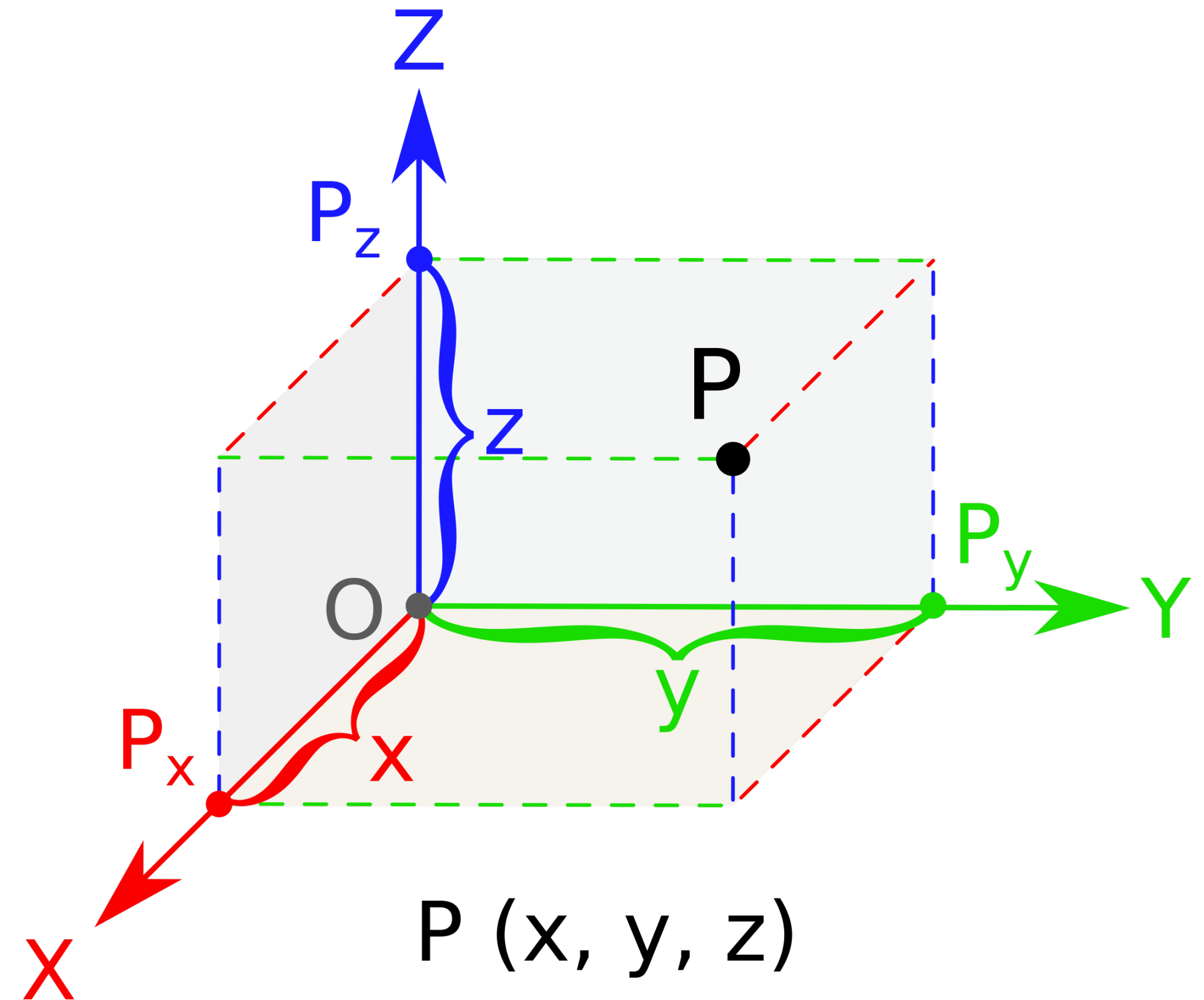
Motivation

What is Analytic Geometry?



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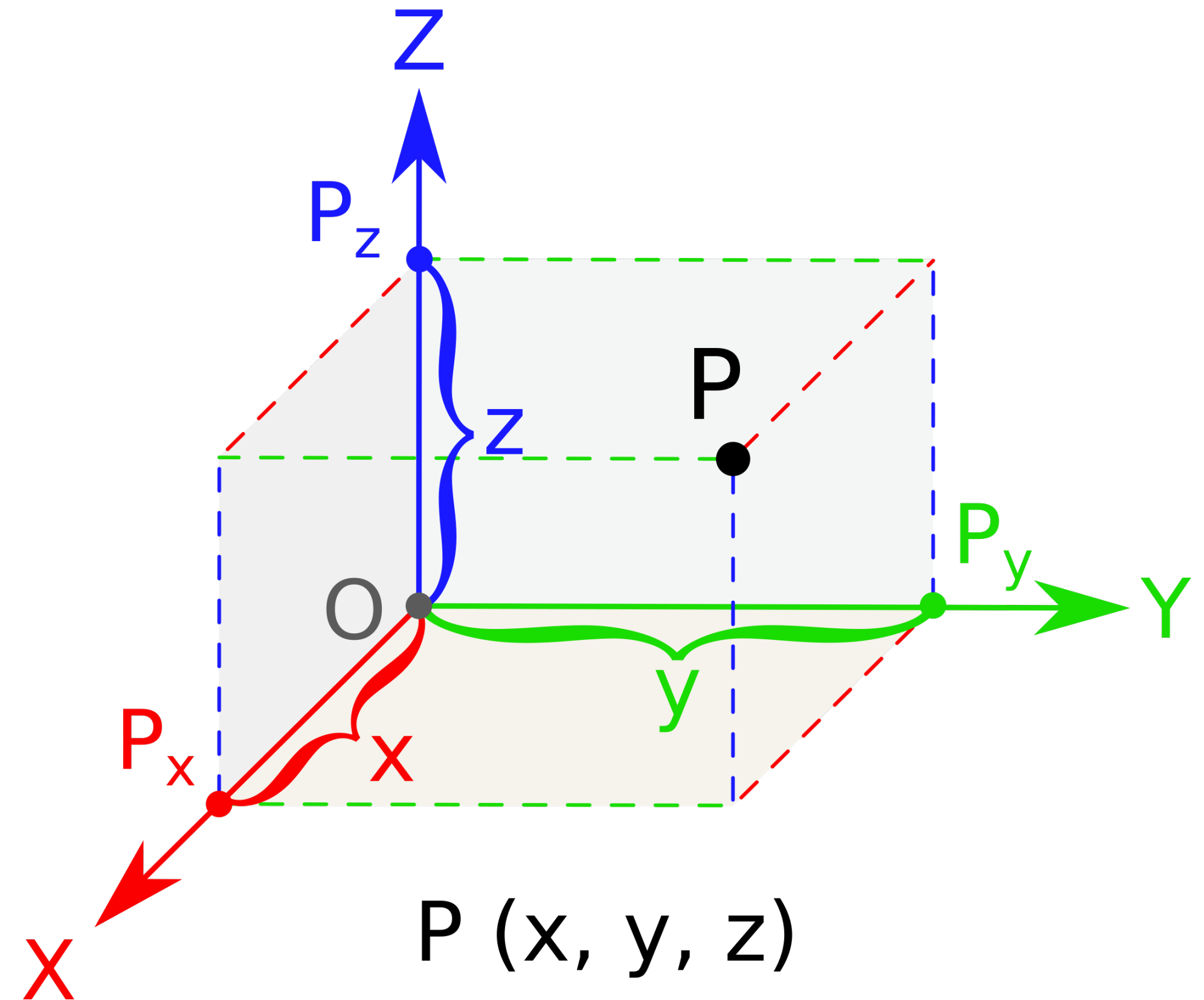
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We're interested in equations about lines, curves, shapes, angles, etc.

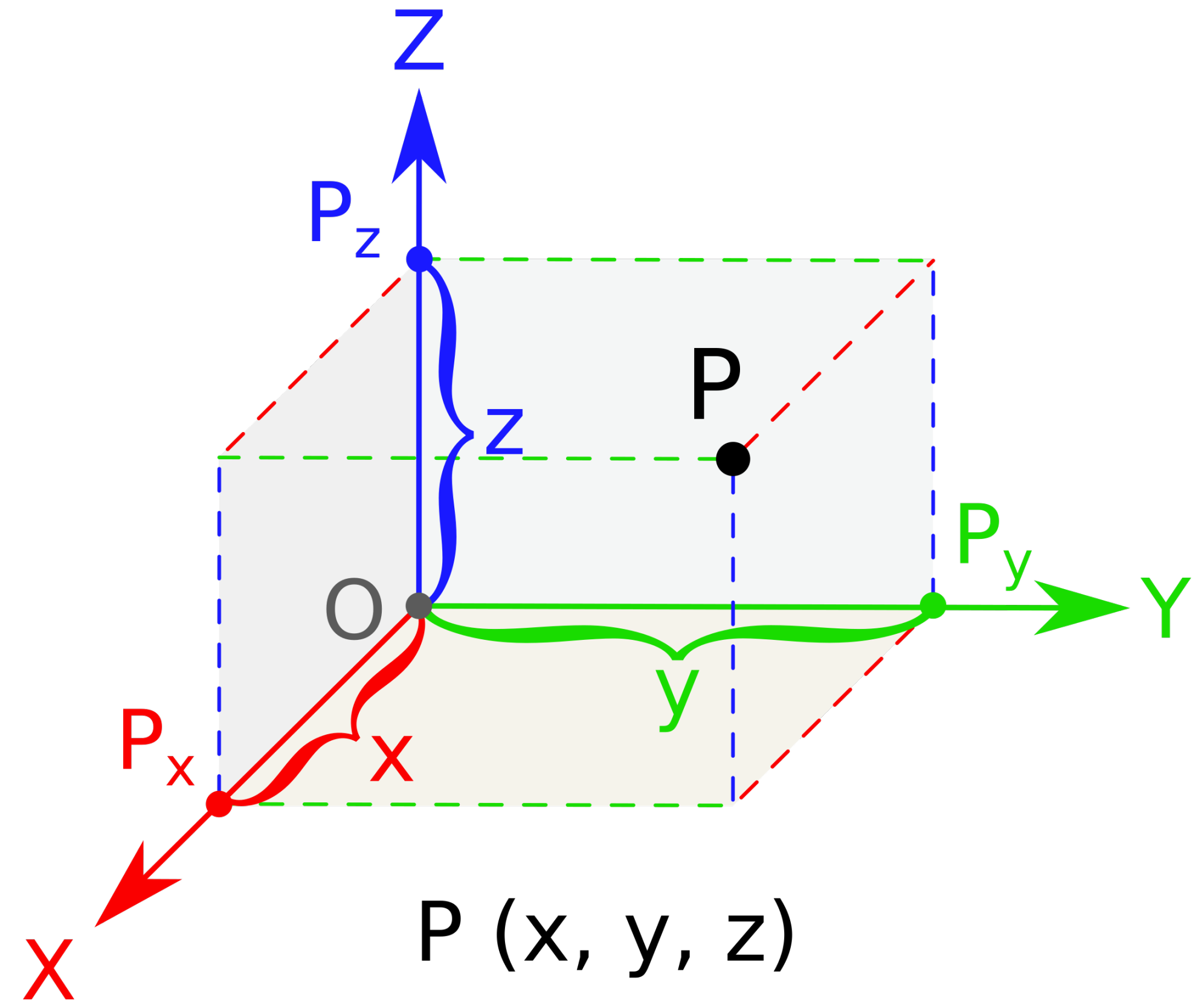


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The fundamental concepts are:



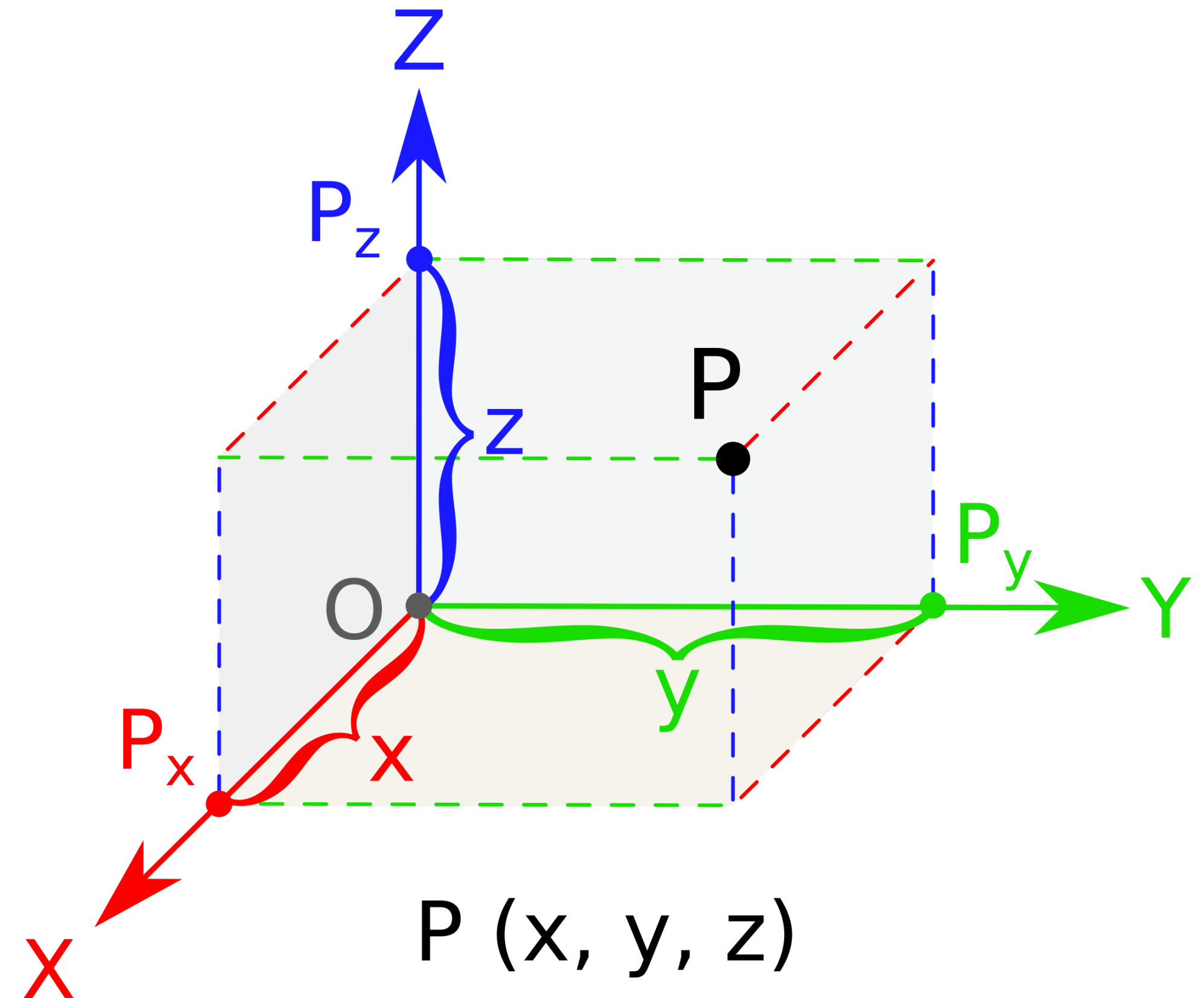
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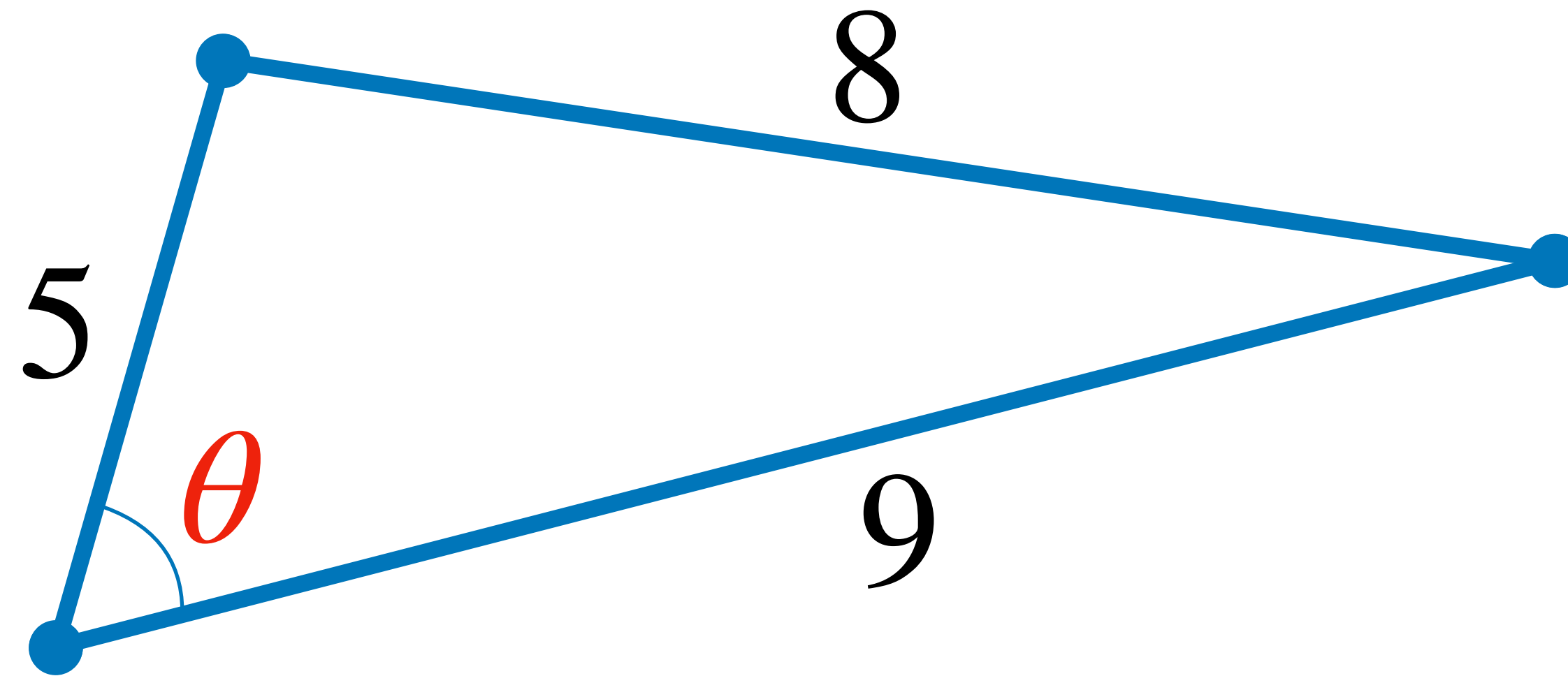
We're interested in equations about lines, curves, shapes, angles, etc.

The fundamental concepts are:

- » distance
- » position
- » area
- » angle

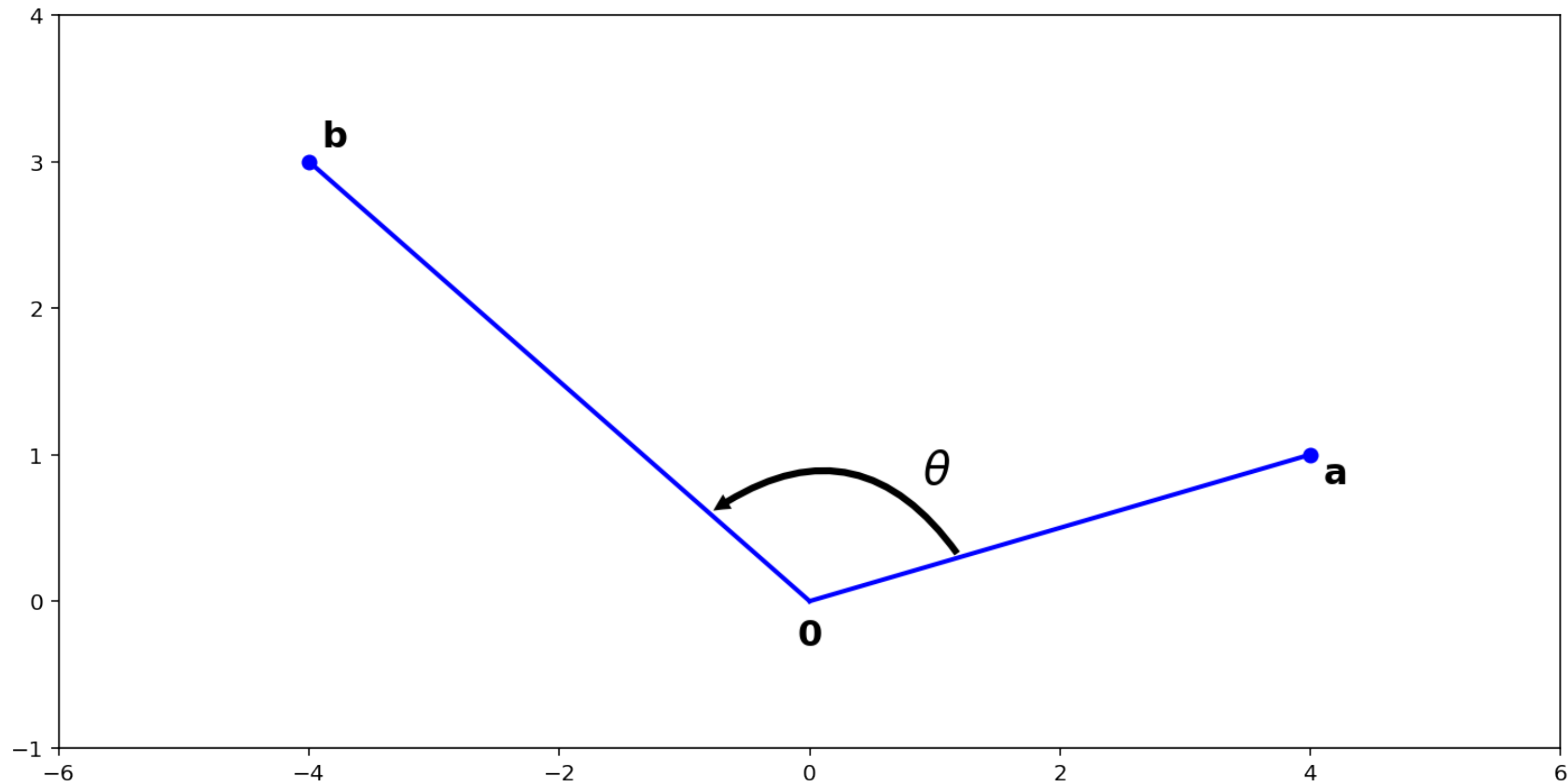


A Potentially Familiar Example



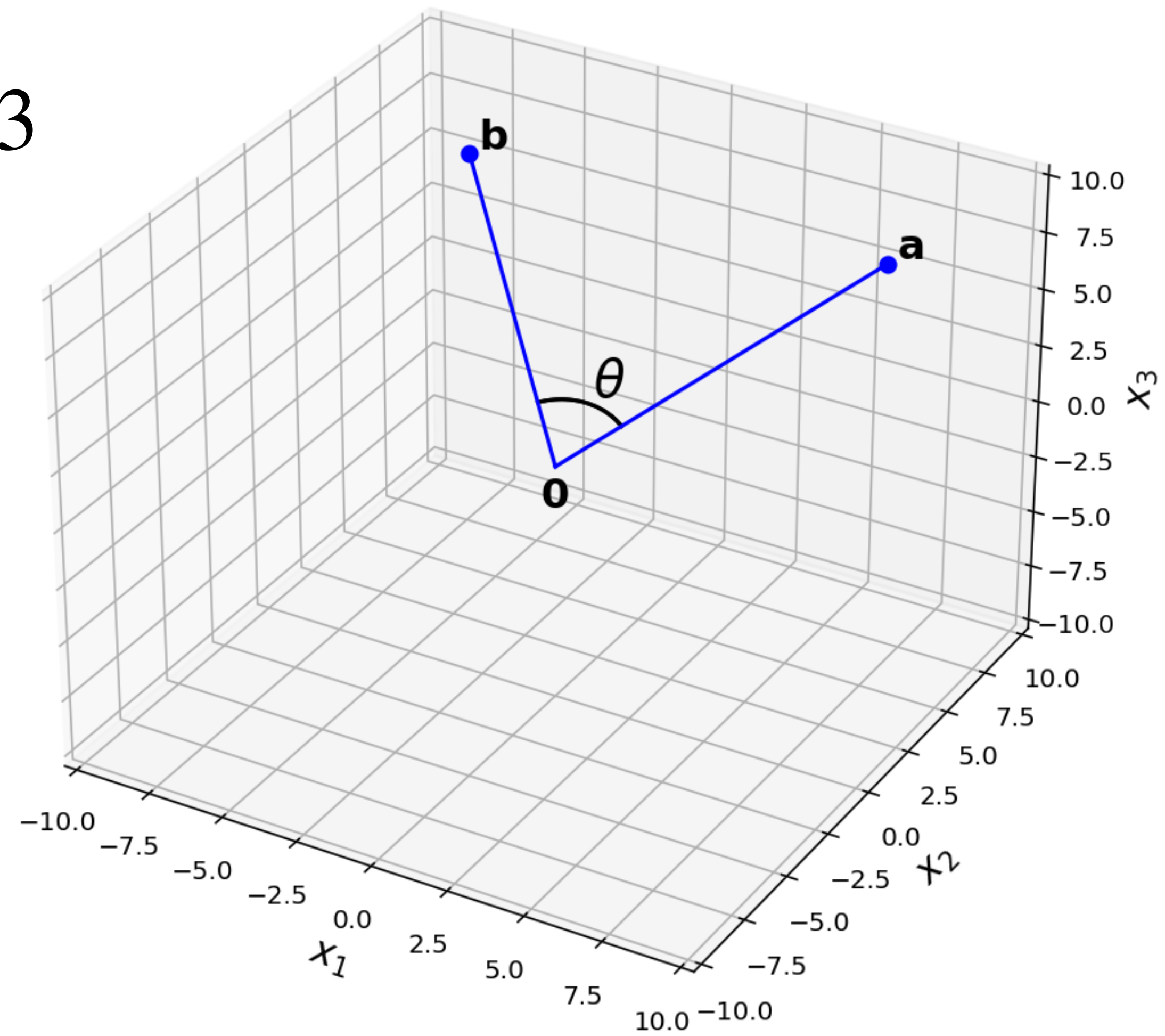
What is the value of θ ?

Angles in \mathbb{R}^2



What is the value of θ ?

Angles in \mathbb{R}^3



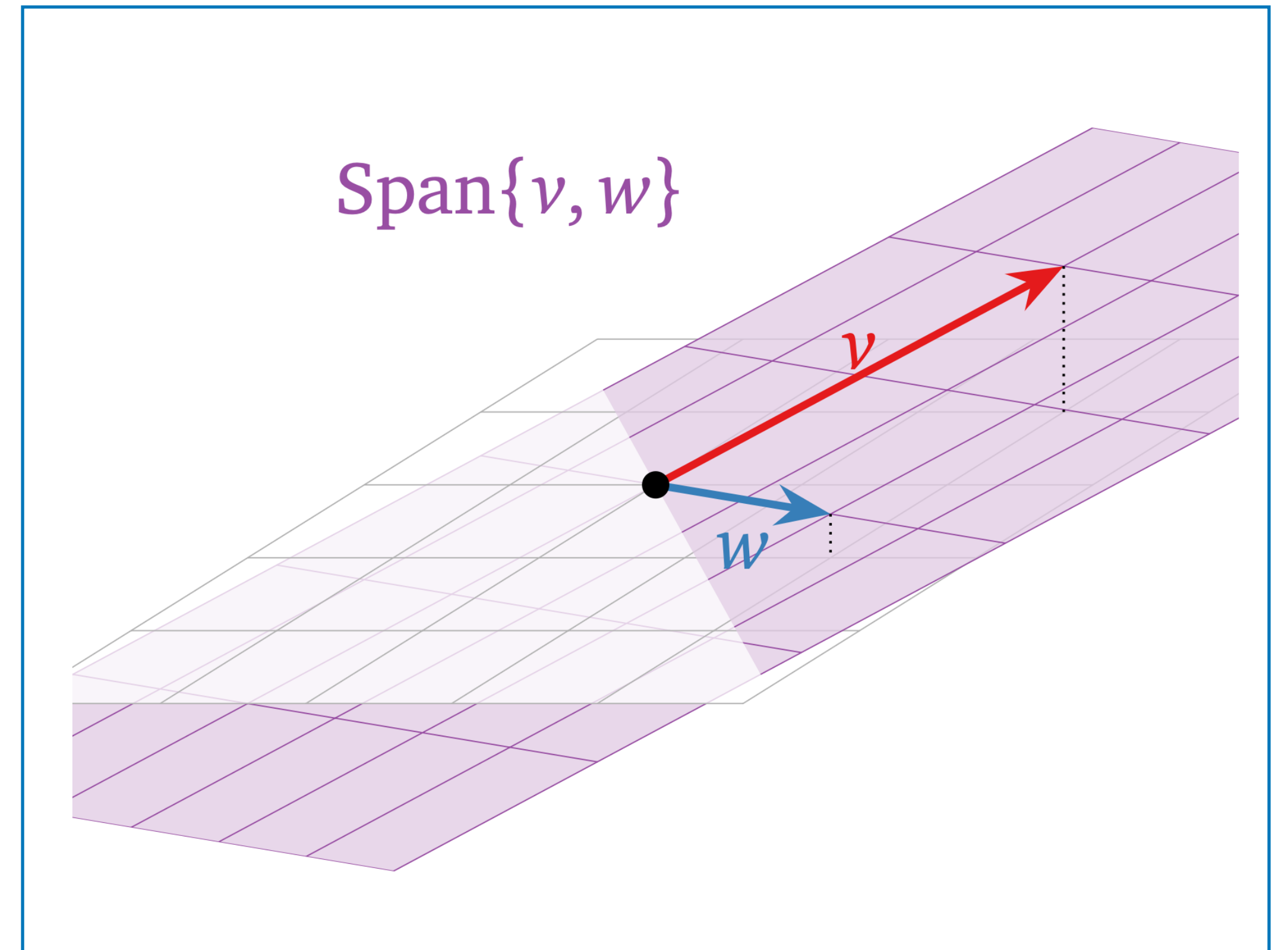
What is the value of θ ?

The First Key Idea

Angles make sense in *any* dimension.

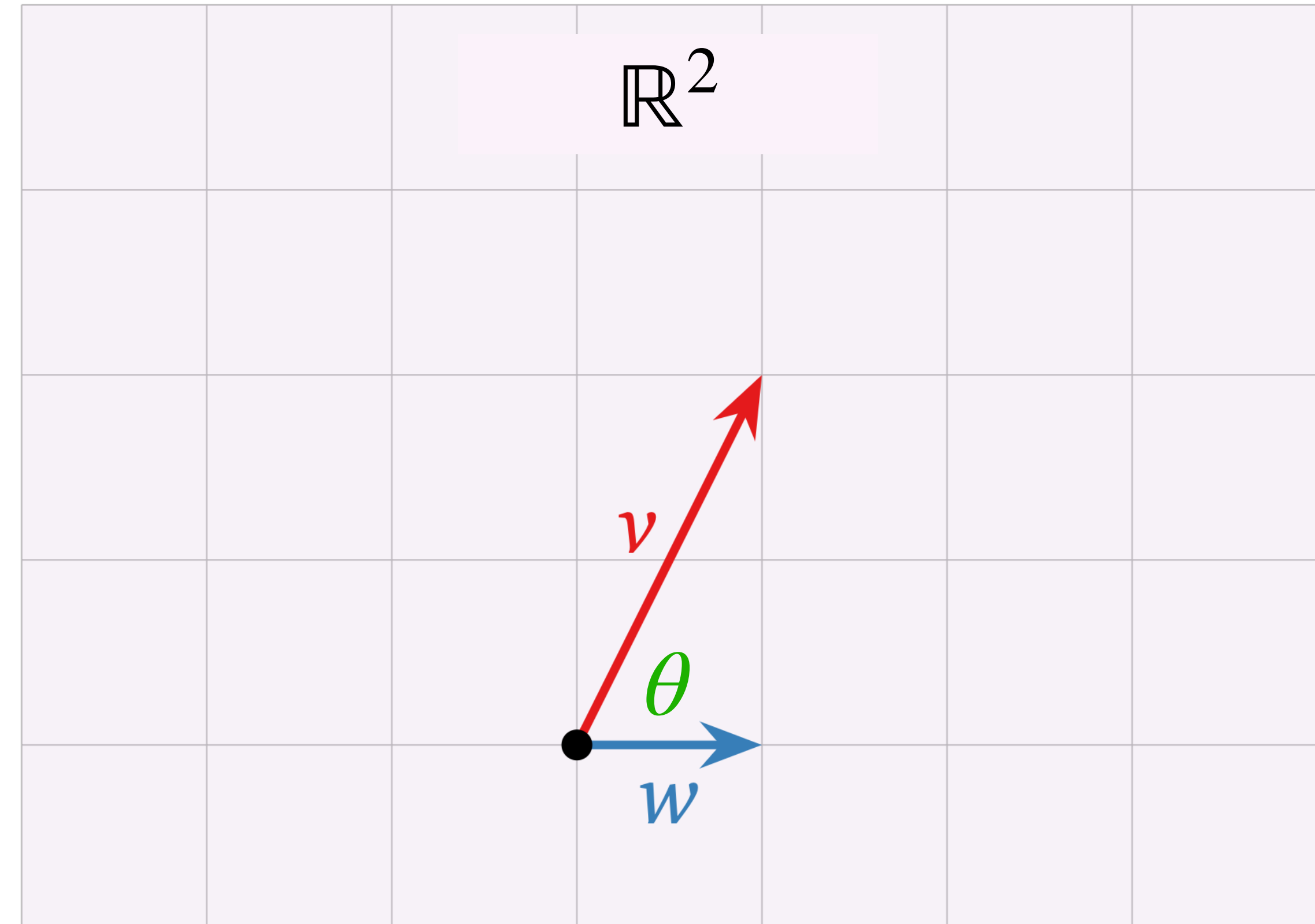
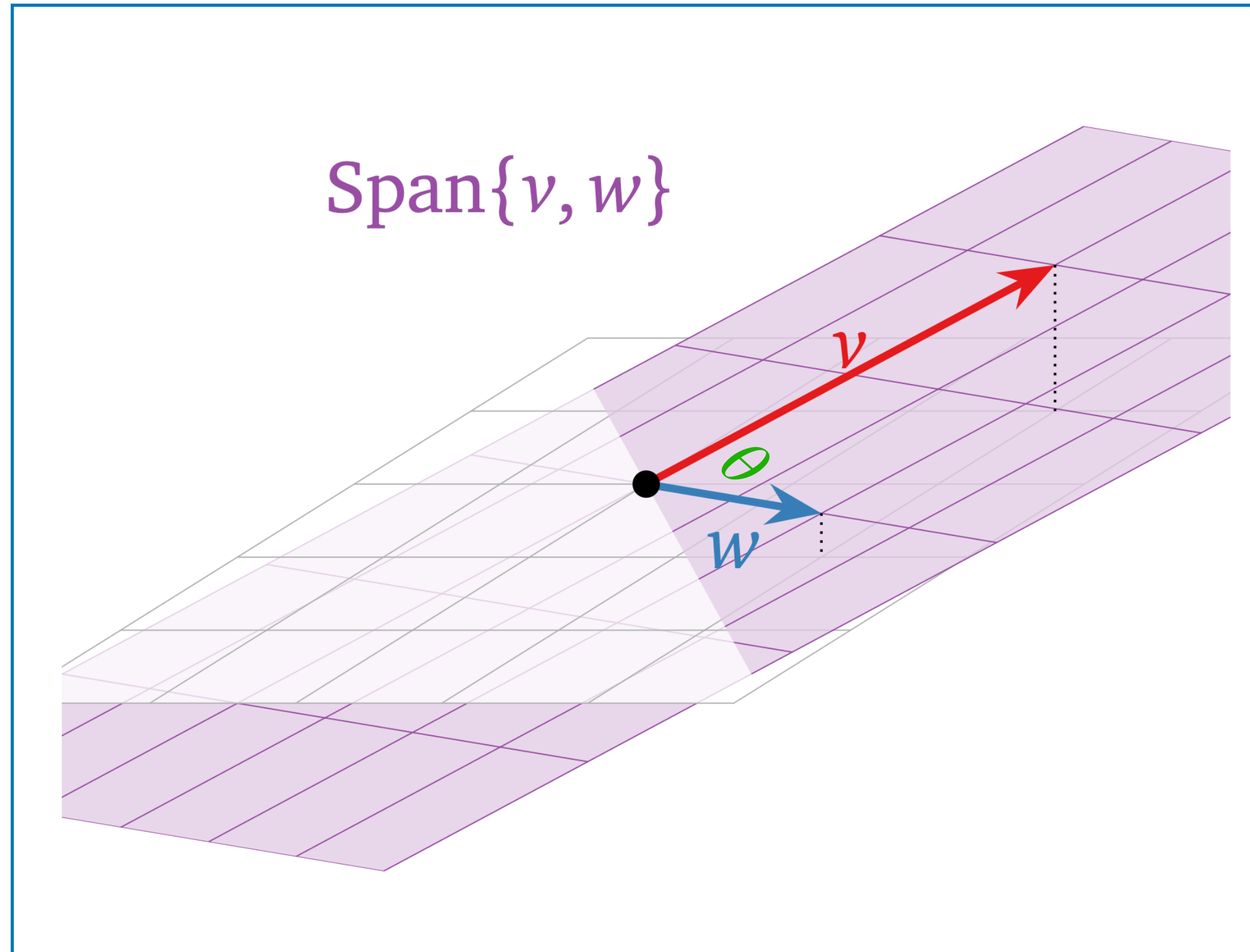
Any pair of vectors in \mathbb{R}^n span a (2D) plane.

(We could formalize this via change of bases)



The Picture

We can do "normal" analytic geometry here



change of basis from $\text{span}\{v, w\}$ to \mathbb{R}^2

A Fundamental Question

Doing this change of basis every time we want to do geometry is a lot of work...

Can we do it directly using ideas we've been learning?

Recall: Inner Products

Recall: Inner Products

$$[u_1 \quad u_2 \quad u_3 \quad u_4] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4$$

Recall: Inner Products

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Definition. The **inner product** of two vectors \mathbf{u} and \mathbf{v} in \mathbb{R}^n is

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}$$

Recall: Inner Products

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Definition. The **inner product** of two vectors \mathbf{u} and \mathbf{v} in \mathbb{R}^n is **a.k.a. dot product**

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All of the basic concepts of analytic geometry can be defined *in terms of inner products*.

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Definition (Advanced). An **inner product space** is a vector space with an inner product function.

Inner product spaces (like \mathbb{R}^n) are places where you can do analytic geometry.

The Fundamental Question

How do we do analytic geometry,
given we have an inner product?

Inner Products

Recall: Inner Products (Again)

$$[u_1 \quad u_2 \quad u_3 \quad u_4] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4$$

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Example

$$\mathbf{u} = \begin{bmatrix} -3 \\ 2 \\ 1 \\ 4 \end{bmatrix}$$

$$\mathbf{v} = \begin{bmatrix} 2 \\ 0.5 \\ -1 \\ 3 \end{bmatrix}$$

Algebraic Properties of Inner Products

- $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ (symmetry)
 - $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = (\mathbf{u} \cdot \mathbf{w}) + (\mathbf{v} \cdot \mathbf{w})$
 - $(\alpha\mathbf{u}) \cdot \mathbf{v} = \alpha(\mathbf{u} \cdot \mathbf{v})$
- } linearity in the first argument
- $\mathbf{u} \cdot \mathbf{u} \geq 0$ (nonnegativity)
 - $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$

Verifying Additivity

$$\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle =$$

$$= \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$$

Homogeneity in the Right Argument

$$\langle \mathbf{v}, c\mathbf{u} \rangle = c \langle \mathbf{v}, \mathbf{u} \rangle$$

Verify:

An Aside: What is this linear transformation?

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \mapsto \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Let's find the matrix for this transformation:

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Nonnegativity

$$\langle \mathbf{v}, \mathbf{v} \rangle = \sum_{i=1}^n v_i^2$$

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Therefore $\langle \mathbf{v}, \mathbf{v} \rangle$ is always nonnegative.

Question. What happens when we scale a vector to make it longer?

Nonnegativity and Scaling

$$\langle c\mathbf{v}, c\mathbf{v} \rangle = c^2 \langle \mathbf{v}, \mathbf{v} \rangle = c^2 \sum_{i=1}^n v_i^2$$

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Increasing the length of a vector increases its inner product with itself.

This means $\langle \mathbf{v}, \mathbf{v} \rangle$ is capturing some notion of magnitude.

The Fundamental Question

How does this all connect back to
distances and angles?

Question

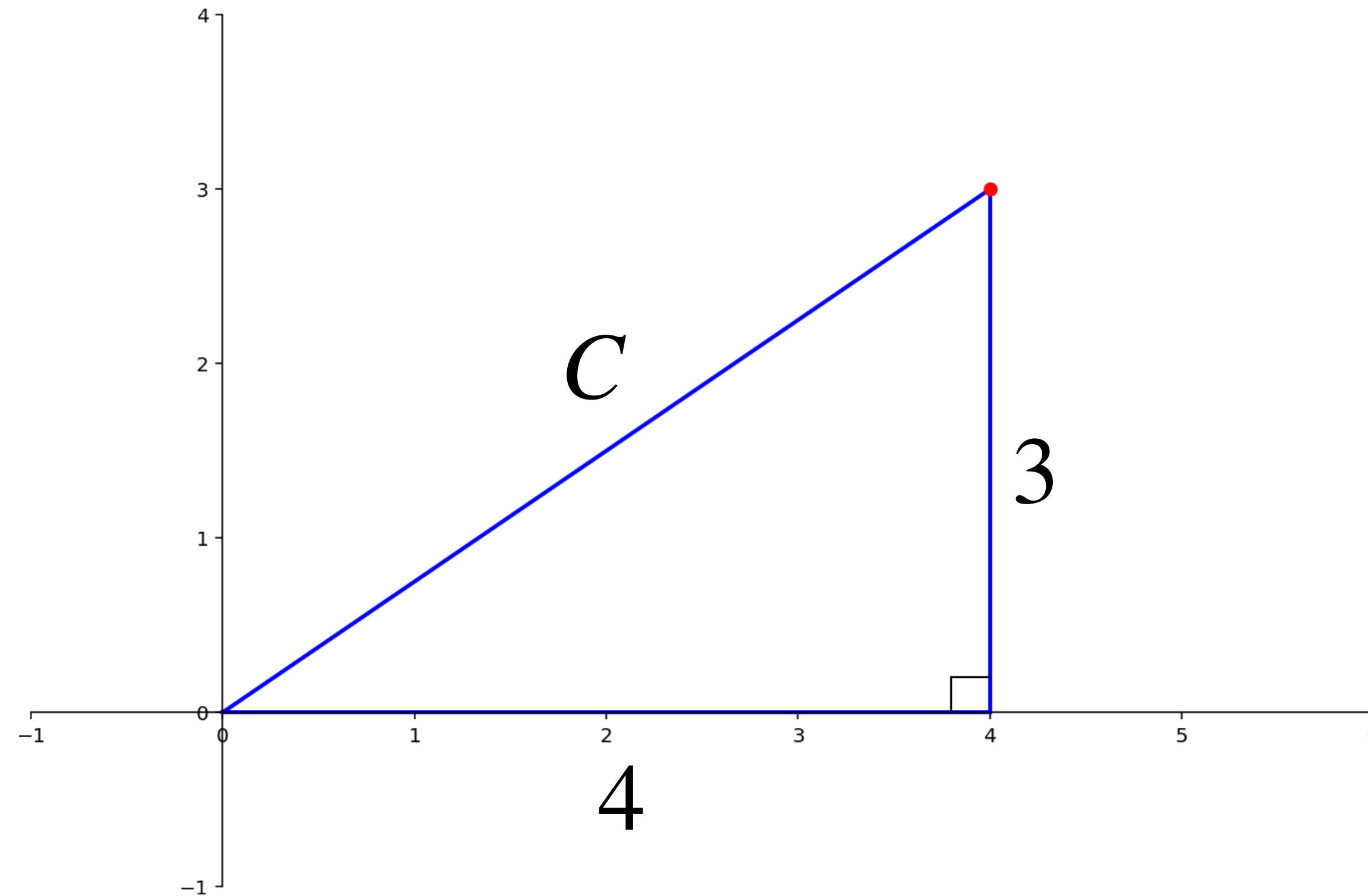
- $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
- $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = (\mathbf{u} \cdot \mathbf{w}) + (\mathbf{v} \cdot \mathbf{w})$

Simplify the expression $\langle \mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle$ using the properties of inner products.

Answer: $\langle \mathbf{u}, \mathbf{u} \rangle - \langle \mathbf{v}, \mathbf{v} \rangle$

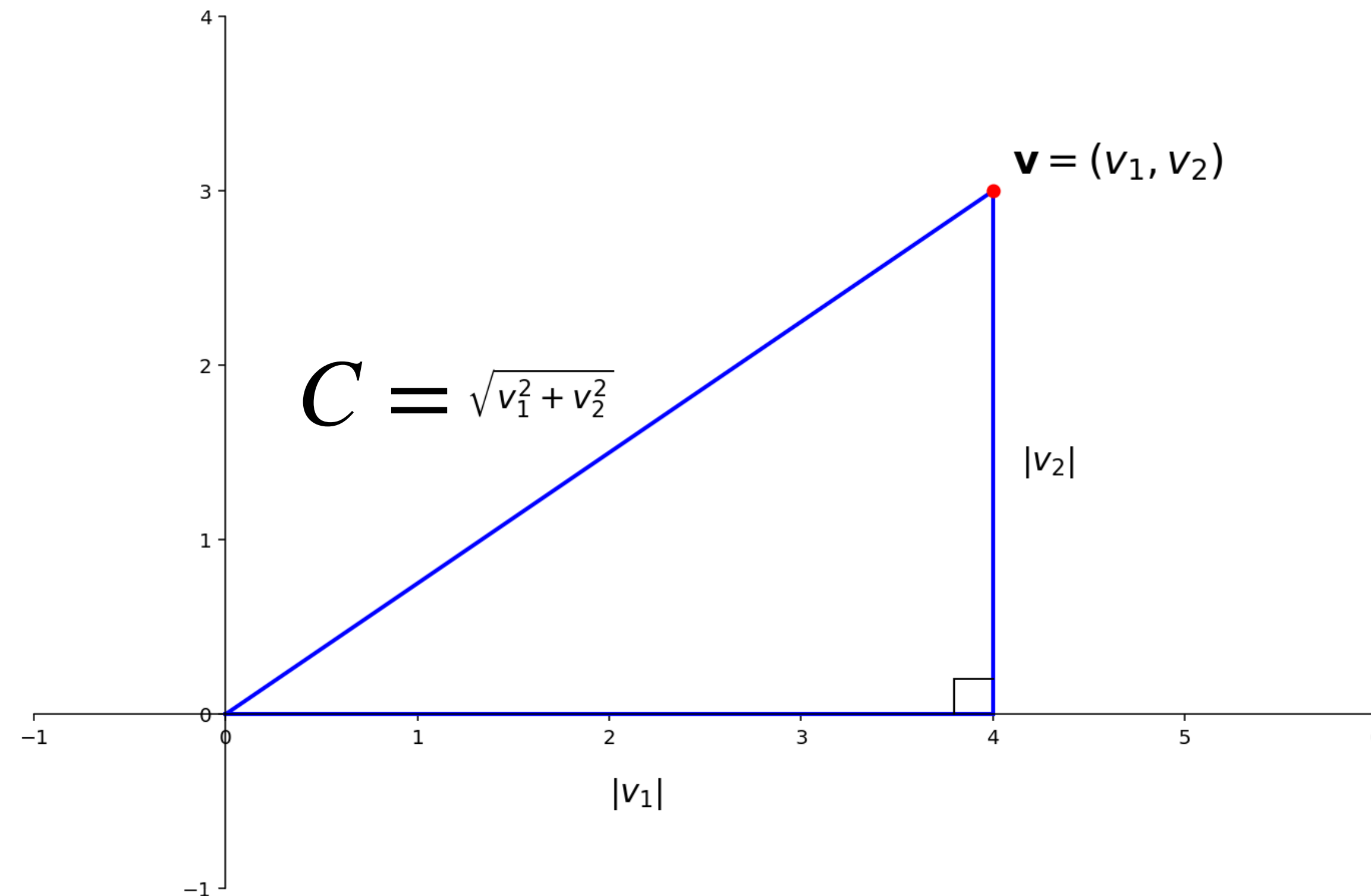
Norms (Lengths/Distances)

Another Potentially Familiar Question



How long is the side C?

Pythagorean Theorem

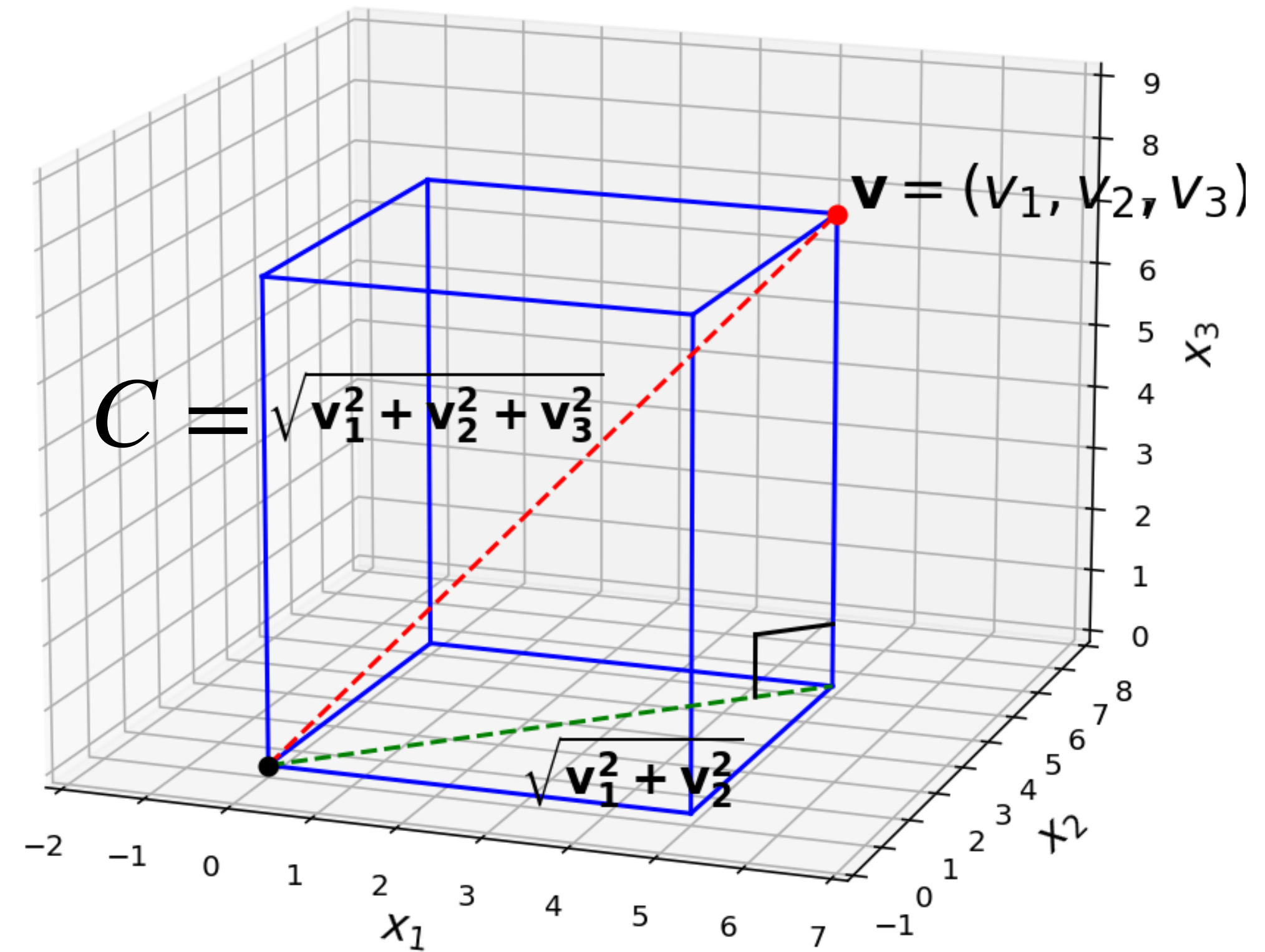


Theorem (Pythagoras). *For a right triangle, the square of the length of the hypotenuse is the sum of the squares of the lengths of the remaining two sides.*

This still works in \mathbb{R}^3

Theorem (Pythagoras). $C = \sqrt{v_1^2 + v_2^2 + v_3^2}$

Verify:



Norm

Definition. The (ℓ^2) norm of a vector \mathbf{v} in \mathbb{R}^n is

$$\|\mathbf{v}\| = \left\| \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \right\| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2} = \sqrt{\sum_{i=1}^n v_i^2}$$

The norm of a vector is the square root of the sum of the squares of its entries.

Norms and Inner Products

Definition. The ℓ^2 norm of a vector \mathbf{v} in \mathbb{R}^n is

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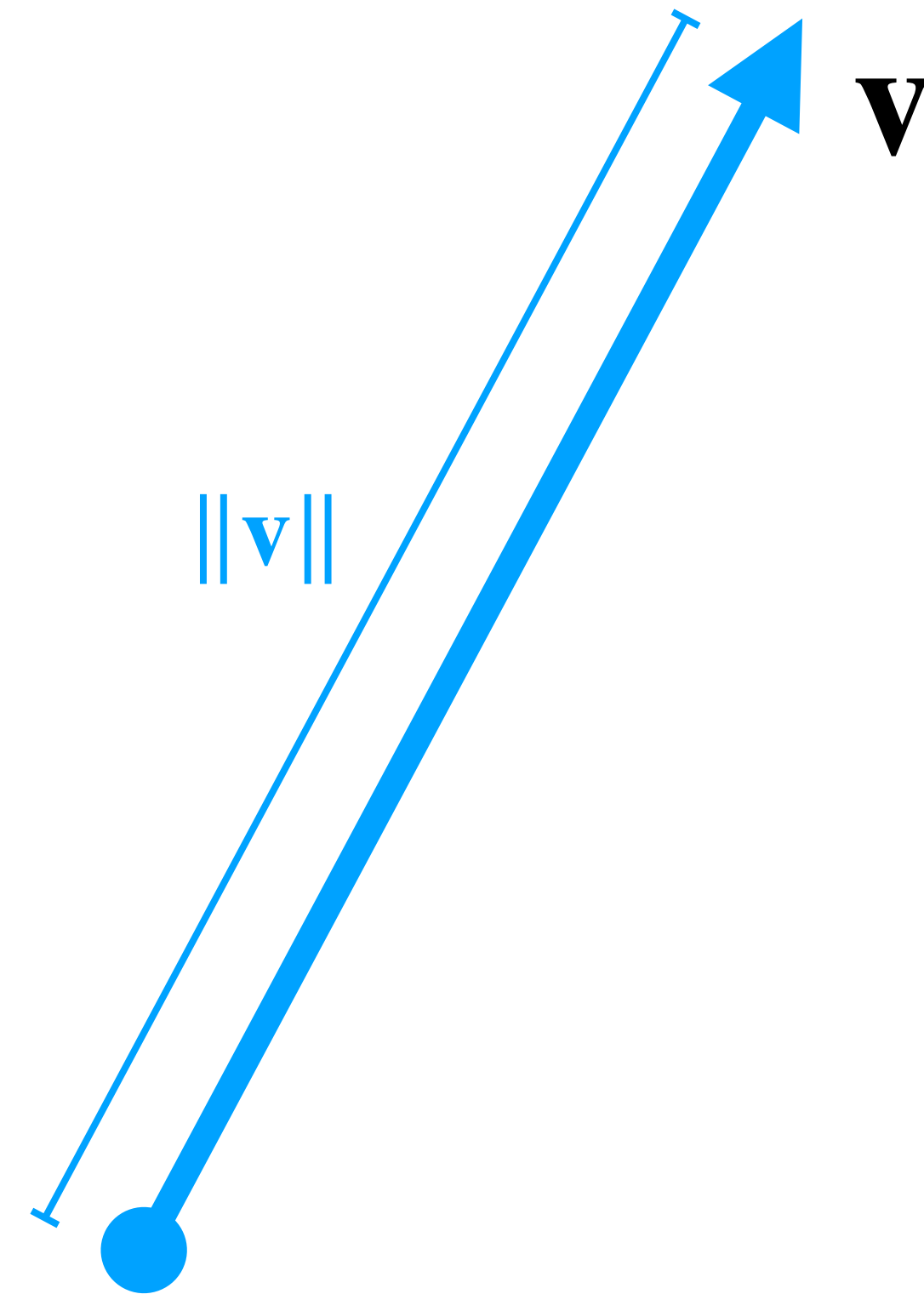
The norm of a vector is the square root of the inner product with itself.

It's important that $\mathbf{v}^T \mathbf{v}$ is nonnegative.

Norms and Distance

Norms give us a notion of length.

In \mathbb{R}^2 and \mathbb{R}^3 this is our existing notion of length.

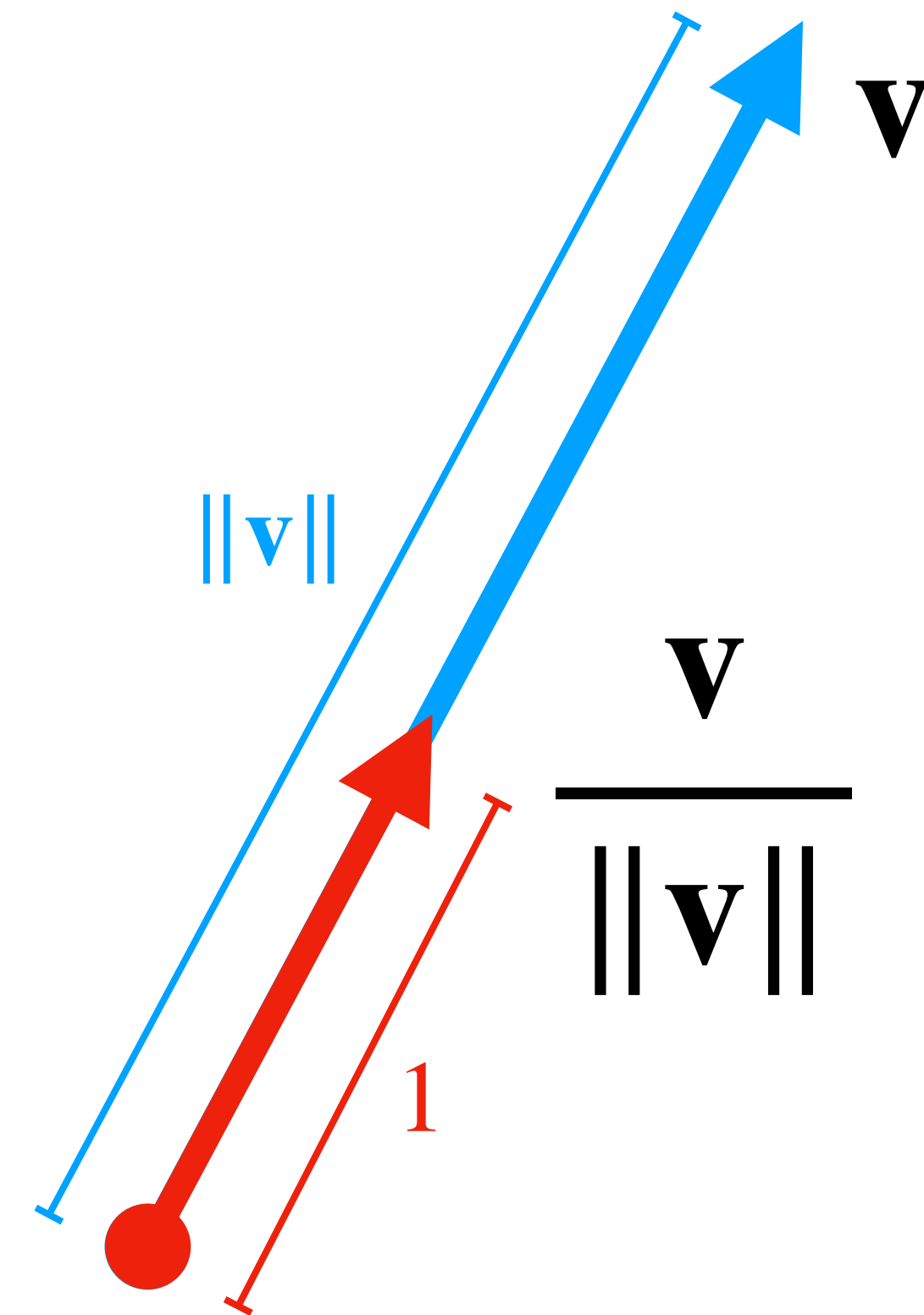


ℓ^2 Normalization

Definition. A **unit vector** is a vector \mathbf{v} such that $\|\mathbf{v}\| = 1$.

We often *normalize* vectors if we only care about their direction:

$$\mathbf{v} \mapsto \frac{\mathbf{v}}{\|\mathbf{v}\|}$$



How To: Normalizing Vectors

Question. Find the unit vector which points in the same direction as \mathbf{u} .

Solution. Compute $\|\mathbf{u}\|$. The unit vector is then

$$\frac{\mathbf{u}}{\|\mathbf{u}\|}$$

Example

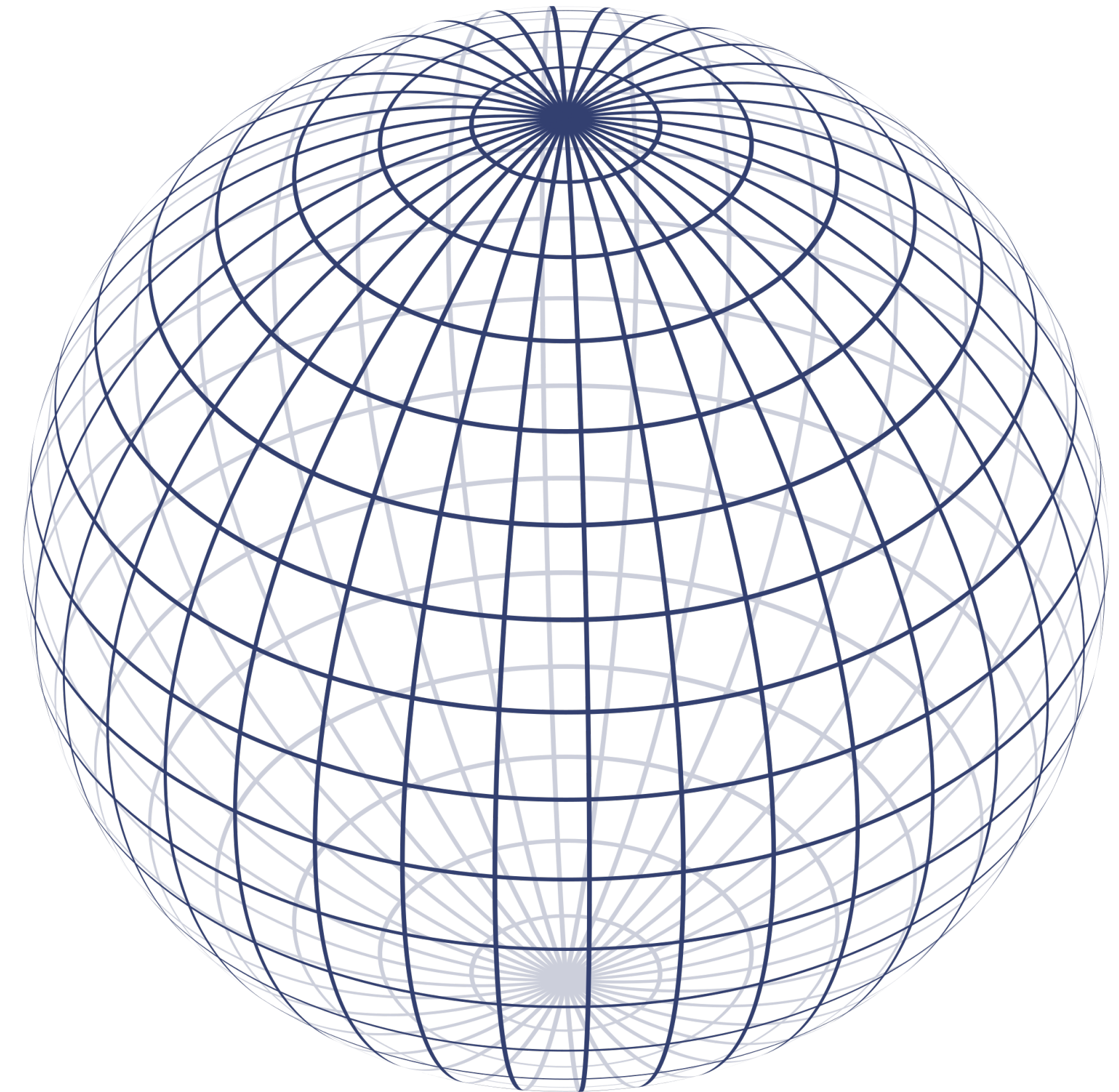
Find the unit vector in the same direction as $\begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix}$

The Unit Sphere

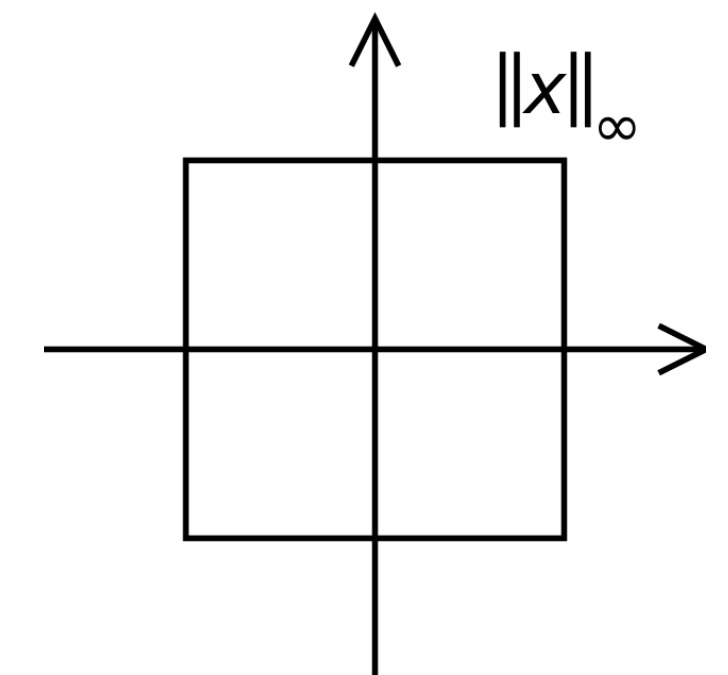
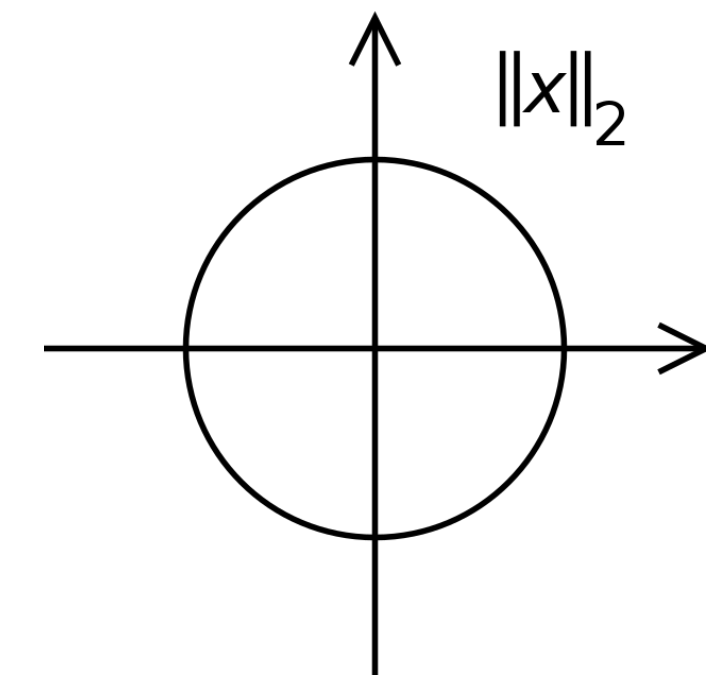
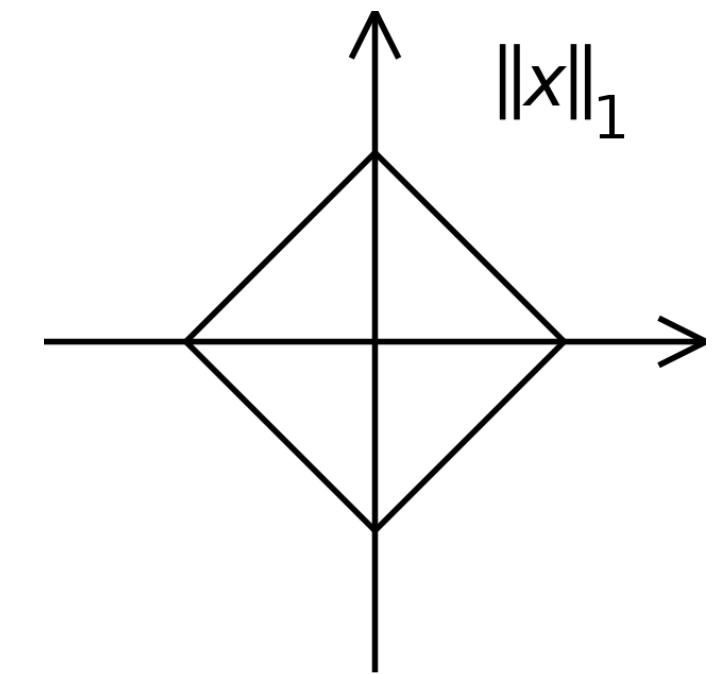
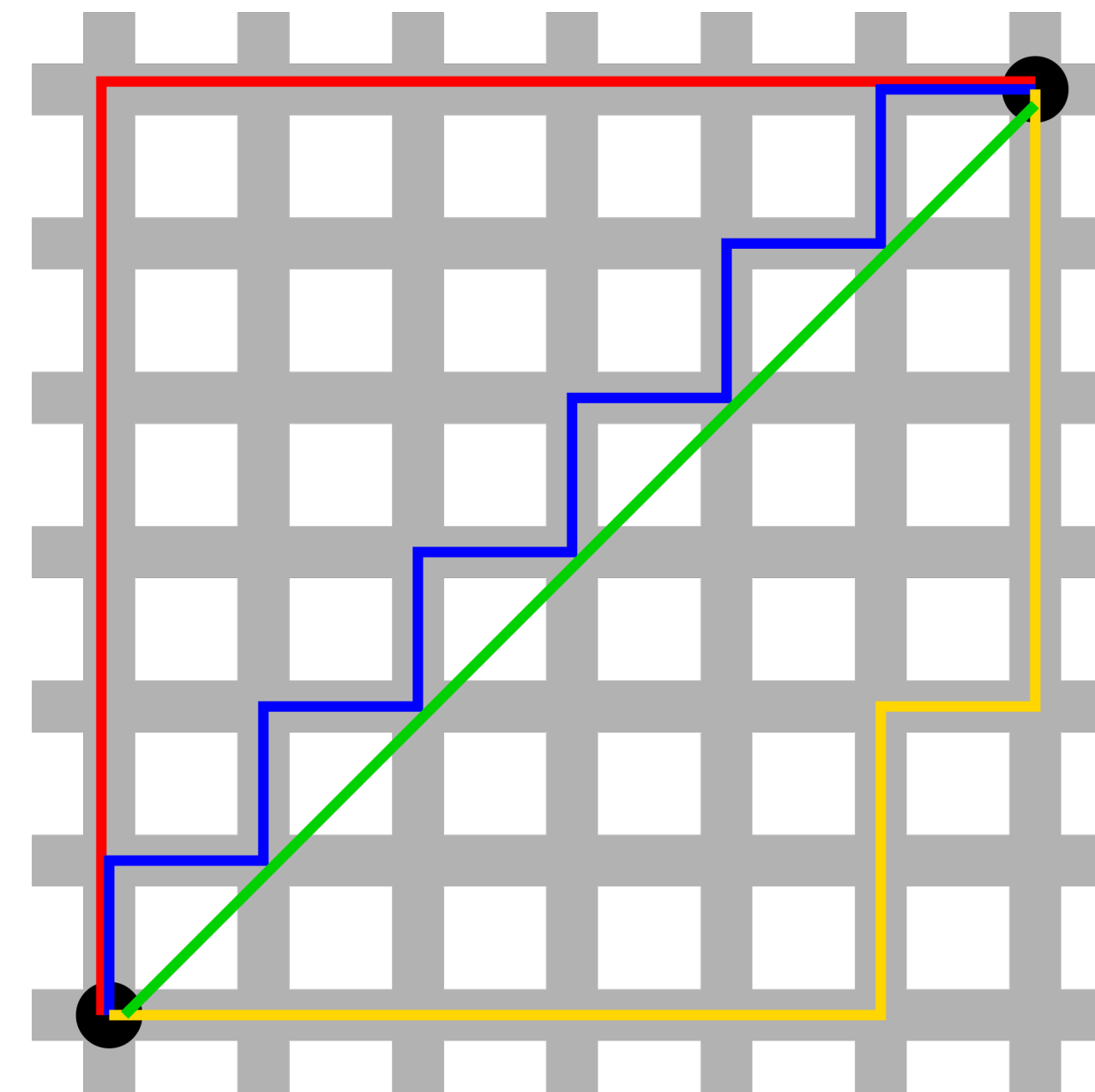
Definition. The unit n -sphere is the collection of all unit vectors in \mathbb{R}^n .

Vector norms allow us to talk about spheres in higher dimensions.

A sphere is a collection of points equidistant from a center point.

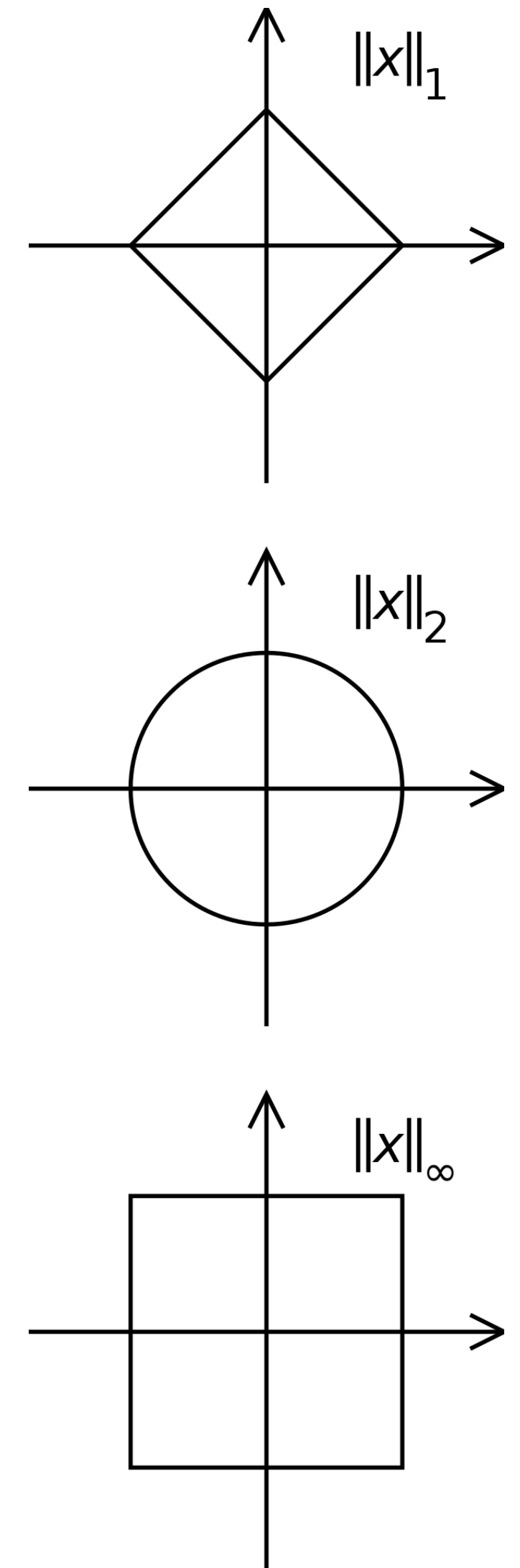
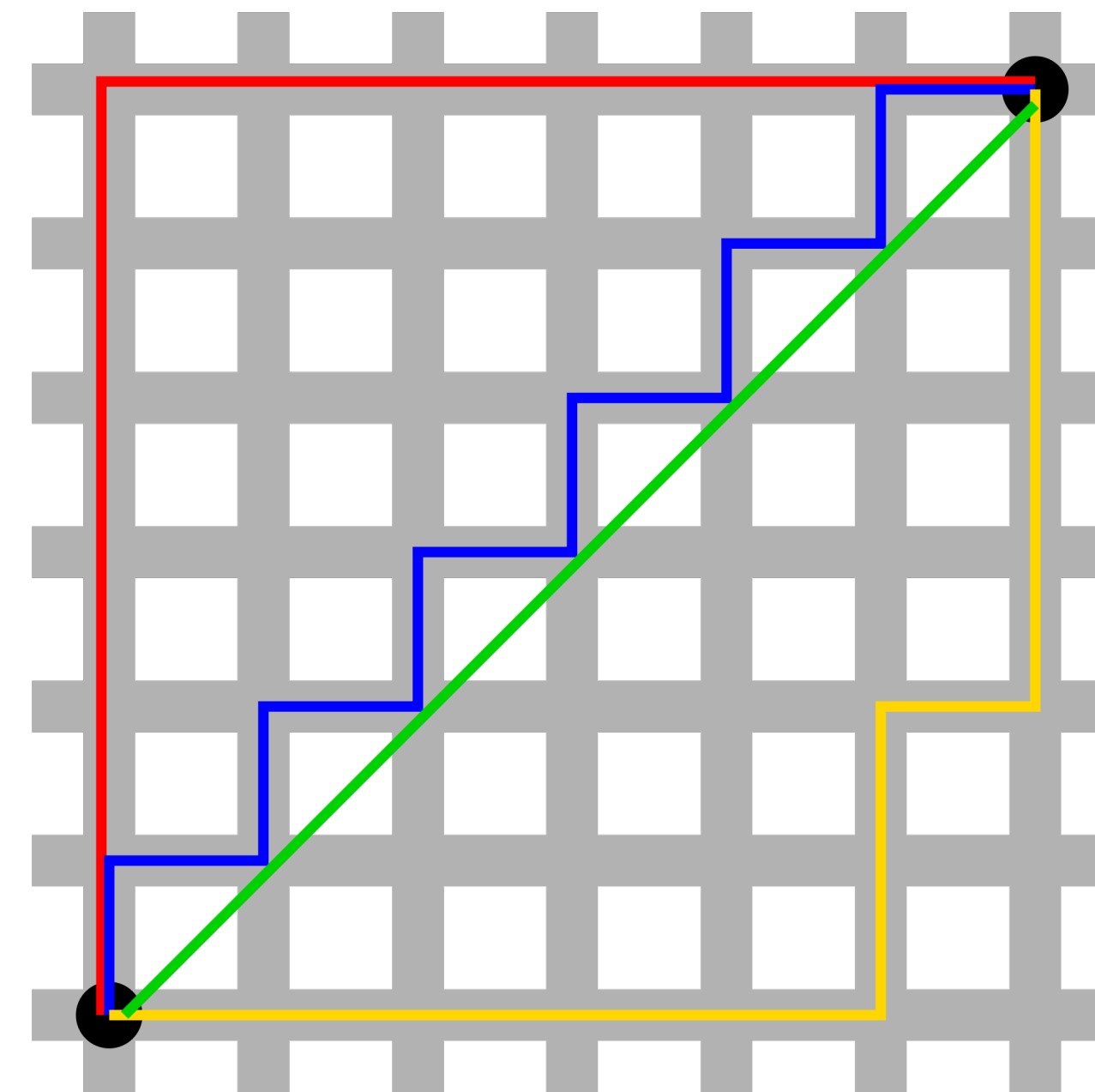


An Aside: Other Notions of Distance



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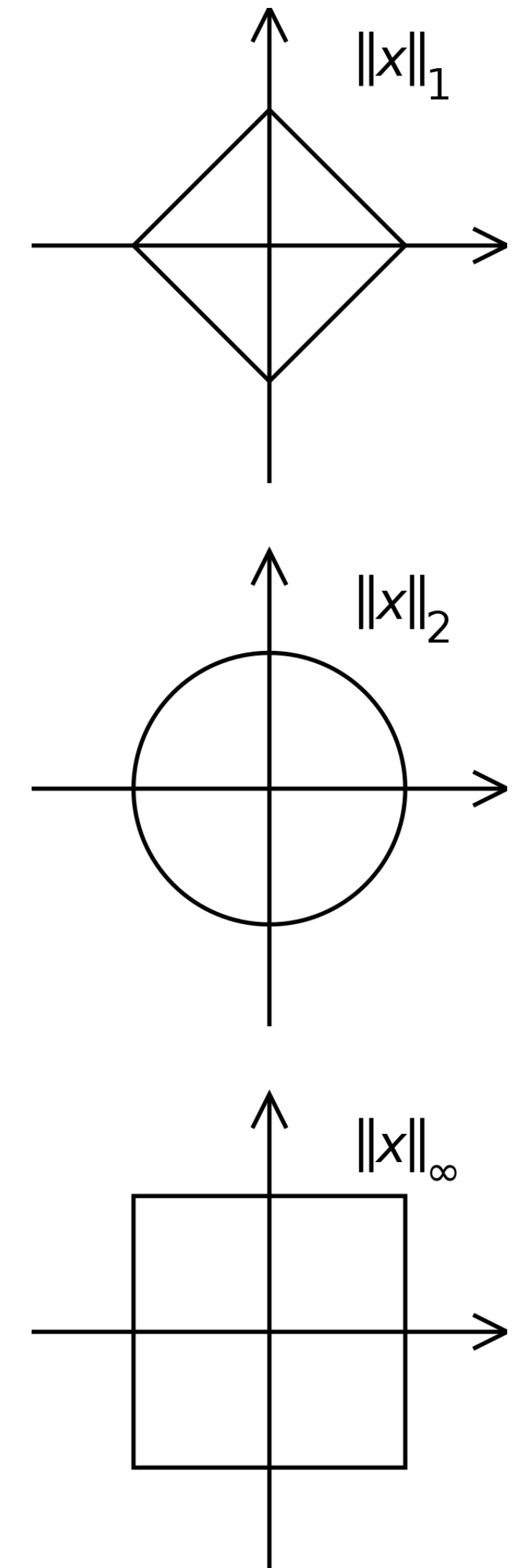
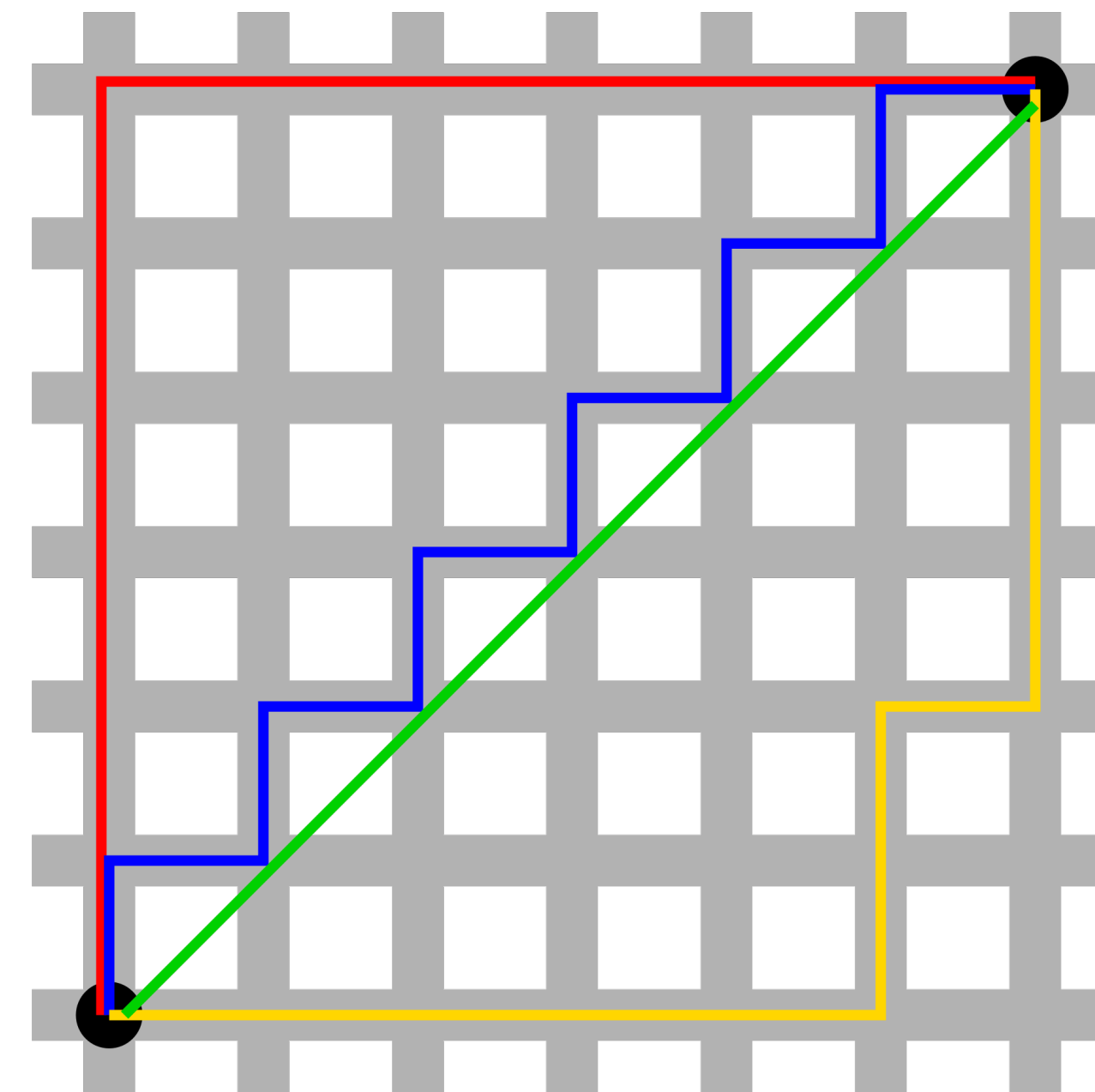
Why are we talking about norms and inner products so generally?



An Aside: Other Notions of Distance

Why are we talking about norms and inner products so generally?

Because there are other inner products and norms.

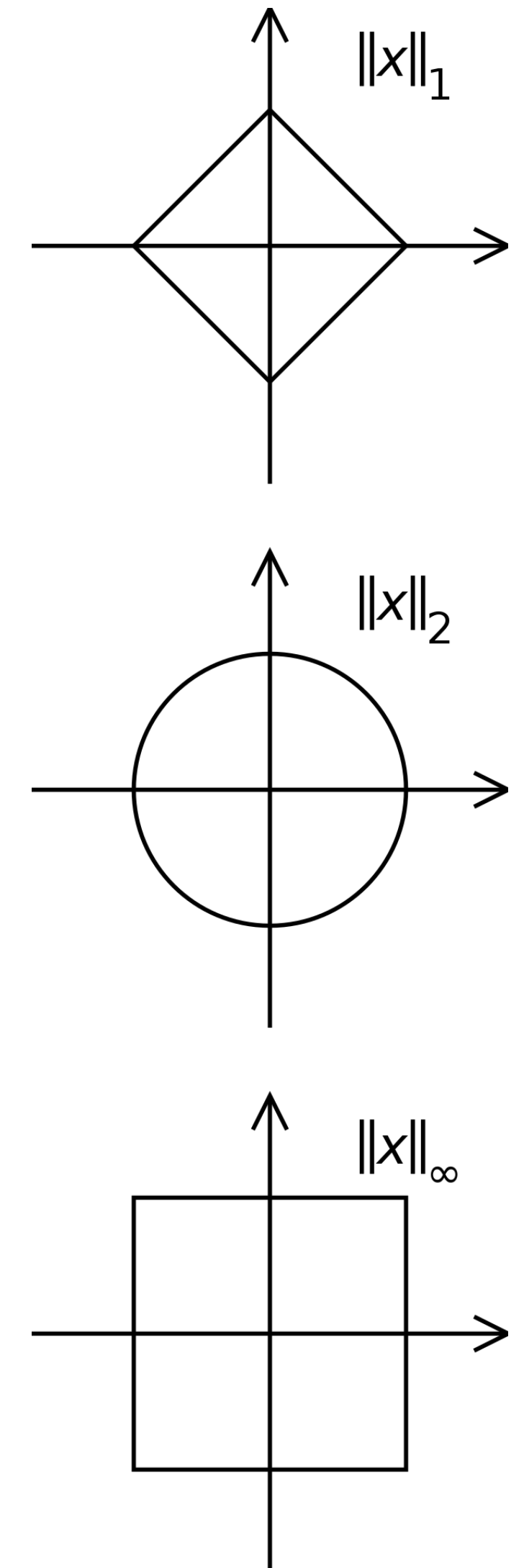
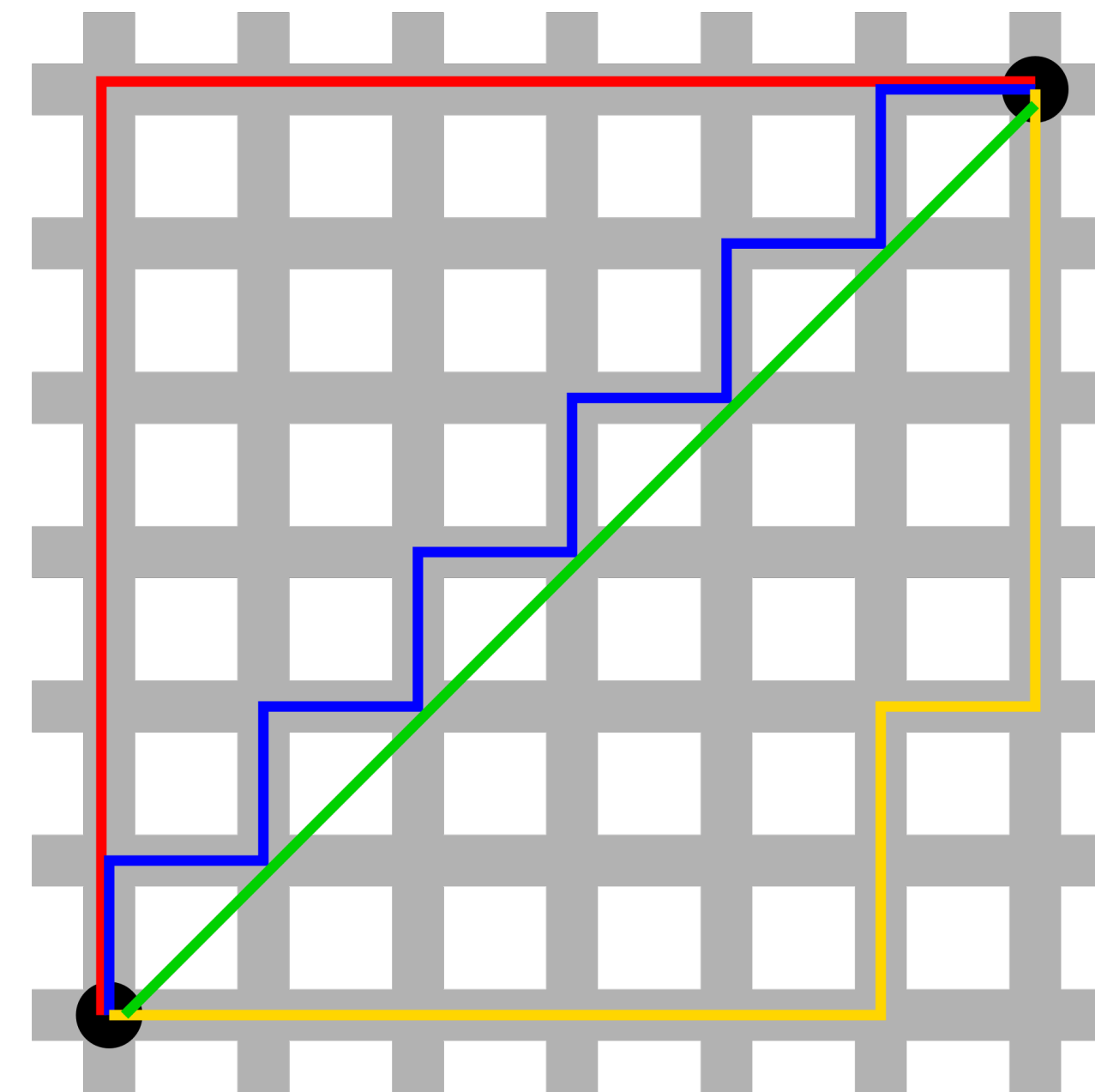


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e.g., Manhattan distance

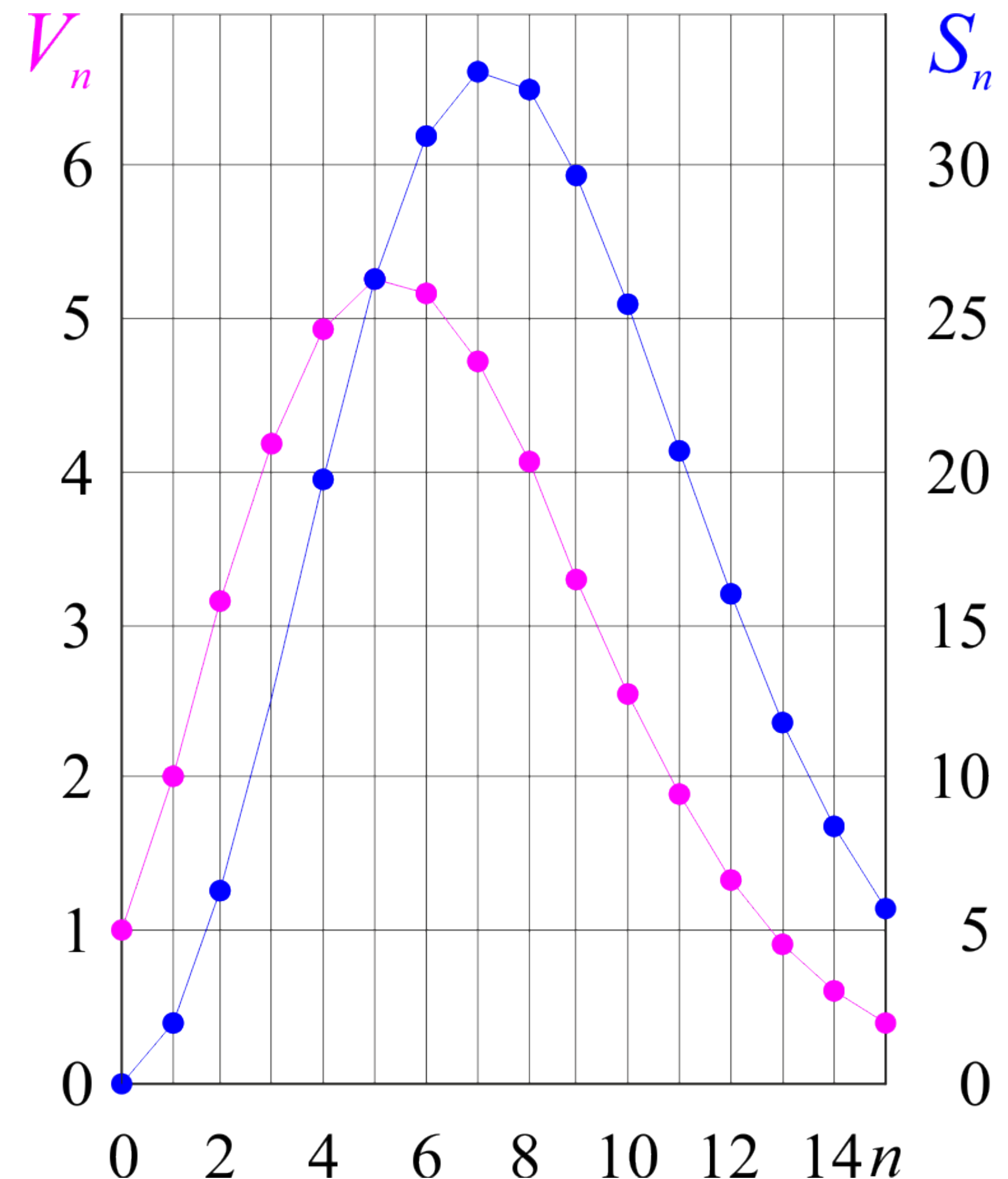


Another Aside: Surface Area and Volume

With a bit of calculus, we can calculate the surface area and volume of the unit n -sphere.

And the result is bizarre...

the infinite dimensional unit sphere has no volume or surface area...



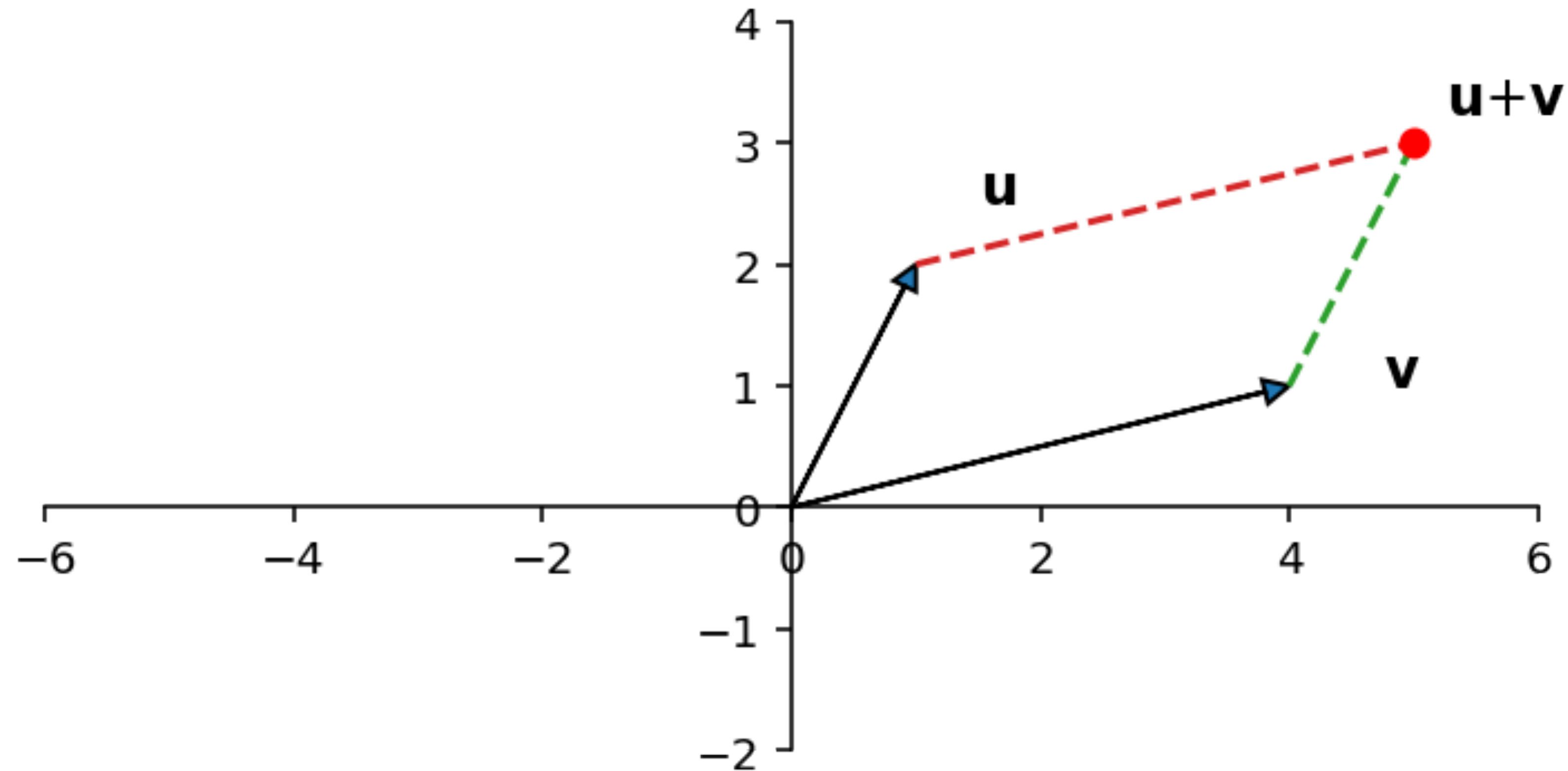
moving on . . .

Distance

If we know how to calculate lengths of vectors, we know how to calculate distances.

Recall: Vector Addition

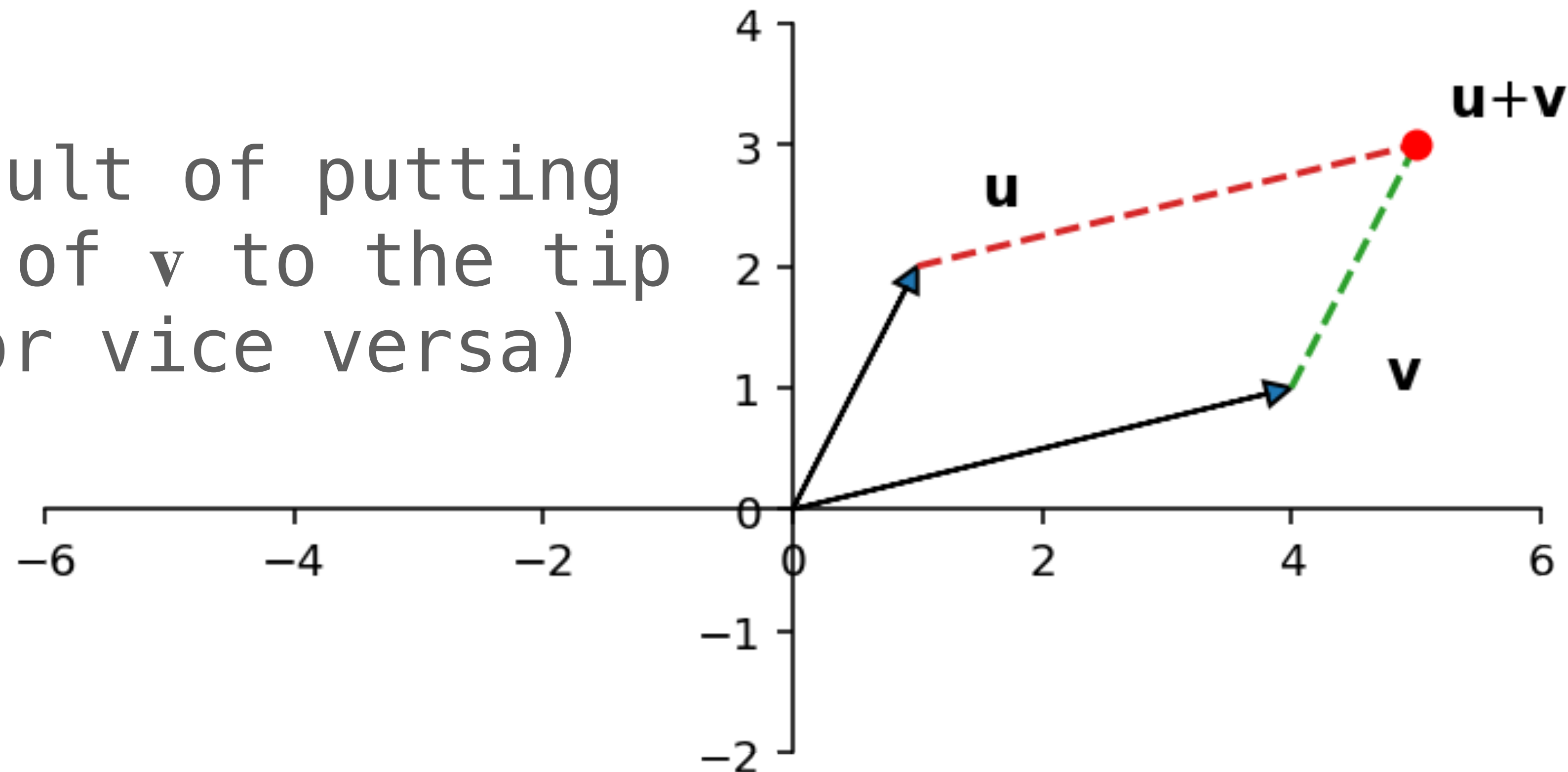
tip-to-tail rule:



Recall: Vector Addition

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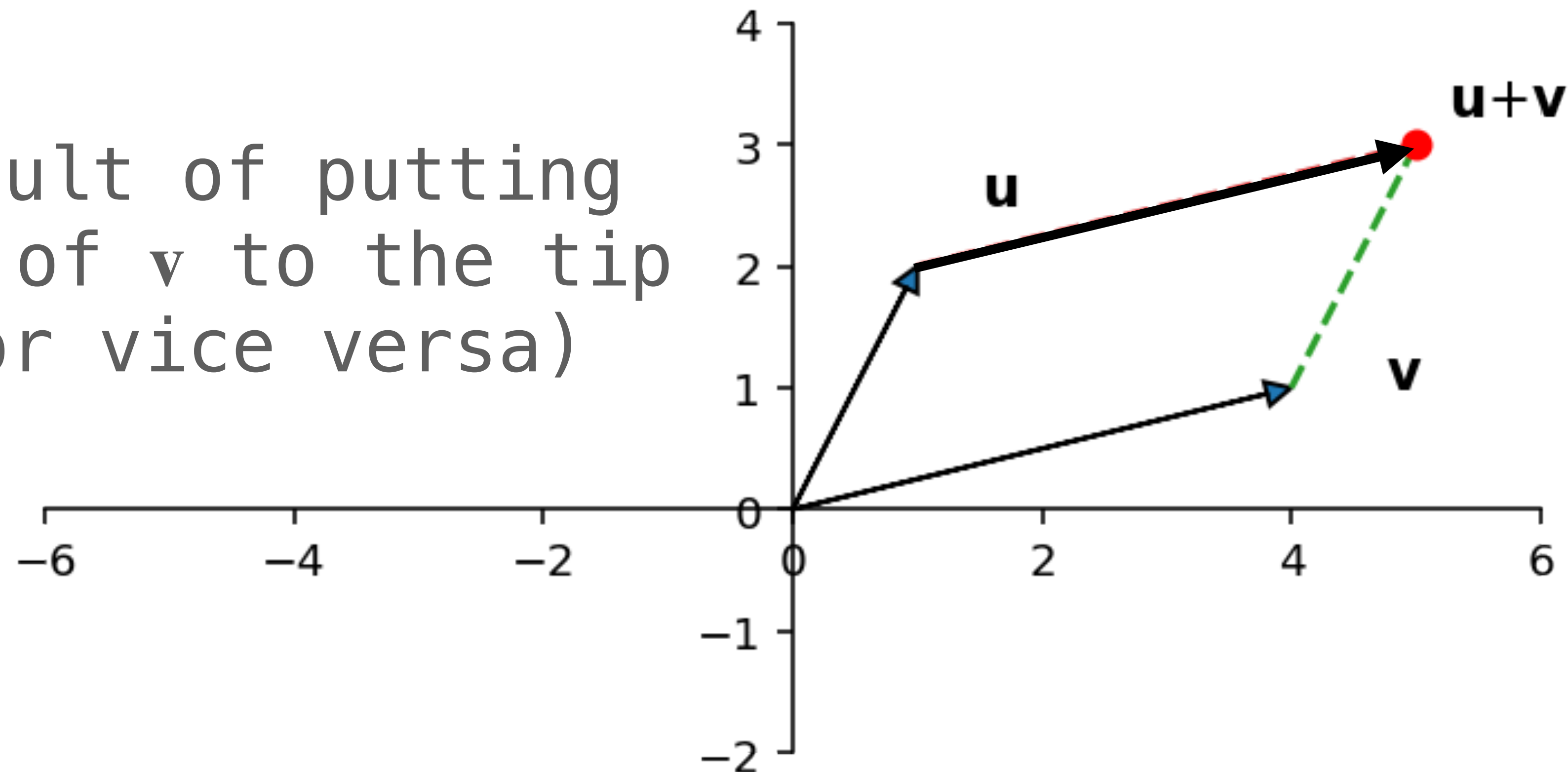
$\mathbf{u} + \mathbf{v}$ result of putting the tail of \mathbf{v} to the tip of \mathbf{u} (or vice versa)



Recall: Vector Addition

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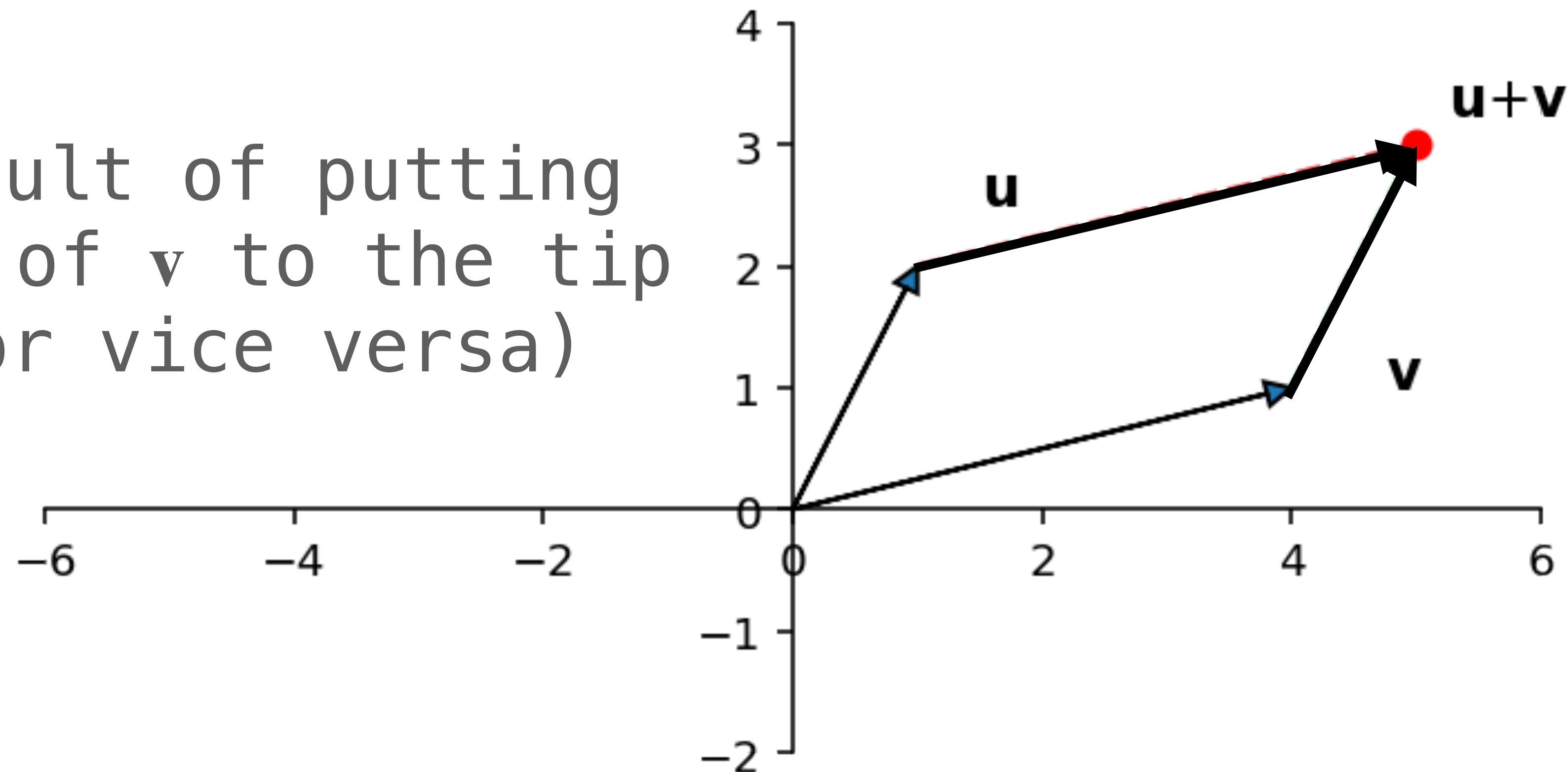
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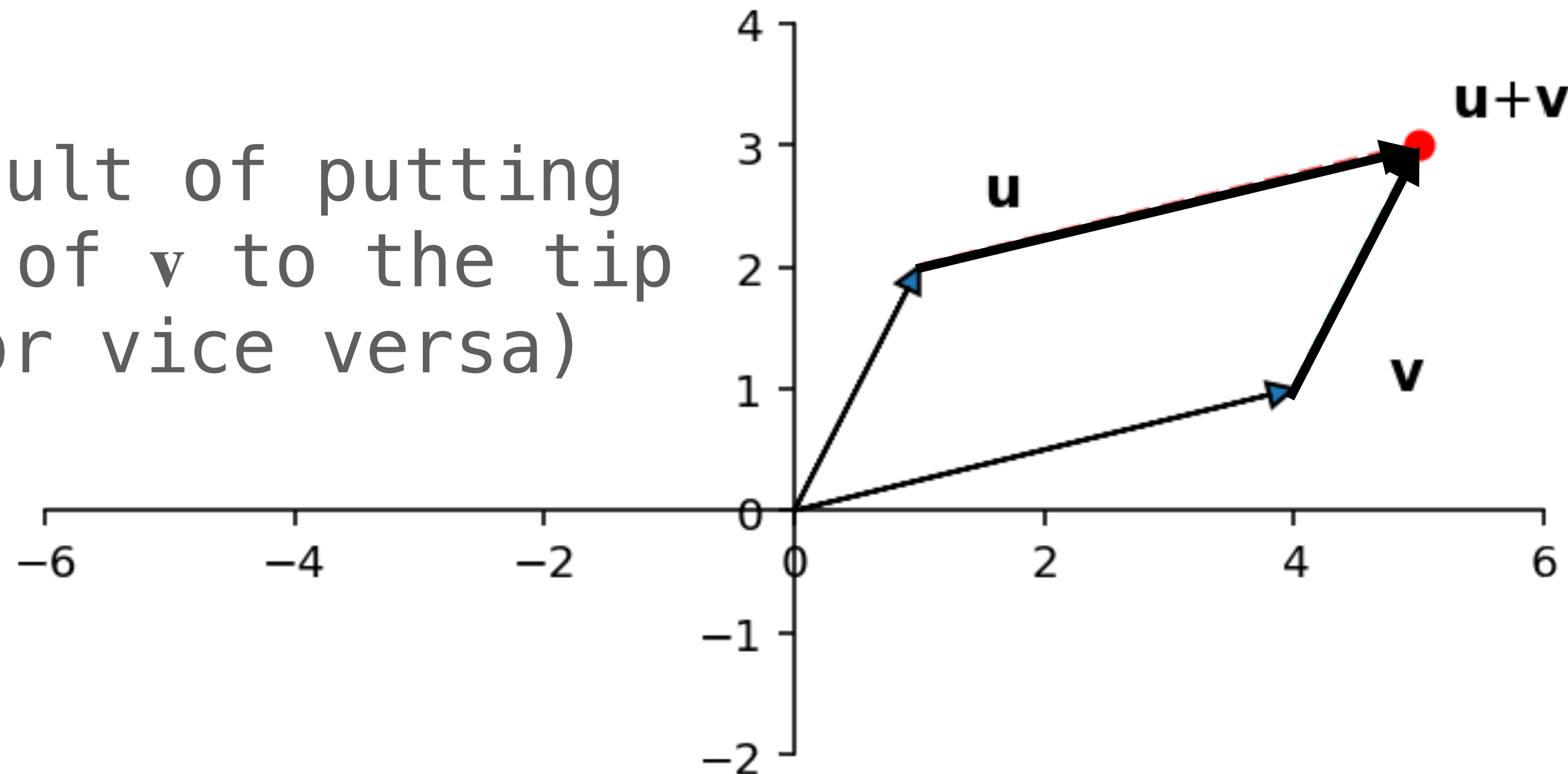


Recall: Vector Addition

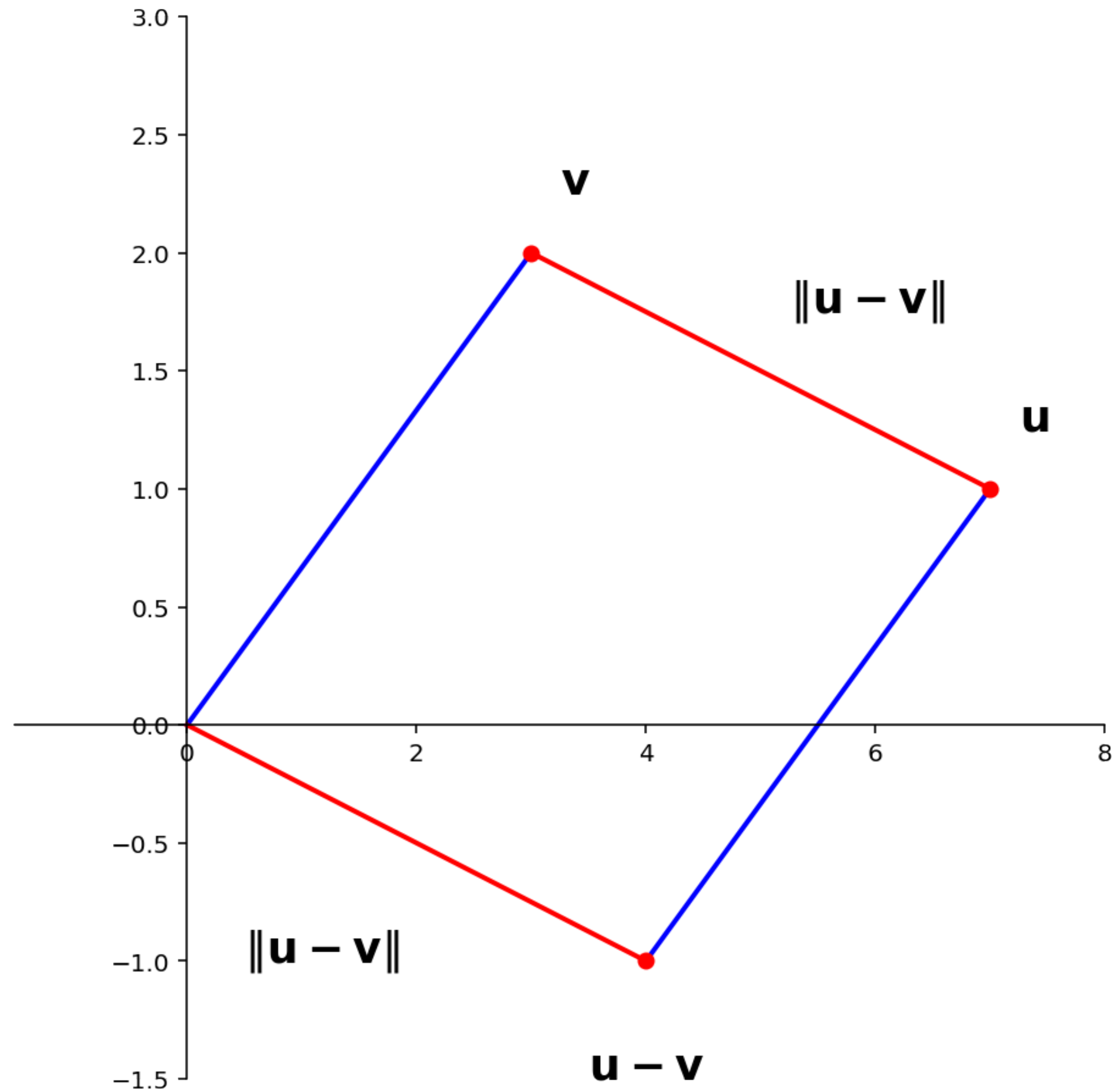
The distance between u and $u+v$ is the length of v

tip-to-tail rule:

$u+v$ result of putting the tail of v to the tip of u (or vice versa)



Distance (Pictorially)



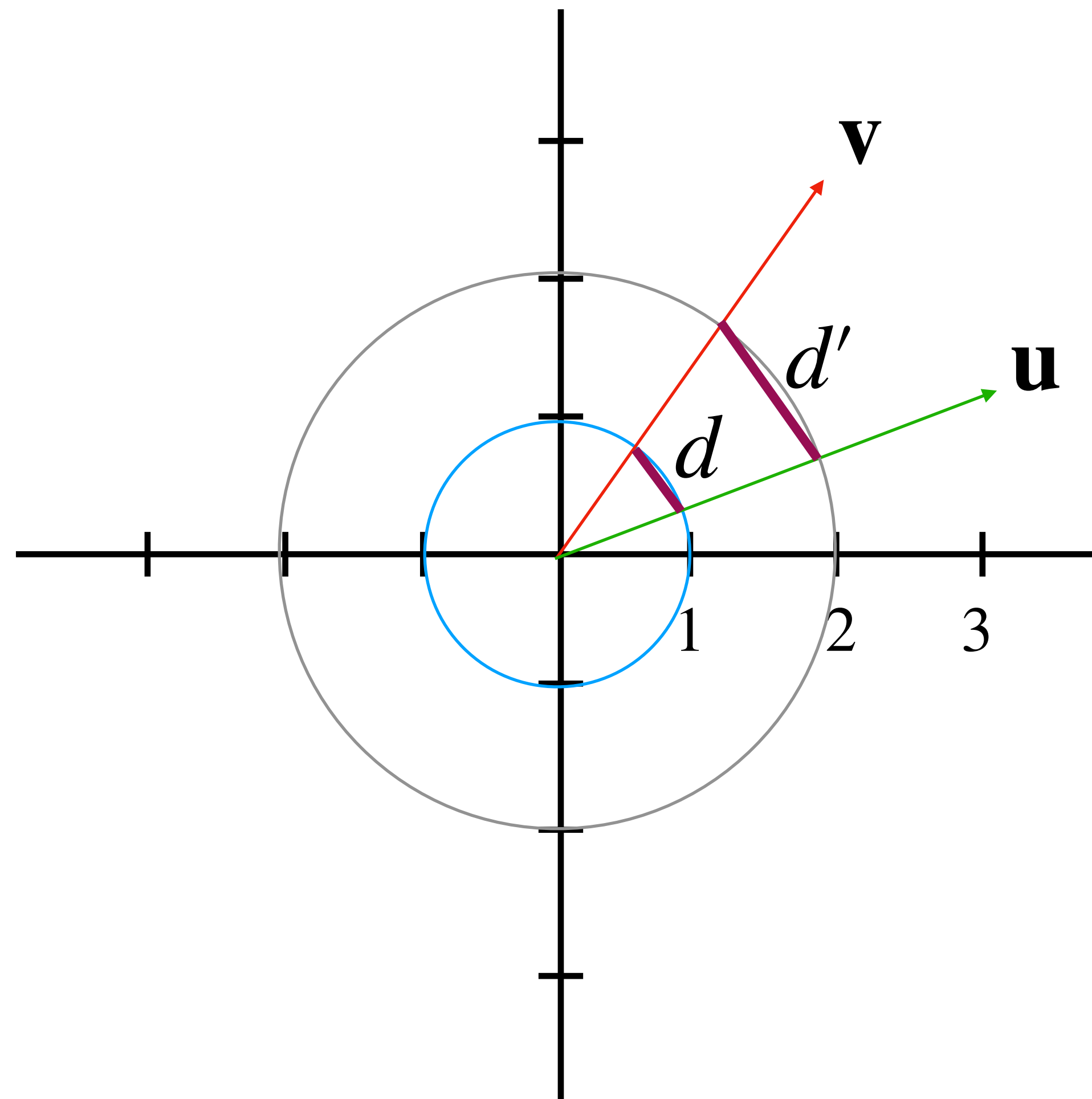
Distance (Algebraically)

Definition. The distance between two vectors \mathbf{u} and \mathbf{v} in \mathbb{R}^n is given by

$$\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

e.g., $\mathbf{u} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Question



Find an expression for the distance d .

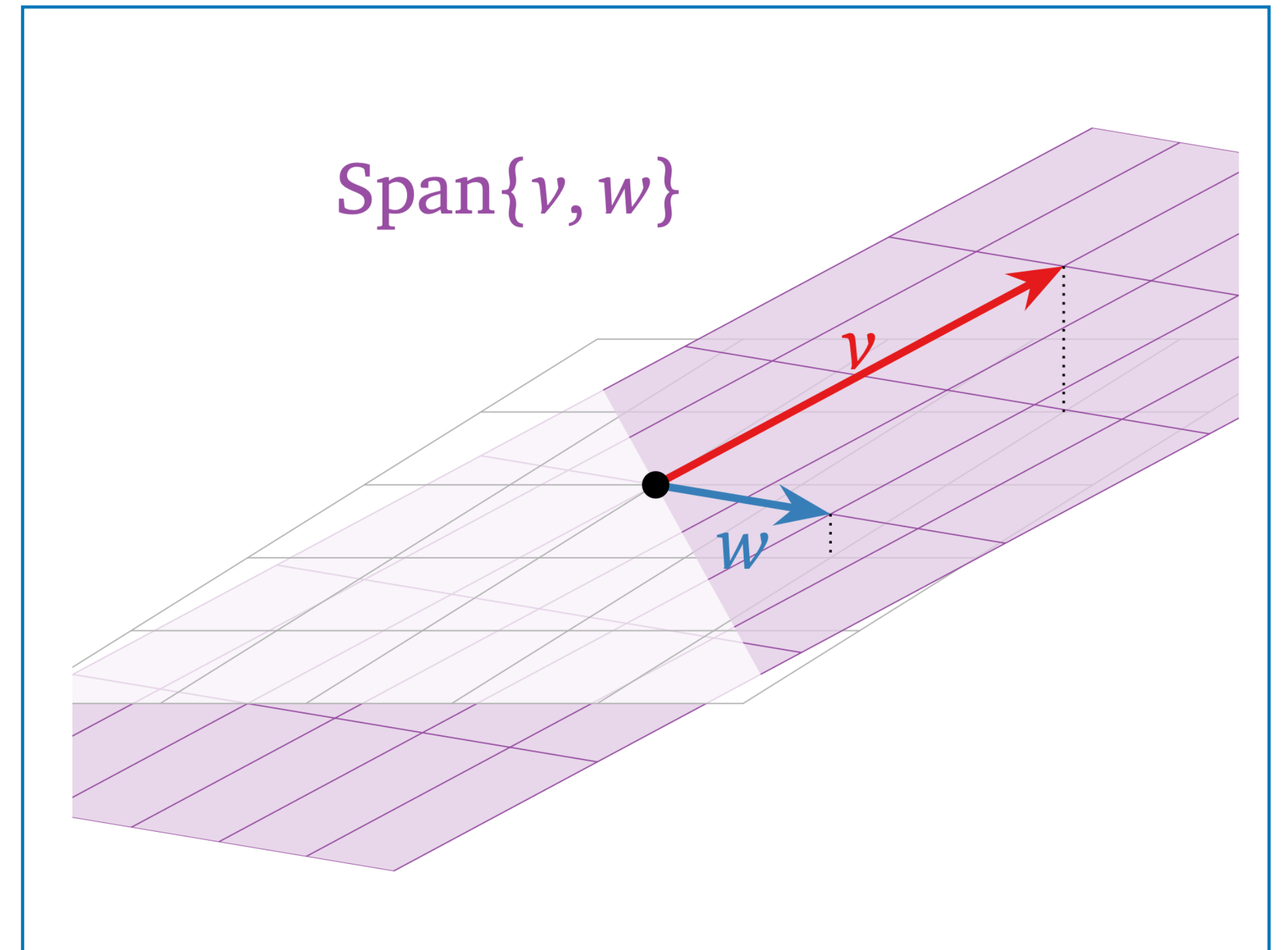
Challenge. Find an expression for d' .

Answer

Angles

Again, Angles still make sense

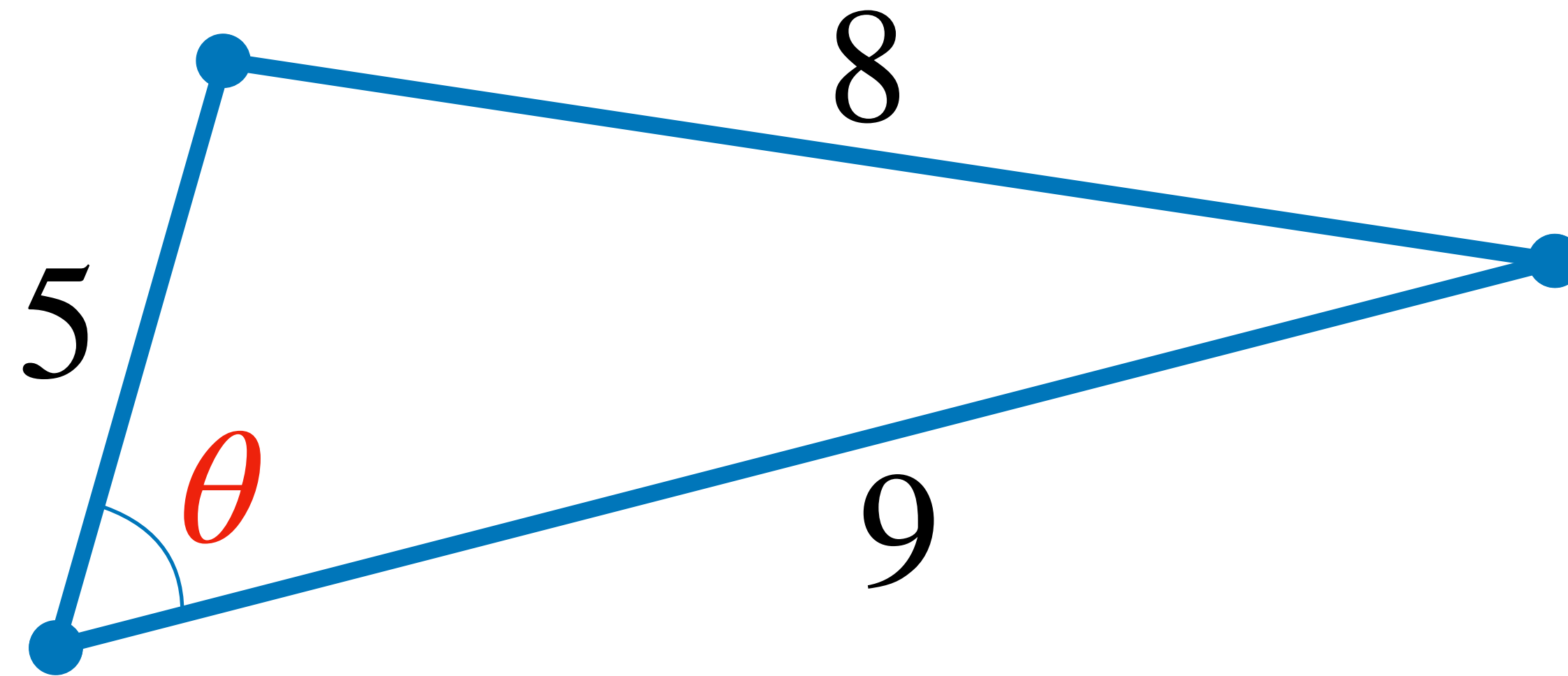
Any pair of vectors in \mathbb{R}^n
span a (2D) plane.



Fundamental Question

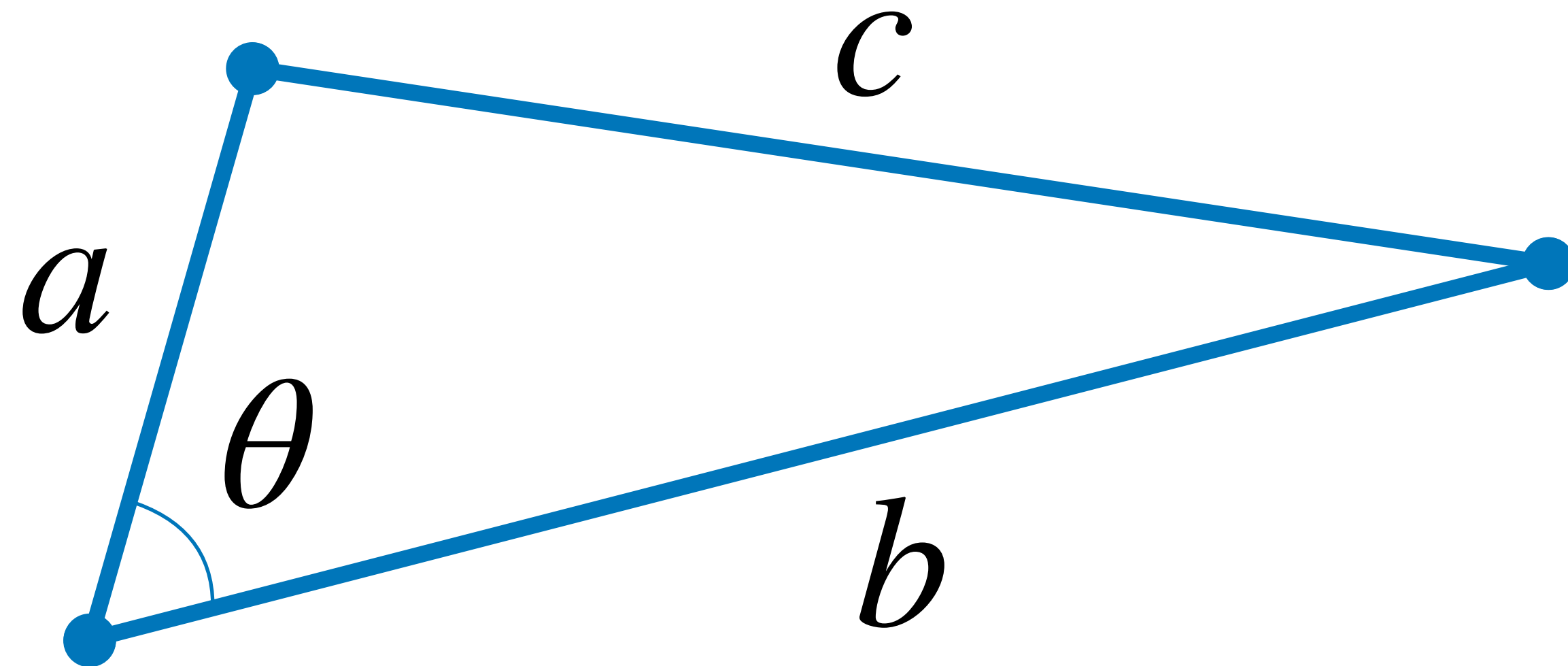
How do we determine the angle
between any two vectors?

Recall: A Potentially Familiar Example



What is the value of θ ?

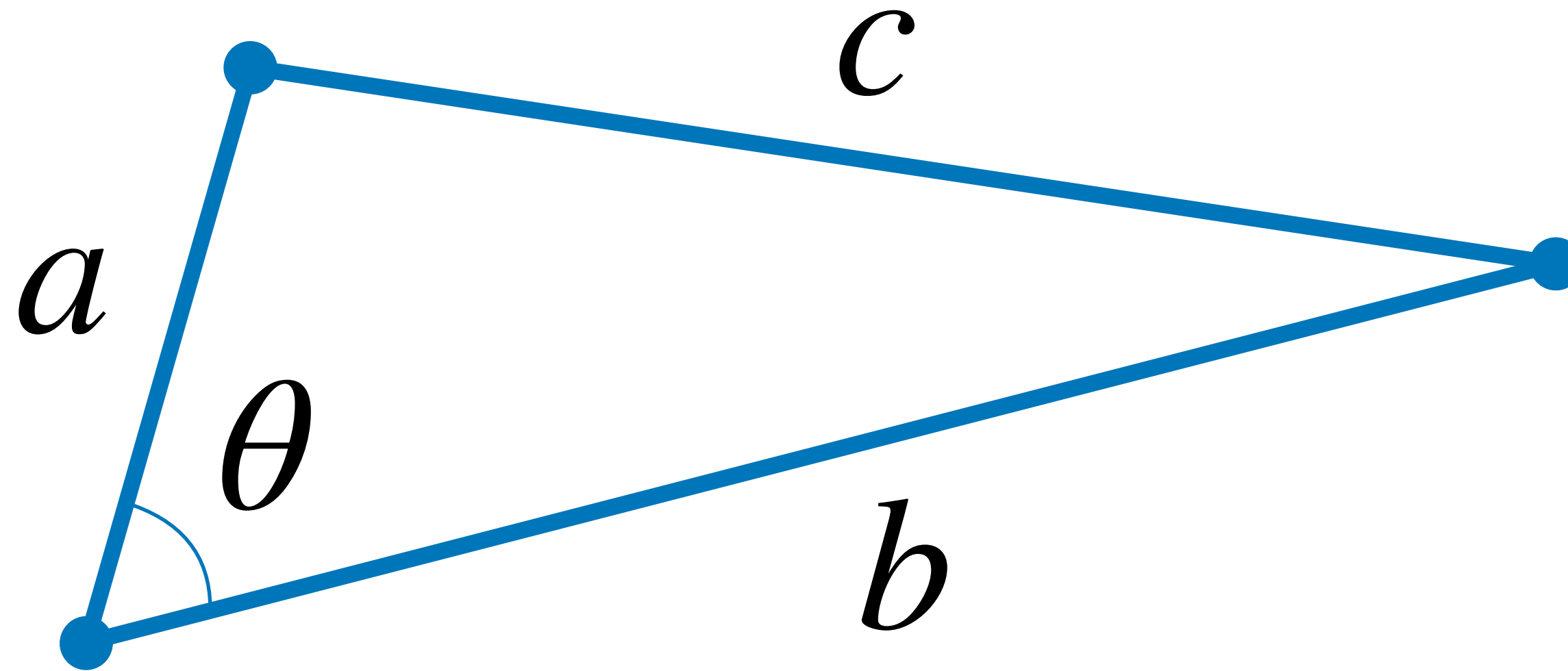
Law of Cosines



Theorem.

$$c^2 = a^2 + b^2 - 2ab \cos \theta$$

Law of Cosines

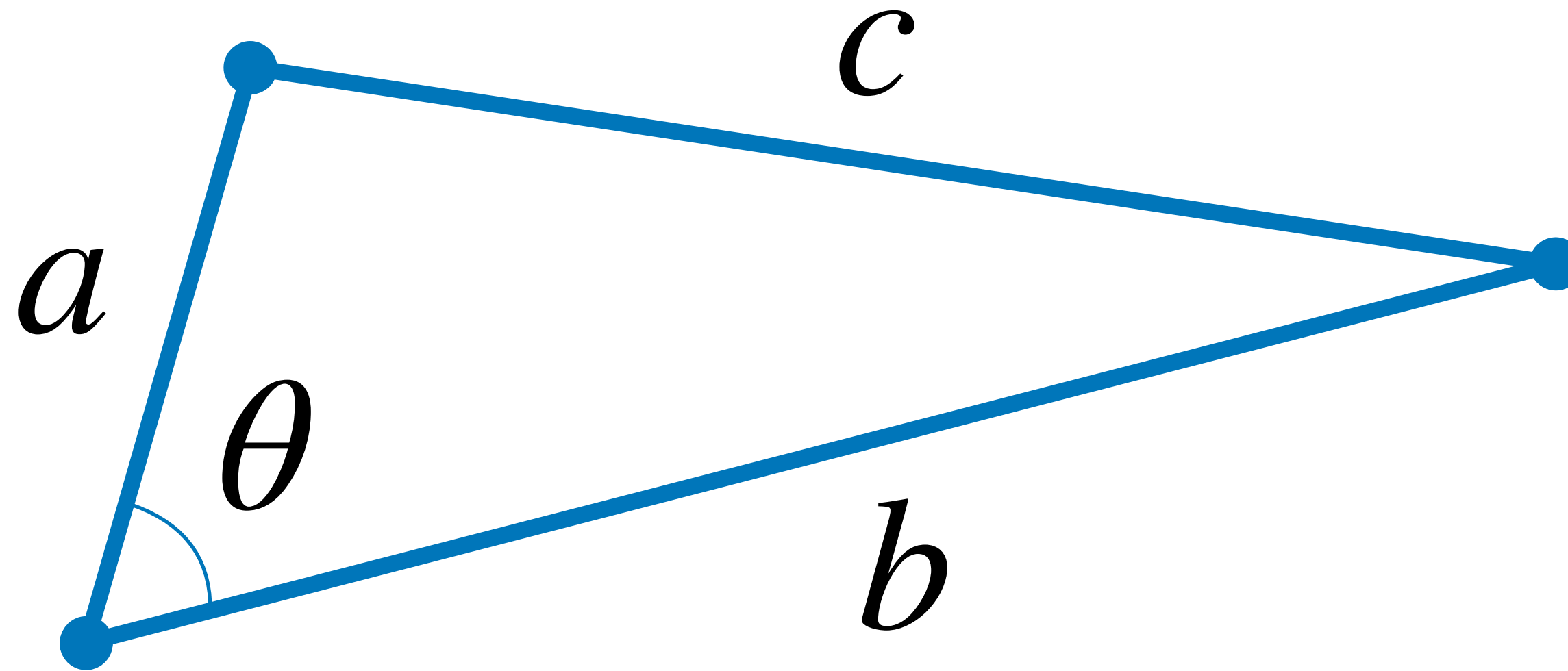


Theorem.

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Generalized the Pythagorean Theorem

Law of Cosines



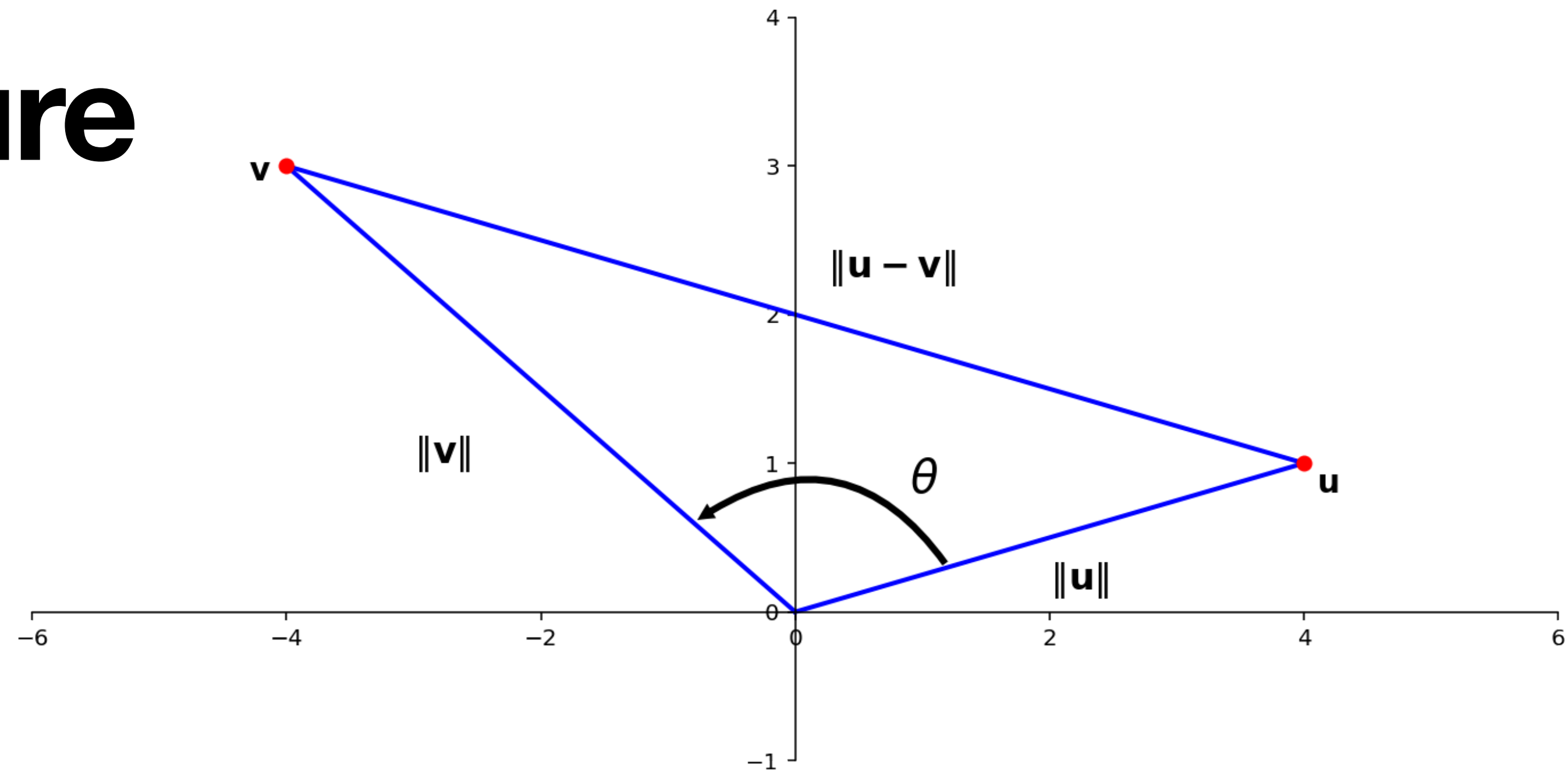
Theorem.

θ exactly when $\theta = 90^\circ$

$$c^2 = a^2 + b^2 - 2ab \cos \theta$$

Generalized the Pythagorean Theorem

The Picture



In more "vector"-y terms:

$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$$

Isolating θ

$$\cos \theta = \frac{c^2 - a^2 - b^2}{2ab}$$

$$\theta = \cos^{-1} \left(\frac{c^2 - a^2 - b^2}{2ab} \right)$$

We might remember these equations...

Isolating θ

$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$$

Let's isolate θ in this equation:

Cosines and Unit Vectors

Theorem. For vectors \mathbf{u} and \mathbf{v} in \mathbb{R}^n with an angle θ between them,

$$\cos \theta = \left\langle \frac{\mathbf{u}}{\|\mathbf{u}\|}, \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\rangle$$

The cosine of the angle between two vectors is the inner product of their ℓ^2 normalizations.

How To: Angles

Question. Find the angle between the two vectors \mathbf{u} and \mathbf{v} .

Solution. Compute $\cos^{-1} \left(\frac{\mathbf{u}}{\|\mathbf{u}\|} \cdot \frac{\mathbf{v}}{\|\mathbf{v}\|} \right)$ (with a calculator).

Example

Find the angle between the vectors

$$\mathbf{u} = \begin{bmatrix} 1 \\ 3 \\ -7 \\ -2 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 8 \\ -2 \\ 4 \\ 6 \end{bmatrix}$$

Example: Step 1

Compute $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$.

$$\|\mathbf{u}\| = \sqrt{1^2 + 3^2 + (-7)^2 + (-2)^2} = 7.93$$

$$\|\mathbf{v}\| = \sqrt{8^2 + (-2)^2 + 4^2 + 6^2} = 10.95$$

Example: Step 2

Normalize the vectors.

$$\frac{\mathbf{u}}{\|\mathbf{u}\|} = \begin{bmatrix} 0.13 \\ 0.38 \\ -0.88 \\ -0.25 \end{bmatrix}$$

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \begin{bmatrix} 0.73 \\ -0.18 \\ 0.36 \\ 0.54 \end{bmatrix}$$

Example: Step 3

Find their inner product.

$$\left\langle \frac{\mathbf{u}}{\|\mathbf{u}\|}, \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\rangle = (0.13 \cdot 0.73) + (0.38 \cdot -0.18) + (-0.88 \cdot 0.36) + (-0.25 \cdot 0.54)$$
$$= -0.44$$

Example: Step 4

Compute the angle.

$$\theta = \cos^{-1}(-0.44) = 116^\circ$$

A Conceptual Question

Why cosine? Why not sine?

Because $\cos 90^\circ = 0$.

This means its an indicator of perpendicularity.

Orthogonality (Perpendicularity)

A Simpler Fundamental Question

How do we determine if angle
between any two vectors is 90° ?

Orthogonality

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Definition (Informal). Two vectors \mathbf{u} and \mathbf{v} in \mathbb{R}^n are orthogonal if the angle between them is 90° .

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(and it's difficult to compute with)

Recall: Cosines and Unit Vectors

Theorem. For vectors \mathbf{u} and \mathbf{v} in \mathbb{R}^n with an angle θ between them,

$$\cos \theta = \left\langle \frac{\mathbf{u}}{\|\mathbf{u}\|}, \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\rangle$$

The cosine of the angle between two vectors is the inner product of their ℓ^2 normalizations.

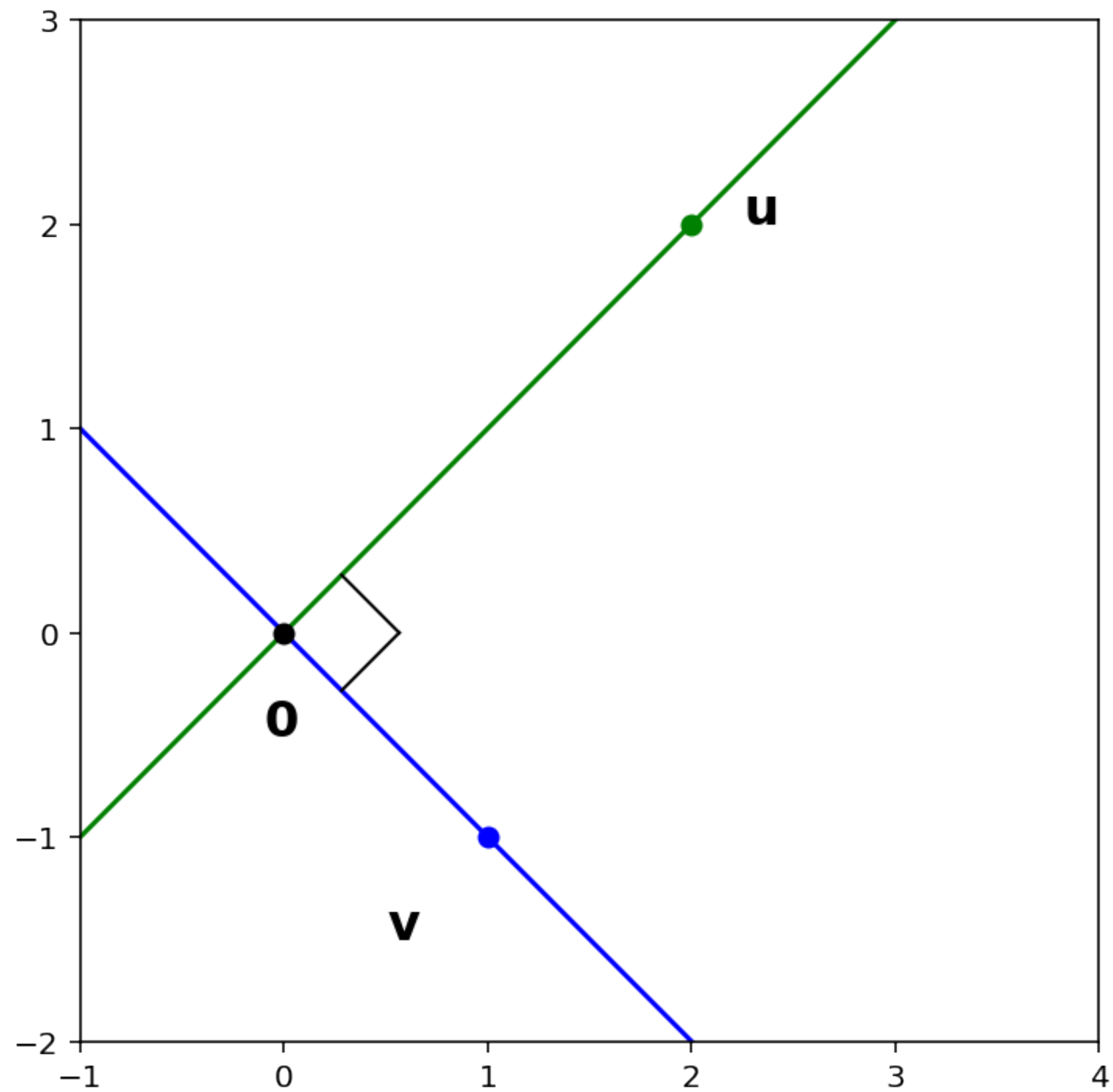
Orthogonality

Definition (Actual). Vectors u and v are **orthogonal** if $\langle u, v \rangle = 0$.

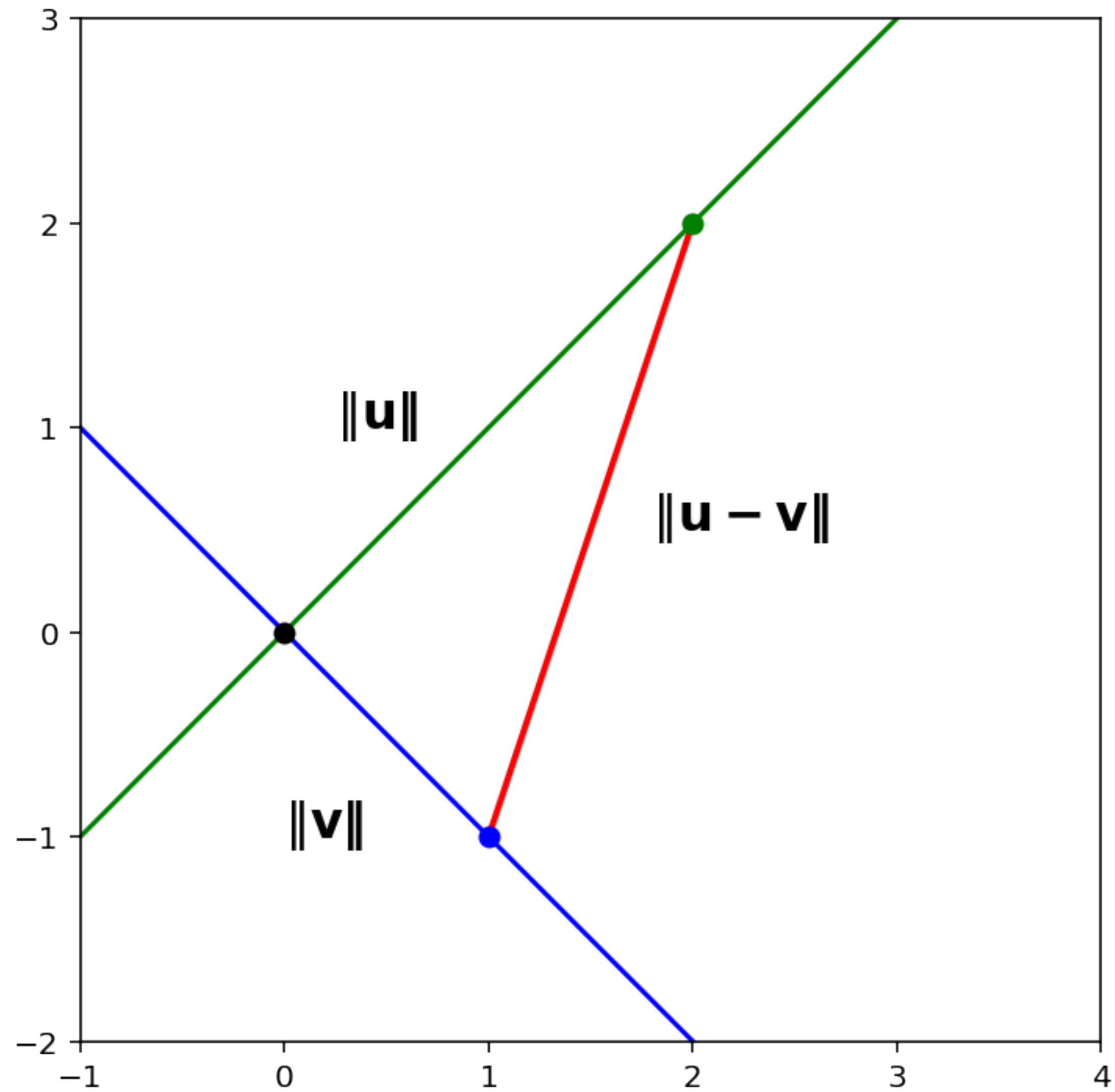
This definition gives an easy computational way to determine orthogonality.

Example.

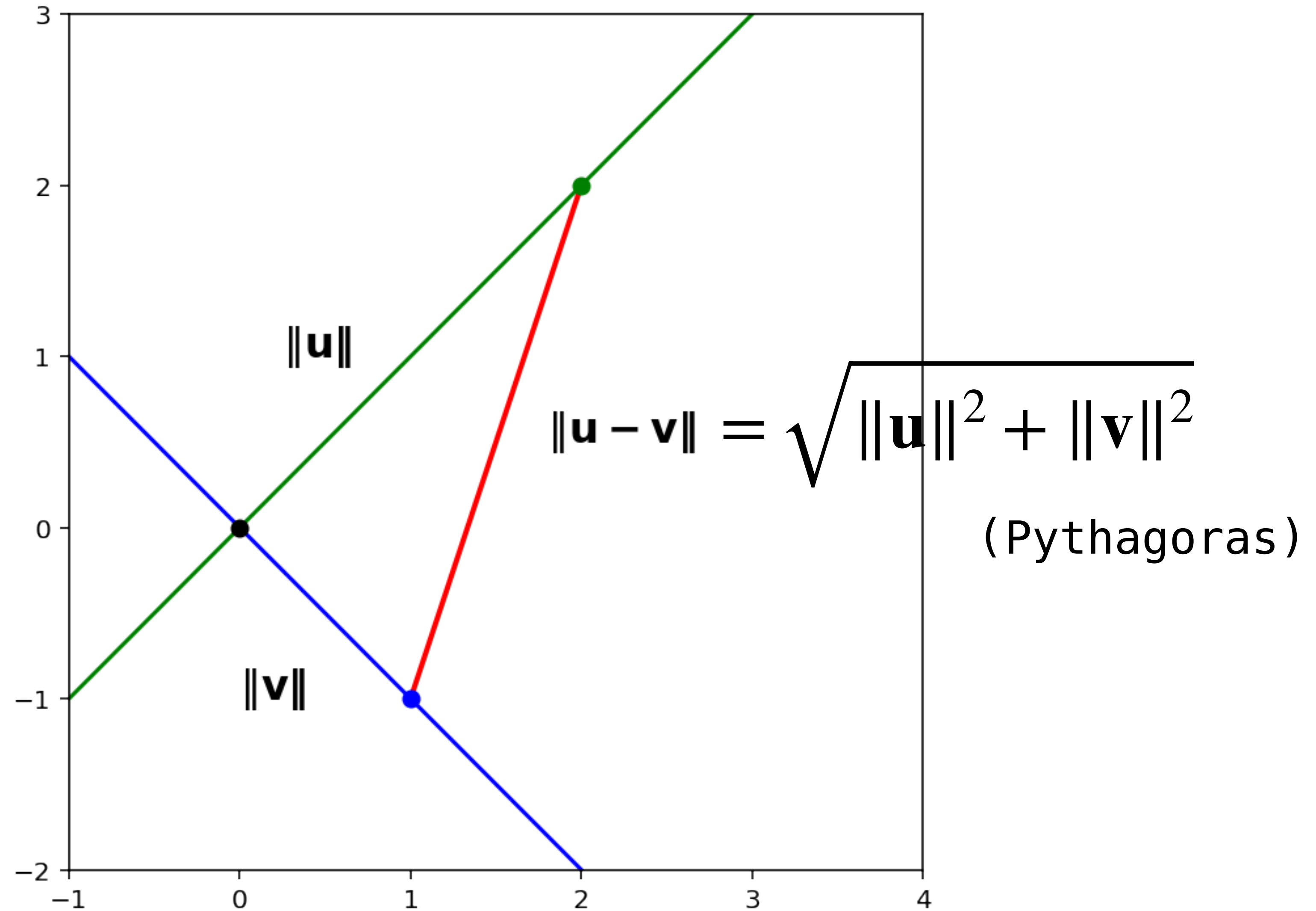
Derivation by Picture



Derivation by Picture



Derivation by Picture



Derivation by Algebra

\mathbf{u} and \mathbf{v} are orthogonal exactly when

$$\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = \|\mathbf{u} - \mathbf{v}\|^2$$

Let's simplify this a bit:

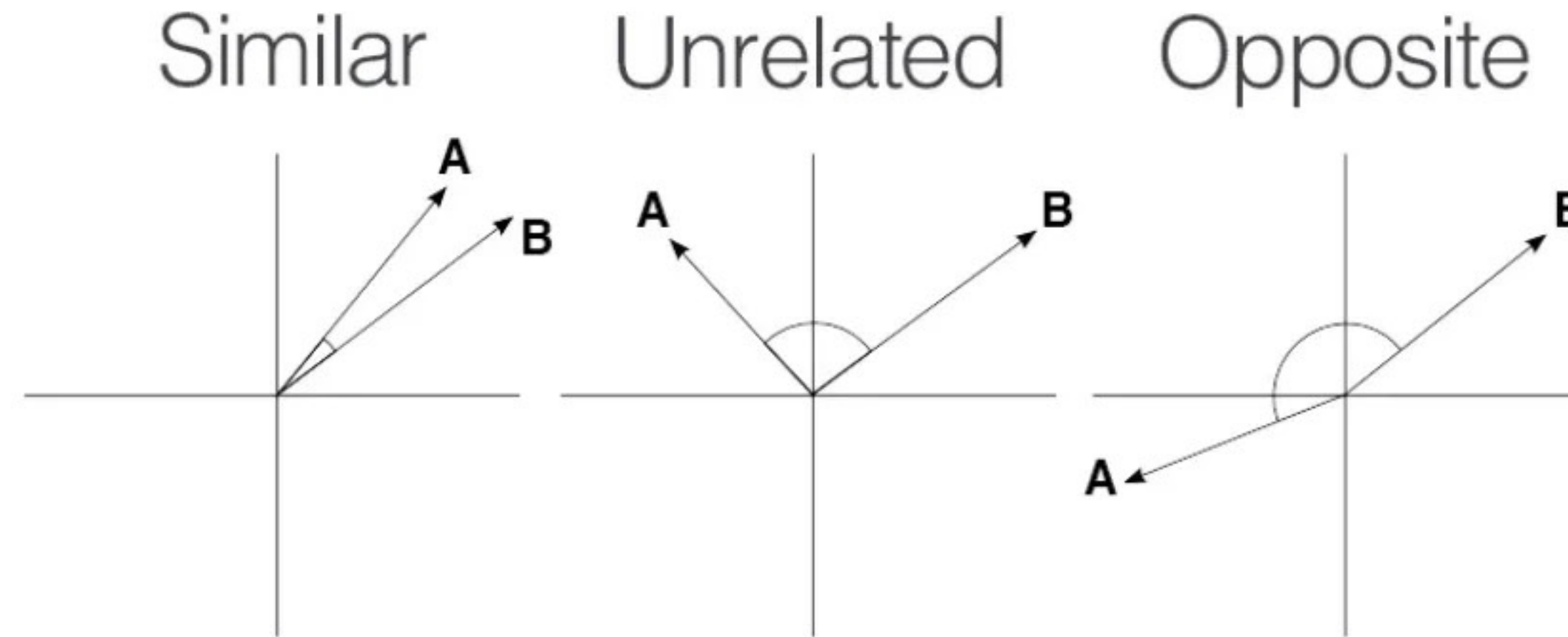
How To: Orthogonality

Question. Determine if \mathbf{u} and \mathbf{v} are perpendicular.

Solution. Determine if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$. If yes, then they are perpendicular. If no, then they are not.

Application: Cosine Similarity

High Level



Data points are very big vectors.

Similar vectors "point in nearly the same direction."

Example: Netflix Users

$$\text{user}_1 = \begin{bmatrix} 2 \\ 10 \\ 1 \\ 3 \end{bmatrix} \quad \text{user}_2 = \begin{bmatrix} 2 \\ 5 \\ 0 \\ 4 \end{bmatrix} \quad \text{user}_3 = \begin{bmatrix} 10 \\ 0 \\ 5 \\ 6 \end{bmatrix} \quad \begin{array}{l} \text{comedy} \\ \text{drama} \\ \text{horror} \\ \text{romance} \end{array}$$

A Netflix user might be represented as a vectors whose i th entry is the number of movies they've watched in a particular genre.

Who are more likely to share similar interests in movies?

Cosine Similarity

Definition. The **cosine similarity** of two vectors is the cosine of the angle between them.

If its close to 0, then two Netflix users watch very different movies.

If its close to 1, then two Netflix users watch very similar movies.

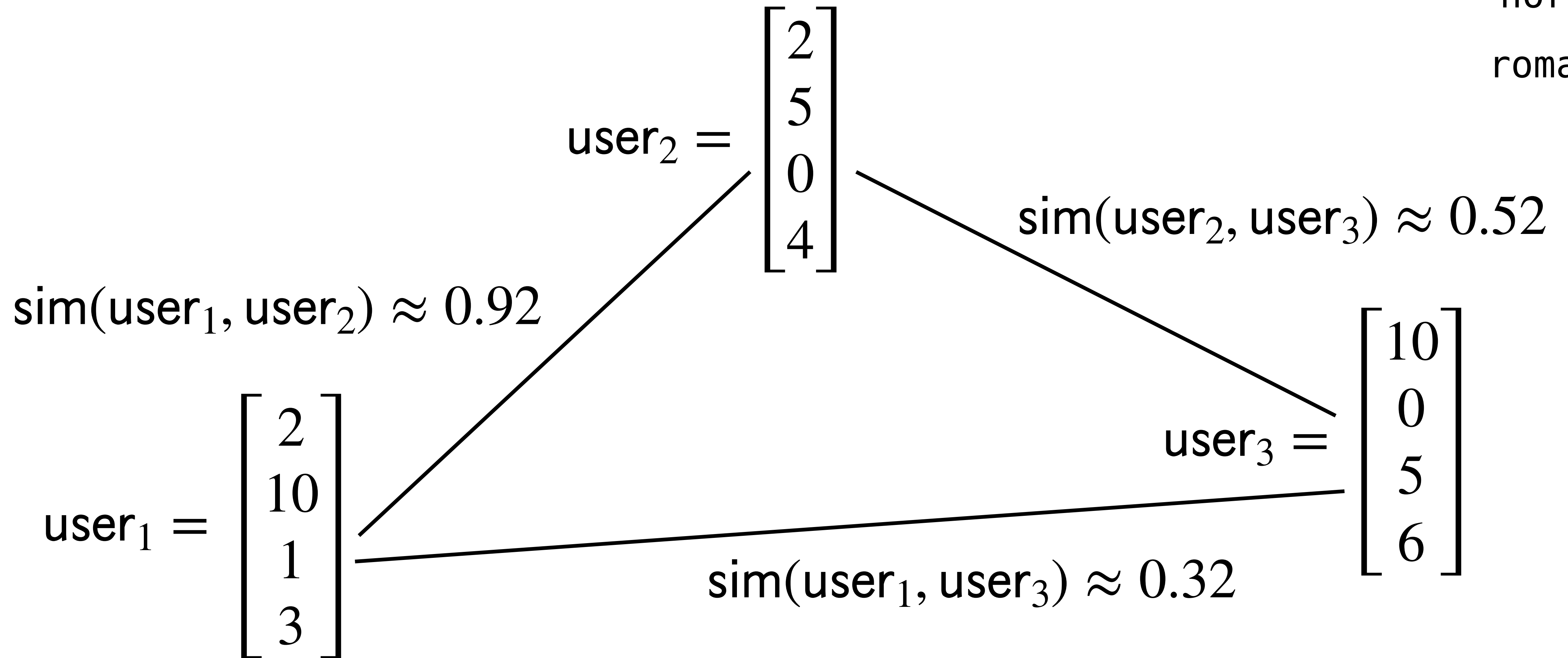
Example: Netflix Users

comedy

drama

horror

romance



Other Examples

- ***Document similarity***
 - Documents \mapsto word count vectors
 - Similar documents should use similar words
- ***Word2Vec***
 - Words \mapsto vector *somehow*
 - This underlies modern natural language processing (NLP)

Summary

We can talk about distances and angles in \mathbb{R}^n .

Every basic geometric concept connects to inner products.

Once we can talk about distances and angles we can talk about similarity.