Orthogonal Sets and Projection

Geometric Algorithms
Lecture 22

CAS CS 132

Introduction

Recap Problem

(Final Review) Find a set of vectors which forms a basis for the hyperplane given by the equation (i.e., the solution set)

$$x_1 + 3x_2 - 4x_3 + 6x_4 = 0$$

Answer

$$\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 4 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad -6 \begin{bmatrix} -6 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$x_1 + 3x_2 - 4x_3 + 6x_4 = 0$$

free

$$\left[\begin{array}{cccc|c} 1 & 3 & -4 & 6 & 0 \end{array}\right]$$

pivot column

$$x_1 = -3x_2 + 4x_3 - 6x_4$$

x_2 is free

x_3 is free

x_4 is free

$$x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -6 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Objectives

1. Recap analytic geometry in R^n.
2. Try to understand why it is useful to work with orthogonal vectors.
3. Get a sense of how to compute orthogonal vectors.
4. Start to connect orthogonality to matrices and linear transformations.

Keywords

orthogonal

orthogonal set

orthogonal basis

orthogonal projection

orthogonal component

orthonormal

orthonormal set

orthonormal basis

orthonormal matrix

orthogonal matrix

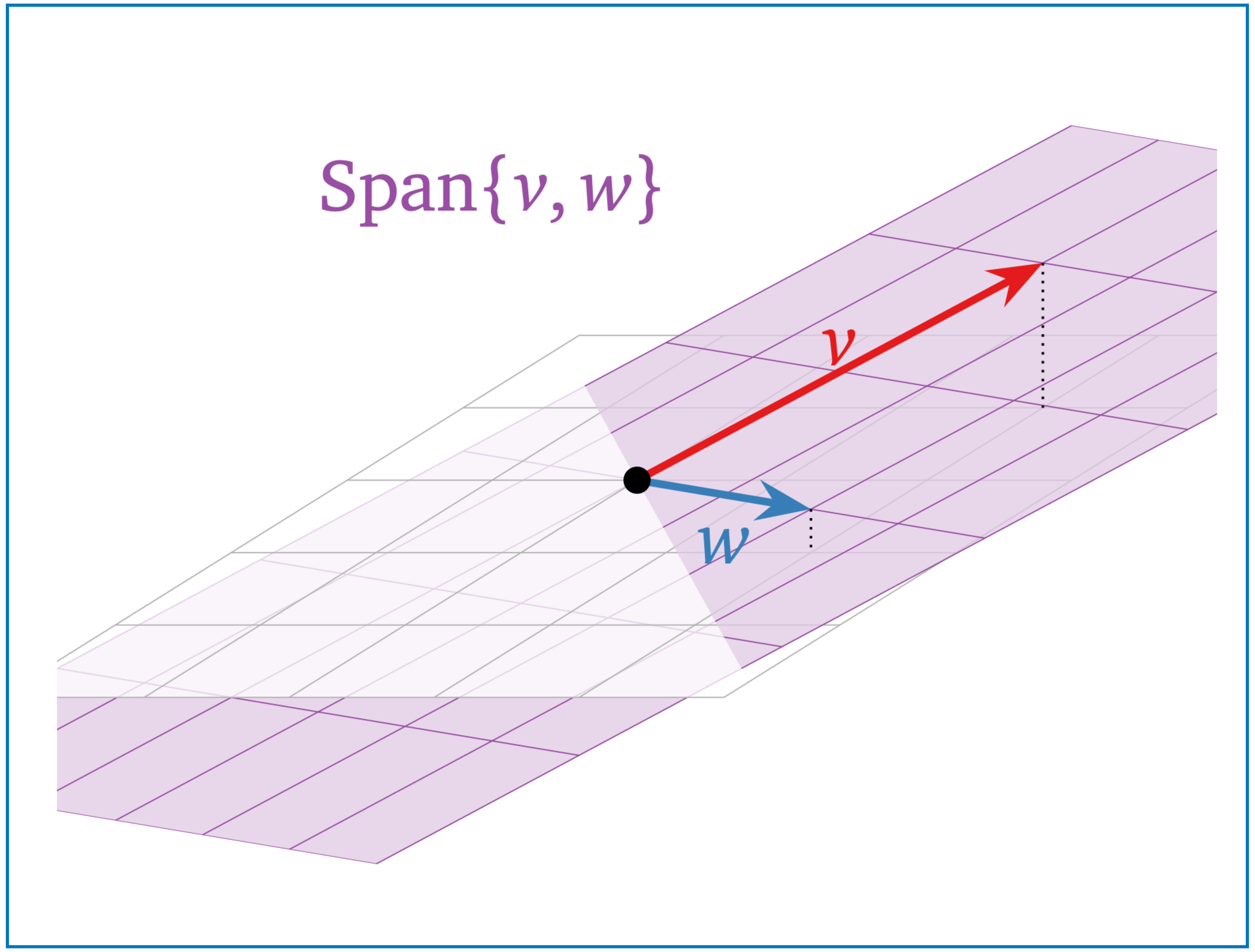
Recap: Analytic Geometry

Recall: The First Key Idea

Angles make sense in *any* dimension.

Any pair of vectors in $\mathbb{R}^n$ span a (2D) plane.

(We could formalize this via change of bases)



Recall: The Second Key Idea

All of the basic concepts of analytic geometry can be defined *in terms of inner products*.

Spaces with inner products (like $\mathbb{R}^n$) are places where you can do analytic geometry.

Recall: Inner Products

$$\begin{bmatrix} u_1 & u_2 & u_3 & u_4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$$

Definition. The **inner product** of two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ is

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}$$

Recall: Inner Products

$$\begin{bmatrix} u_1 & u_2 & u_3 & u_4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$$

Definition. The **inner product** of two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ is

a.k.a. dot product

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}$$

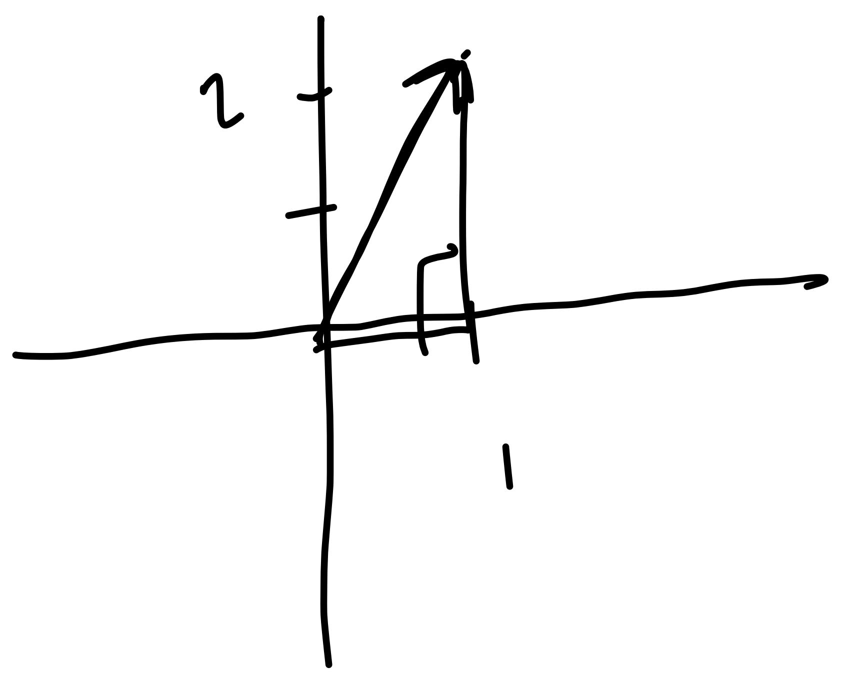
Recall: Norms and Inner Products

Definition. The ℓ^2 **norm** of a vector $\mathbf{v}$ in $\mathbb{R}^n$ is

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$$

The norm of a vector is the square root of the inner product with itself.

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$



$$\sqrt{1^2 + 2^2} = \sqrt{\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

Recall: Norms and Inner Products

Definition. The ℓ^2 **norm** of a vector $\mathbf{v}$ in $\mathbb{R}^n$ is

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$$

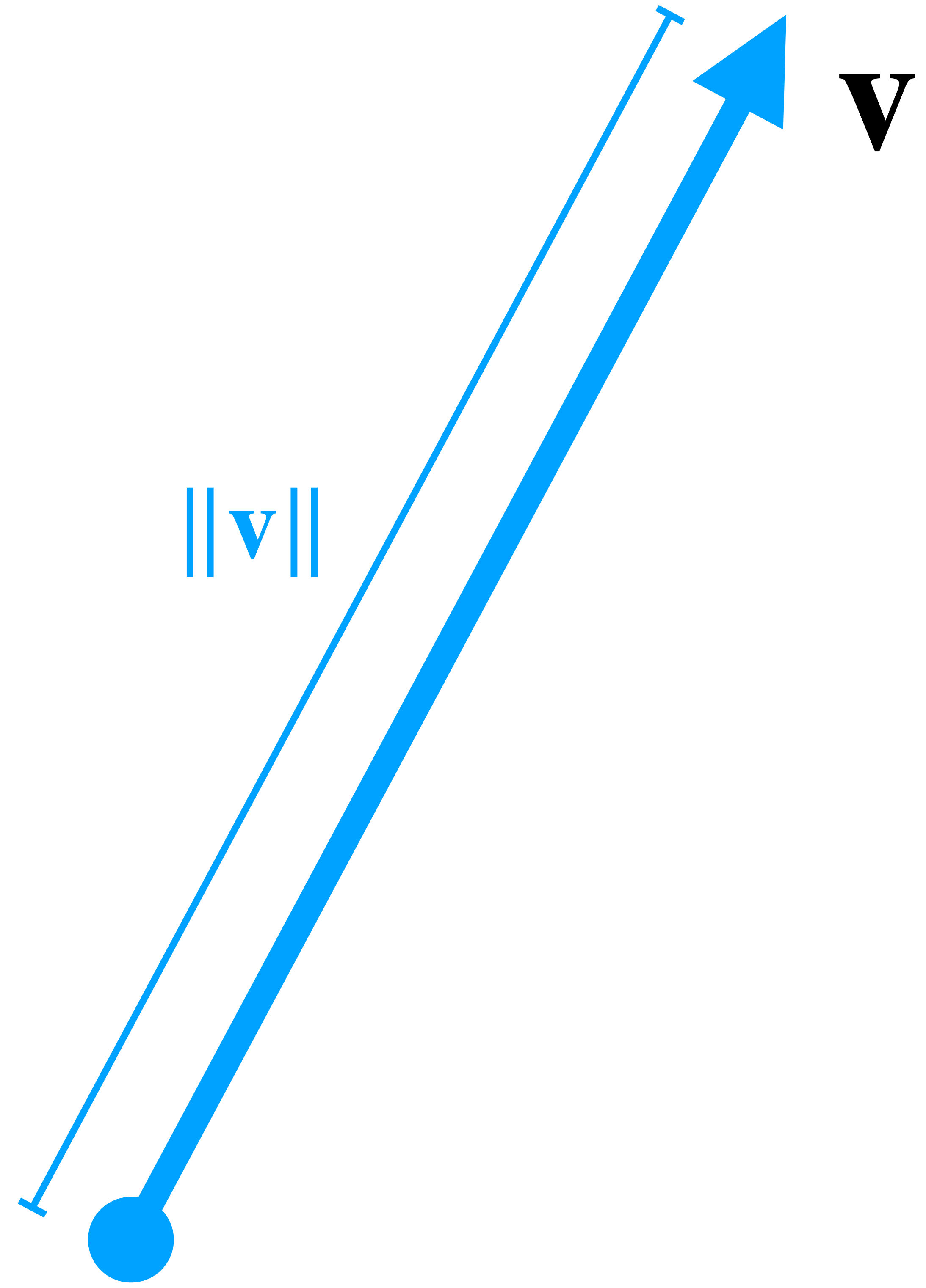
The norm of a vector is the square root of the inner product with itself.

It's important that $\mathbf{v}^T\mathbf{v}$ is nonnegative.

Recall: Norms and Length

Norms give us a notion of length.

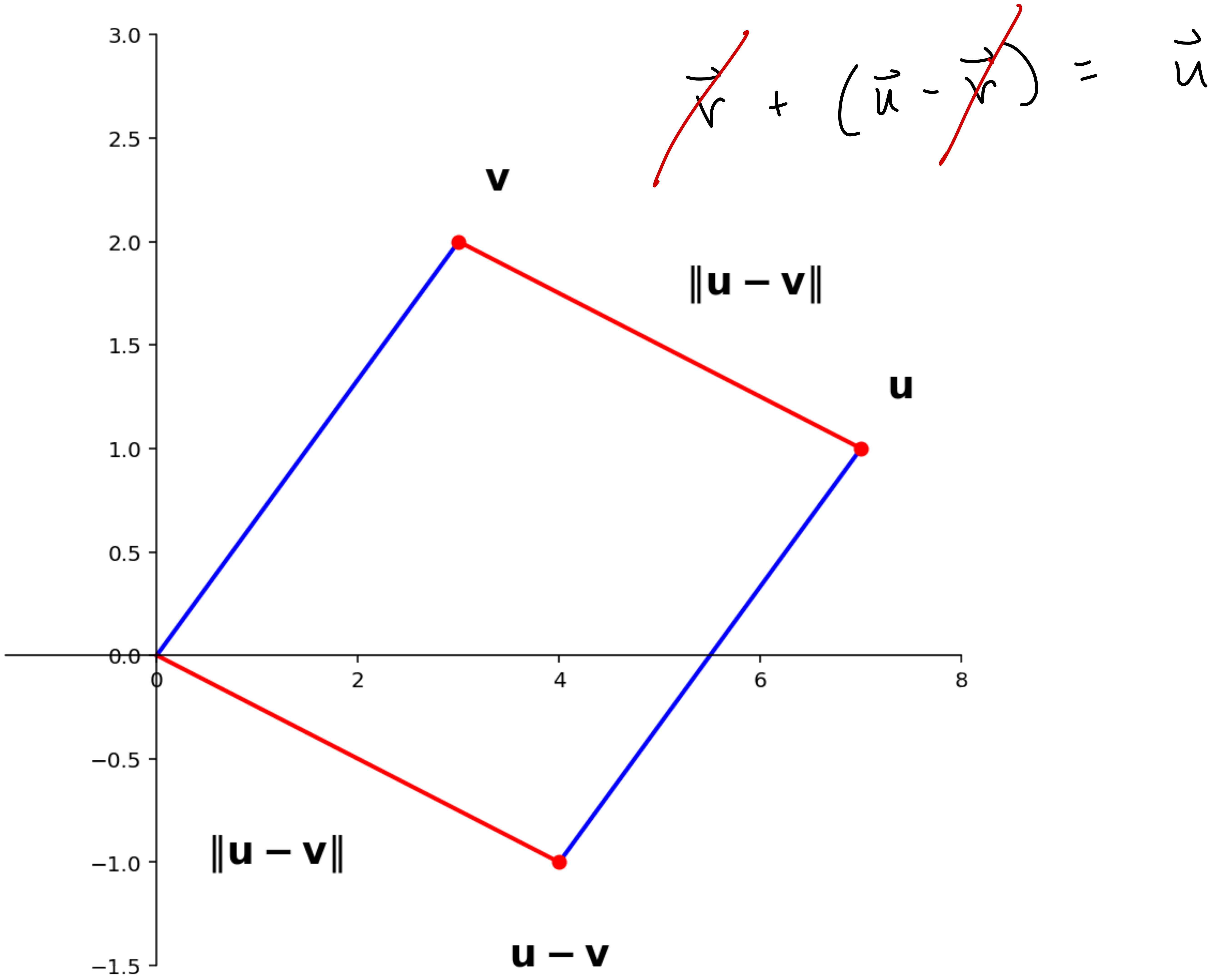
In $\mathbb{R}^2$ and $\mathbb{R}^3$ this is our existing notion of length.



Recall: Distance

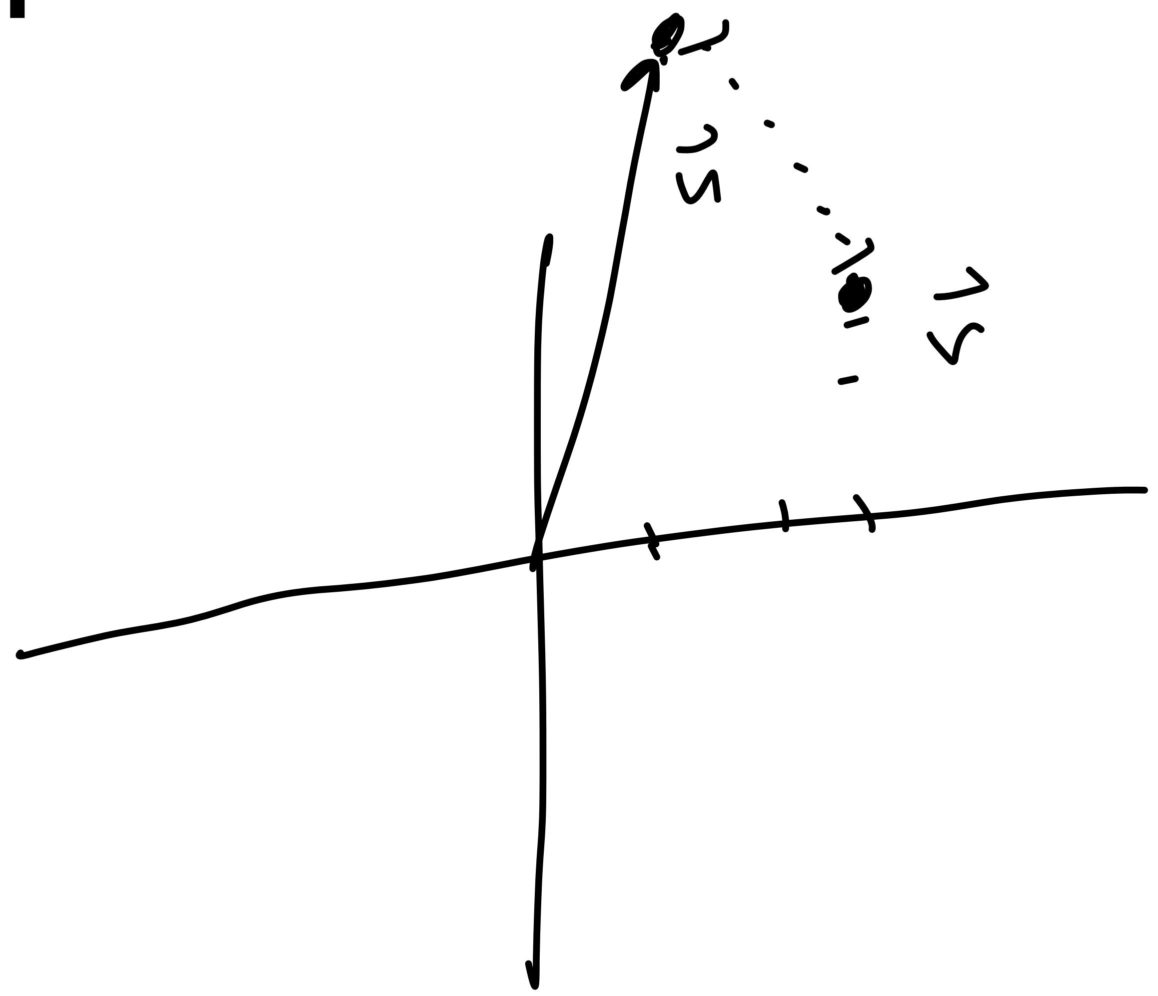
If we know how to calculate lengths of vectors, we know how to calculate distances.

Recall: Distance (Pictorially)



Recall: Distance (Algebraically)

Definition. The distance between two points $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ is given by



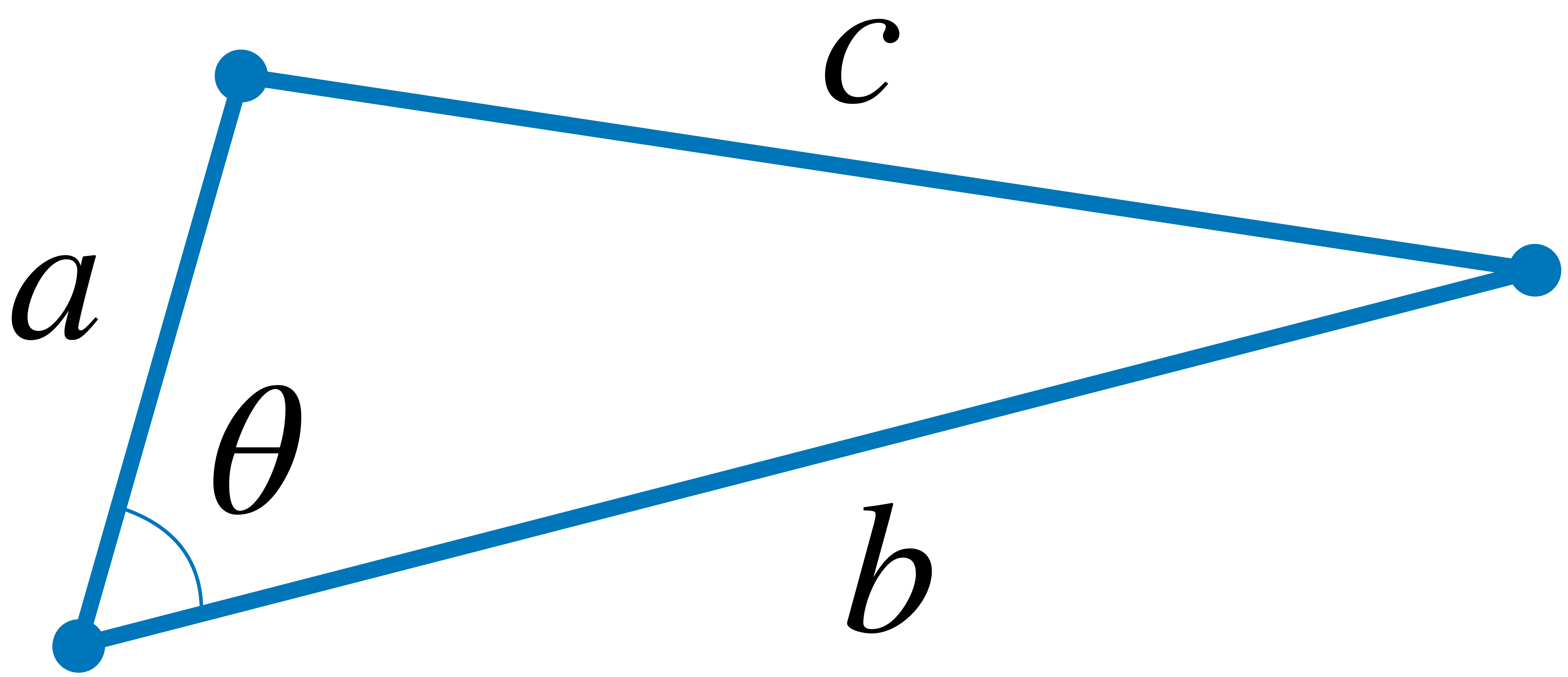
$$\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

e.g., $\mathbf{u} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$$\sqrt{(7-3)^2 + (1-2)^2} = \sqrt{17}$$

$$\sqrt{16 + 1}$$

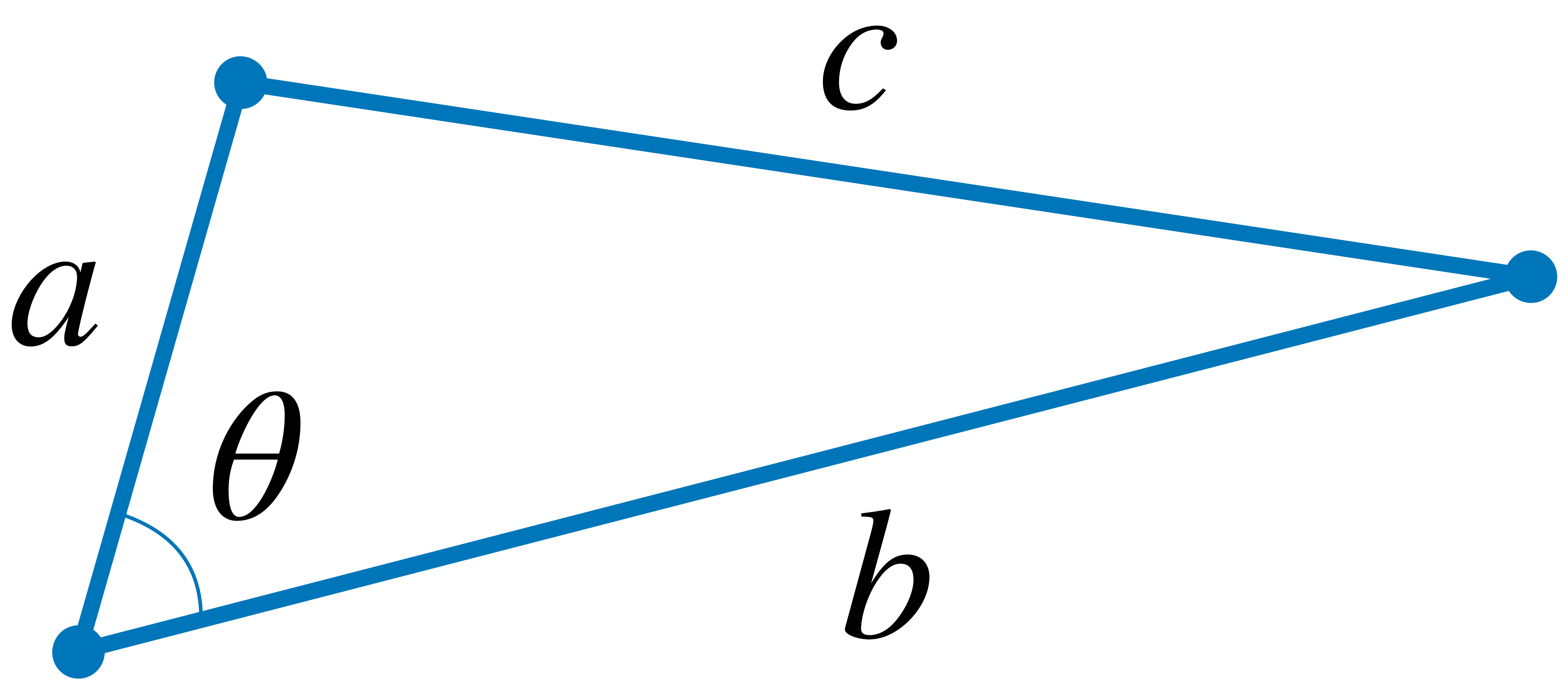
Recall: Law of Cosines



Theorem.

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

Recall: Law of Cosines

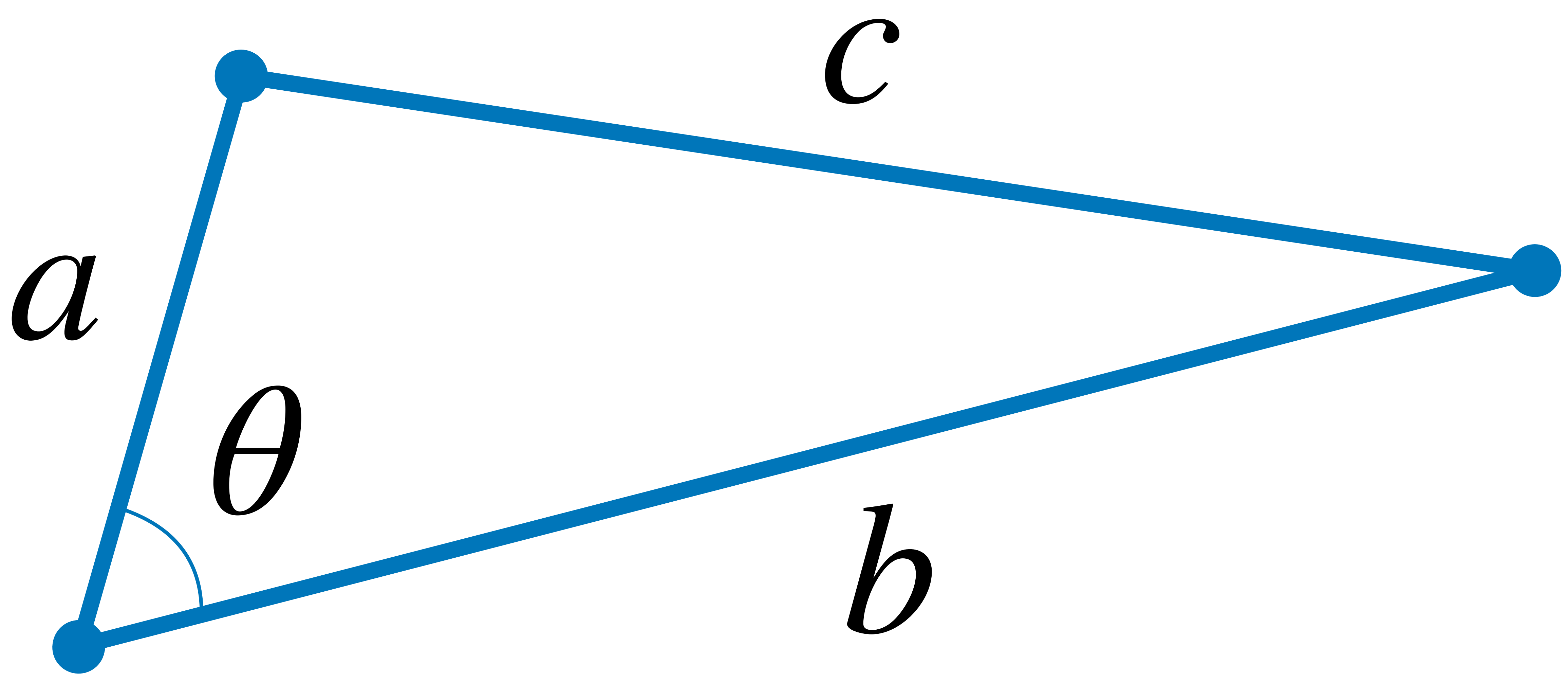


Theorem.

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

Generalized the Pythagorean Theorem

Recall: Law of Cosines



Theorem.

0 exactly when $\theta = 90°$

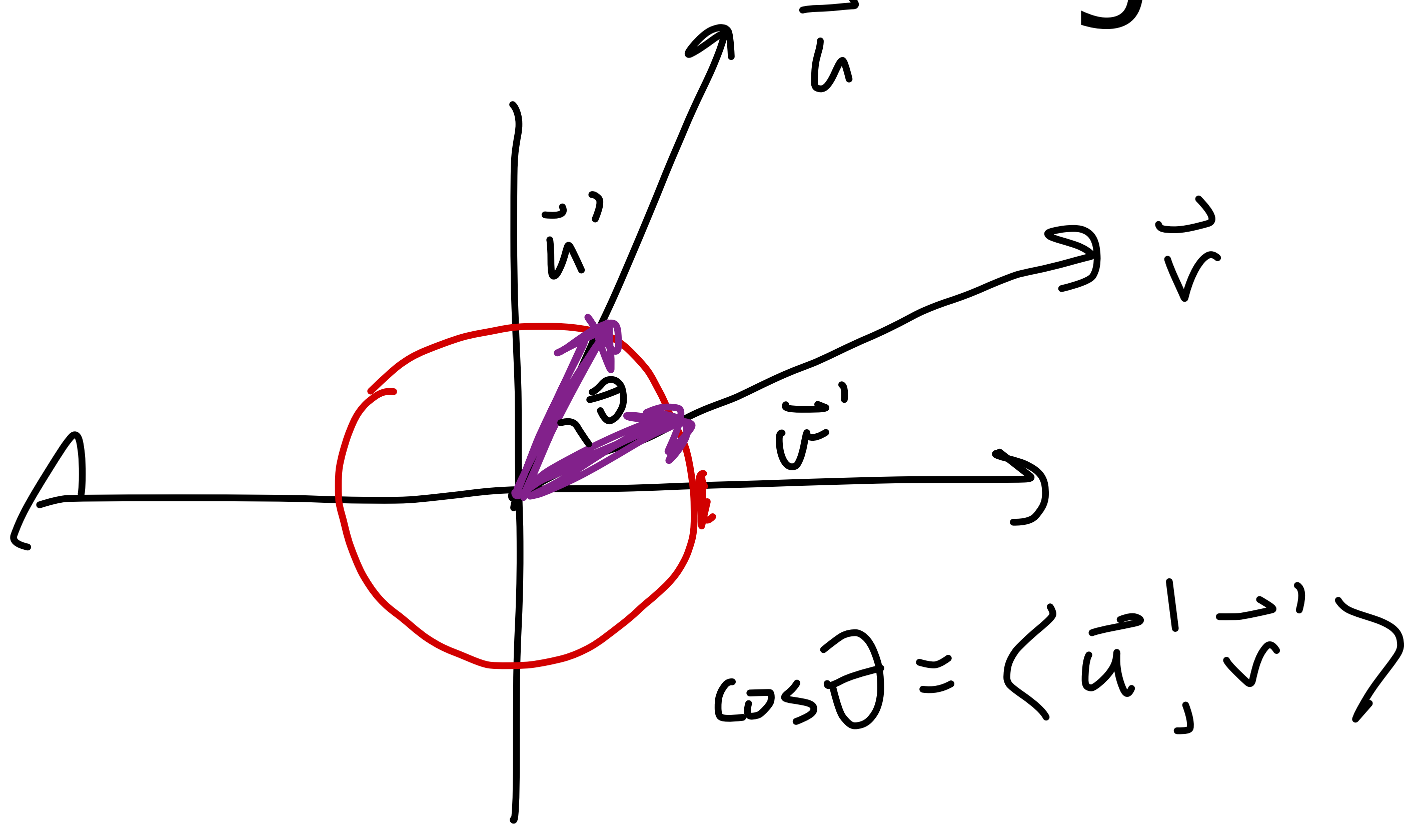
$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

Generalized the Pythagorean Theorem

Recall: Cosines and Unit Vectors

Theorem. For vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ with an angle θ between them,

$$\cos\theta = \left\langle \frac{\mathbf{u}}{\|\mathbf{u}\|}, \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\rangle$$



The cosine of the angle between two vectors is the inner product of their ℓ^2 normalizations.

Recall: Orthogonality

Recall: Orthogonality

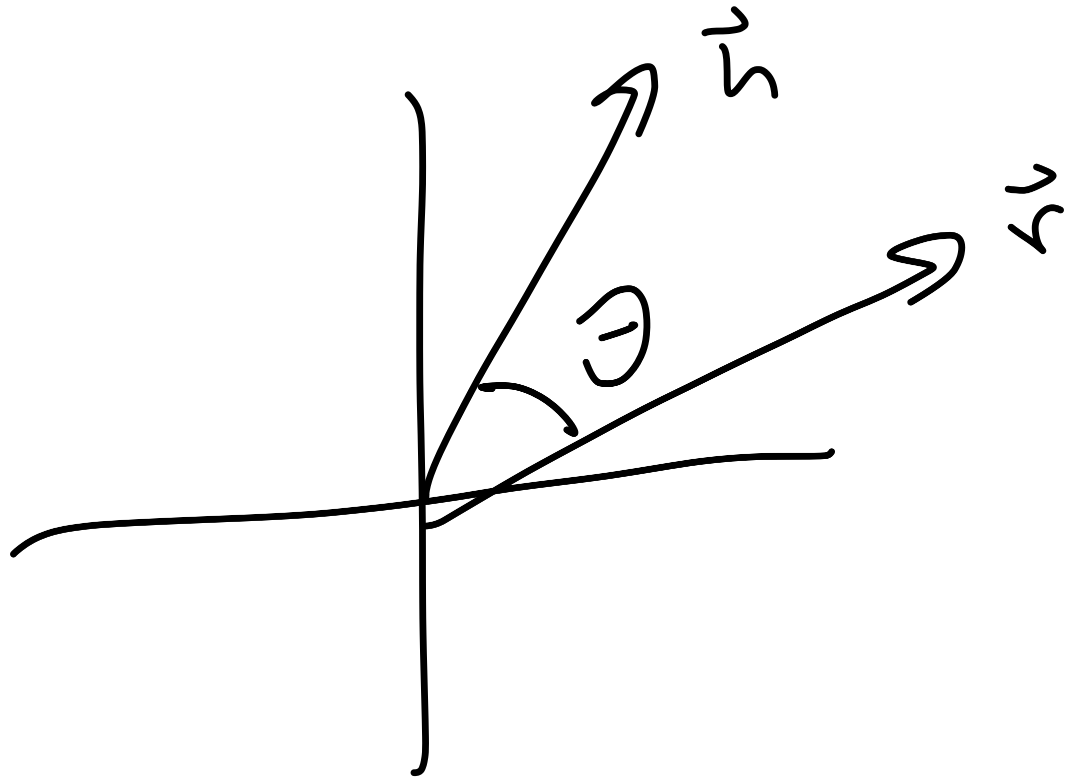
Definition (Informal). Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are **orthogonal** if the angle between them is 90°.

Recall: Orthogonality

Definition (Informal). Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are **orthogonal** if the angle between them is $90°$.

Orthogonal and perpendicular are the same thing.

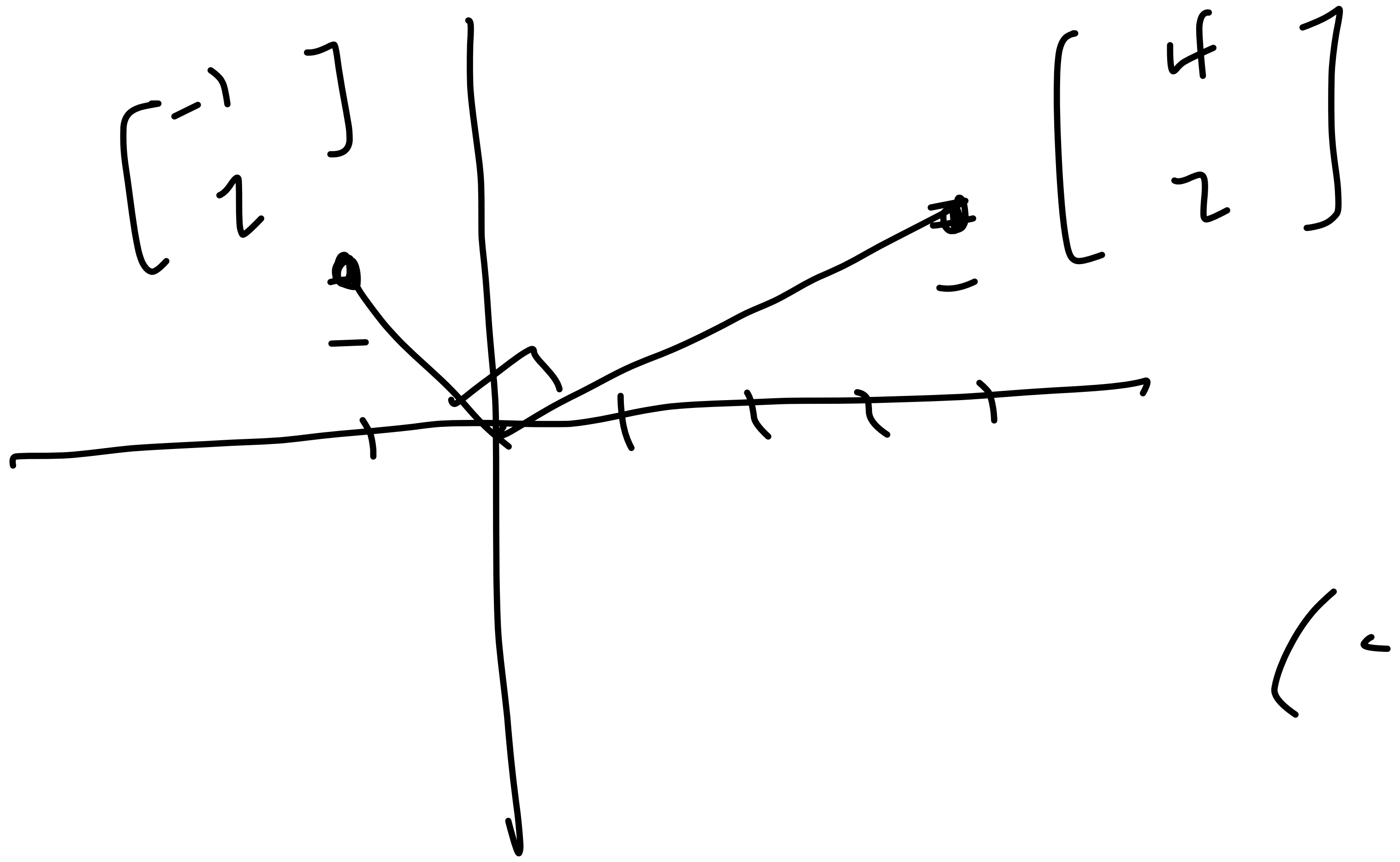
Recall: Orthogonality



Definition (Actual). Vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

Verify: $\cos\theta = \left\langle \frac{\vec{u}}{\|u\|}, \frac{\vec{v}}{\|\vec{v}\|} \right\rangle = 0$ if and only if $\theta = 90^\circ$

Example:



$\left\langle \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \end{bmatrix} \right\rangle =$

$(-1)(4) + 2(2) = 0$ ✓

In All

With inner products:

- Given a vector we can determine its length
- Given two points (vectors) we can determine the distance between them
- Given two vectors we can determine the angle between them

Orthogonal Sets

Orthogonal Sets

Definition. A set $\{u_1, u_2, \ldots, u_p\}$ of vectors from R^n is an **orthogonal set** if every pair of distinct vectors is orthogonal: if $i \neq j$ then

$$\langle u_i, u_j \rangle = 0$$

Each vector is pairwise/mutually perpendicular.

Example

$$u_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \qquad u_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$$

Verify:

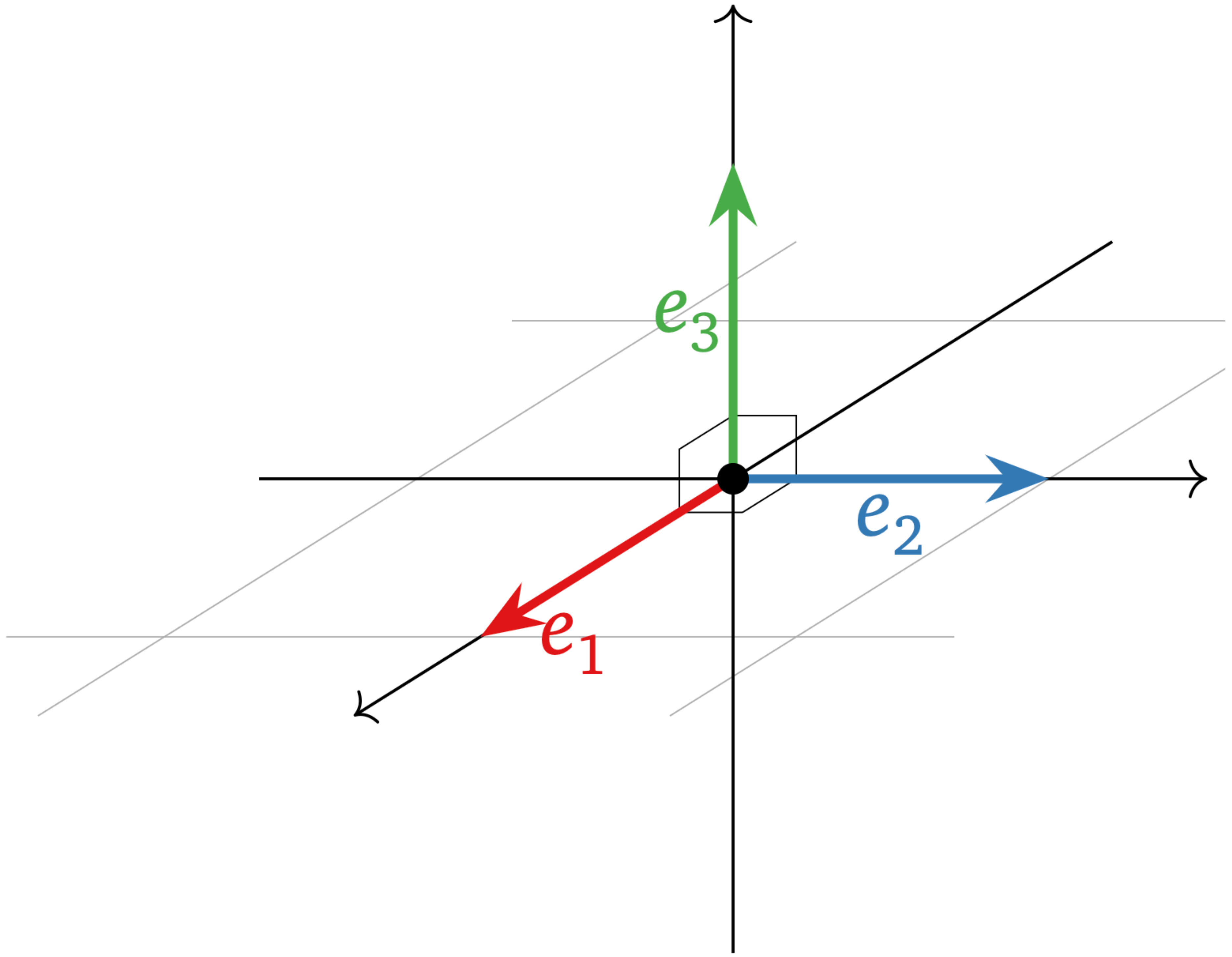
$$\langle u_1, u_2 \rangle = 3(-1) + 1(2) + 1 = 0$$

$$\langle u_1, u_3 \rangle = 3\left(-\tfrac{1}{2}\right) + 1(-2) + 1\left(\tfrac{7}{2}\right) = 0$$

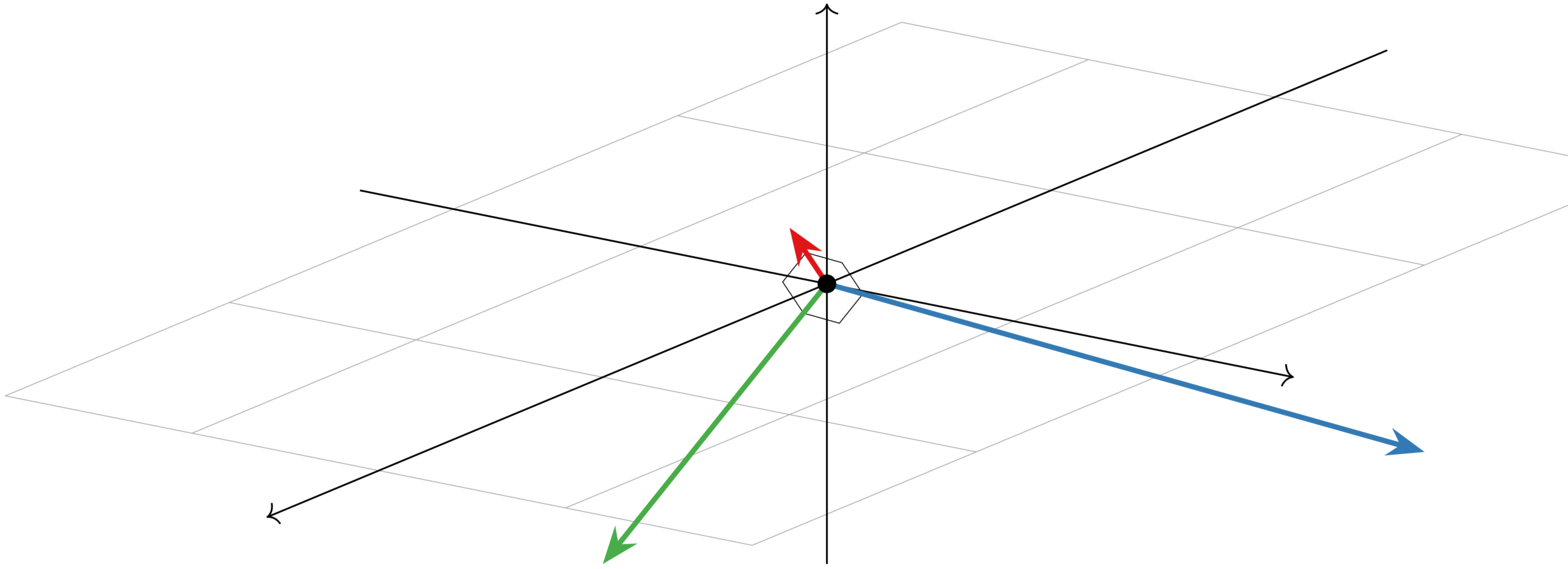
$$-3/2 \qquad -4/2 \qquad + 7/2$$

What do orthogonal sets look like?

The Picture



the standard basis forms a
"centered" orthogonal set



an orthogonal set is like
the standard basis *after*
some rotations and scalings

https://textbooks.math.gatech.edu/ila/orthogonal-sets.html

Orthogonal Sets and Independence

Theorem. If $\{u_1, u_2, \ldots, u_k\}$ is an orthogonal set of *nonzero* vectors from R^n, then it is <u>linearly independent</u>.

Verify:

$$c_1\vec{u}_1 + c_2\vec{u}_2 + \ldots + c\vec{u}_k = \vec{0}$$

has to be zero.

$$\langle c_1\vec{u}_1 + c_2\vec{u}_2 + \ldots + c\vec{u}_k, \vec{u}_1 \rangle =$$

$$c_1\langle \vec{u}_1, \vec{u}_1\rangle + c_2\langle \vec{u}_2, \vec{u}_1\rangle + \ldots + c_k\langle \vec{u}_k, \vec{u}_1\rangle$$

$\langle \vec{u}_2, \vec{u}_1\rangle = 0$, $\langle \vec{u}_k, \vec{u}_1\rangle = 0$

has to be nonzero

$$c_1 \|\vec{u}_1\|^2 = \langle \vec{0}, \vec{u}_1\rangle = 0$$

The Takeaway

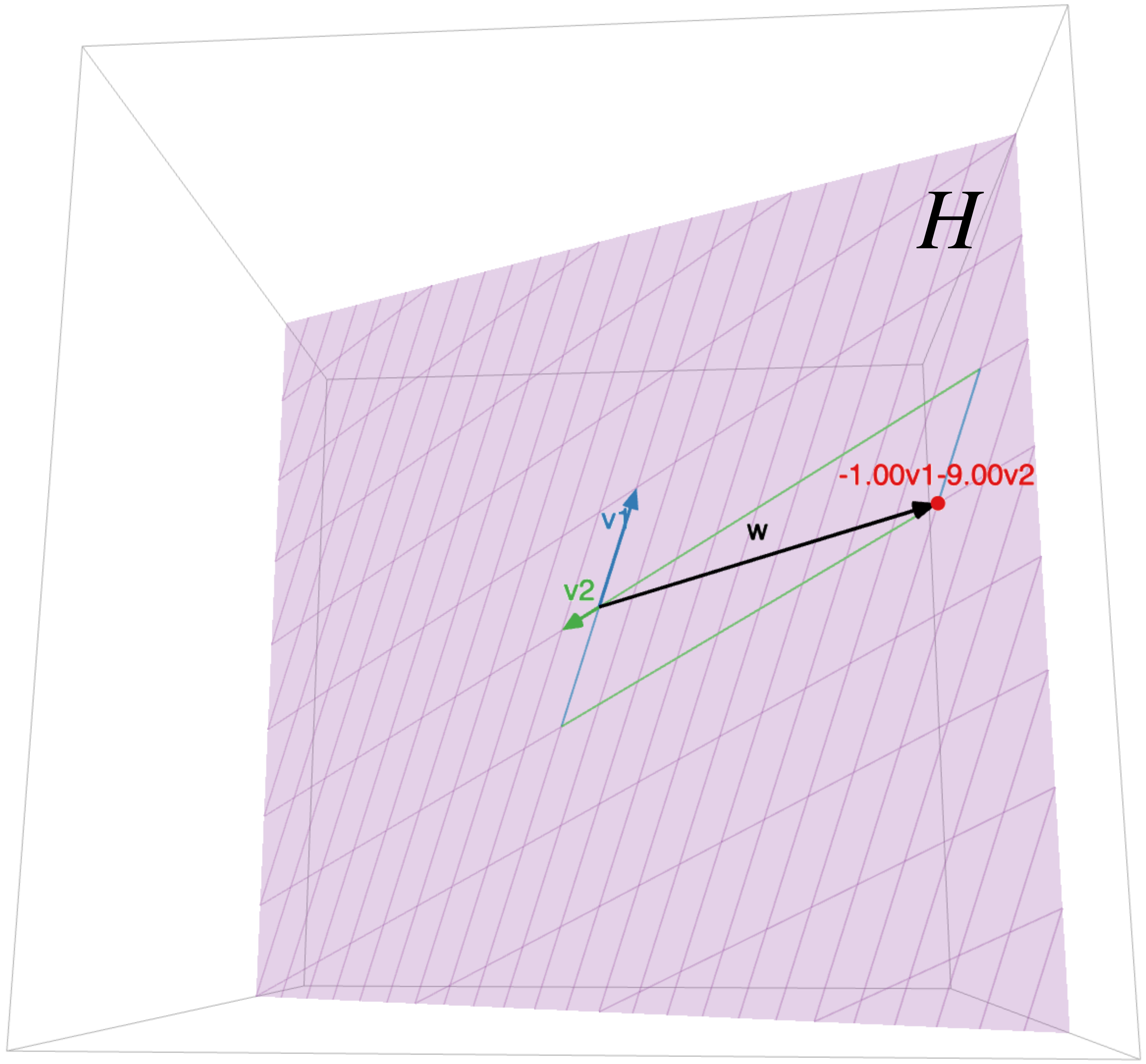
If $\{u_1, u_2, \ldots, u_k\}$ is an orthogonal set, then it is a **basis** for $span\{u_1, u_2, \ldots, u_k\}$.

Orthogonal Basis

Definition. An **orthogonal basis** for a subspace W of R^n is a basis for W which is also an orthogonal set.

Orthogonal Basis

Definition. An **orthogonal basis** for a subspace W of R^n is a basis for W which is also an orthogonal set.

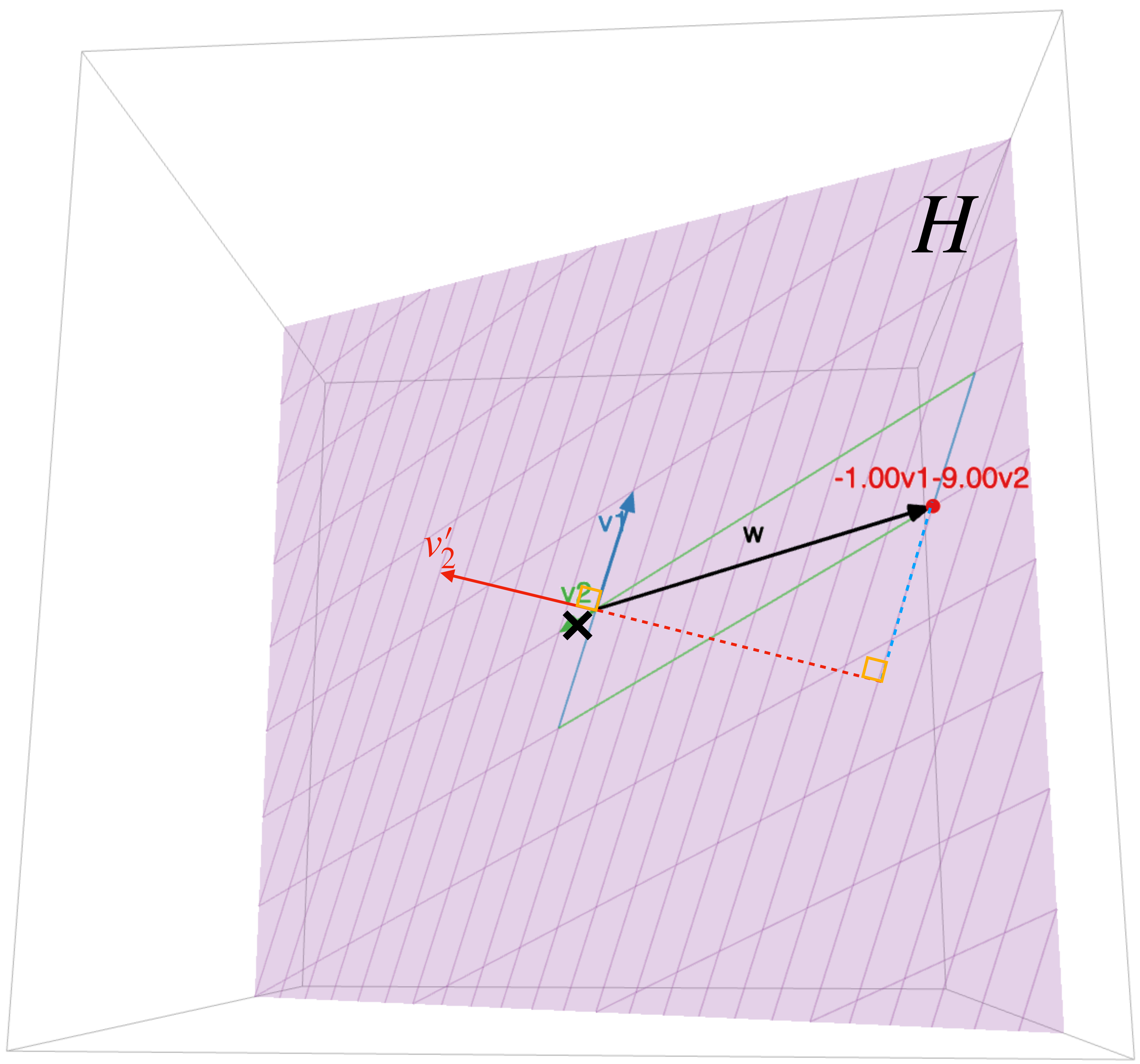


v_1 and v_2 form a basis of H

https://textbooks.math.gatech.edu/ila/spans.html

Orthogonal Basis

Definition. An **orthogonal basis** for a subspace W of R^n is a basis for W which is also an orthogonal set.



v_1 and v_2 form a basis of H

v_1 and v_2' form an **orthogonal** basis of H

https://textbooks.math.gatech.edu/ila/spans.html

What's nice about an orthogonal basis?

Recall: How To: Bases

Recall: How To: Bases

Question. Given a basis $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ for a subspace W of R^n and a vector $\mathbf{w}$ in W, weights $c_1, c_2, \ldots, c_p$ such that

$$\mathbf{w} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots c_p\mathbf{u}_p$$

Recall: How To: Bases

Question. Given a basis $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ for a subspace W of R^n and a vector $\mathbf{w}$ in W, weights $c_1, c_2, \ldots, c_p$ such that

$$\mathbf{w} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots c_p\mathbf{u}_p$$

Solution. Solve the vector equation

$$x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + \ldots x_p\mathbf{u}_p = \mathbf{w}$$

by Gaussian elimination, matrix inversion, etc.

Recall: How To: Bases

Question. Given a basis $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ for a subspace W of R^n and a vector $\mathbf{w}$ in W, weights $c_1, c_2, \ldots, c_p$ such that

$$\mathbf{w} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots c_p\mathbf{u}_p$$

Solution. Solve the vector equation

$$x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + \ldots x_p\mathbf{u}_p = \mathbf{w}$$

by Gaussian elimination, matrix inversion, etc.

This takes work

Orthogonal Bases and Linear Combinations

Theorem. For an orthogonal set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$, if $\mathbf{y} = c_1\mathbf{u}_1 + \ldots + c_p\mathbf{u}_p$ then for $j = 1, \ldots, p$

$$c_j = \frac{\mathbf{y}^T\mathbf{u}_j}{\mathbf{u}_j^T\mathbf{u}_j}$$

Verify: $\langle c_1\vec{u}_1 + \ldots + c\vec{u}_k, \vec{u}_1 \rangle = \langle c_1\vec{u}_1, \vec{u}_1\rangle + \ldots + \langle c_k\vec{u}_k, \vec{u}_1\rangle$ $\quad 0 =$

$$\frac{\langle \vec{y}, \vec{u}_1\rangle}{\langle \vec{u}_1, \vec{u}_1\rangle} = c_1 \frac{\langle \vec{u}_1, \vec{u}_1\rangle}{\langle \vec{u}_1, \vec{u}_1\rangle}$$

How To: Orthogonal Bases

How To: Orthogonal Bases

Question. Given an **orthogonal** basis $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ for a subspace W of R^n and a vector $\mathbf{w}$ in W, weights $c_1, c_2, \ldots, c_p$ such that

$$\mathbf{w} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots c_p\mathbf{u}_p$$

How To: Orthogonal Bases

Question. Given an **orthogonal** basis $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ for a subspace W of R^n and a vector $\mathbf{w}$ in W, weights $c_1, c_2, \ldots, c_p$ such that

$$\mathbf{w} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots c_p\mathbf{u}_p$$

Solution. $c_j = \dfrac{\mathbf{w} \cdot \mathbf{u}_j}{\mathbf{u}_j \cdot \mathbf{u}_j}$

How To: Orthogonal Bases

Question. Given an **orthogonal** basis $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ for a subspace W of R^n and a vector $\mathbf{w}$ in W, weights $c_1, c_2, \ldots, c_p$ such that

$$\mathbf{w} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots c_p\mathbf{u}_p$$

Solution. $c_j = \dfrac{\mathbf{w} \cdot \mathbf{u}_j}{\mathbf{u}_j \cdot \mathbf{u}_j}$

Much easier to compute.

Question

Express $[6 \;\; 1 \;\; (-8)]^T$ *as a linear combination of vectors in* $\{u_1, u_2, u_3\}$ *where*

$$u_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \qquad u_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$$

Answer: $u_1 - 2u_2 - 2u_3$

$\vec{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ $\quad u_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ $\quad \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix}$

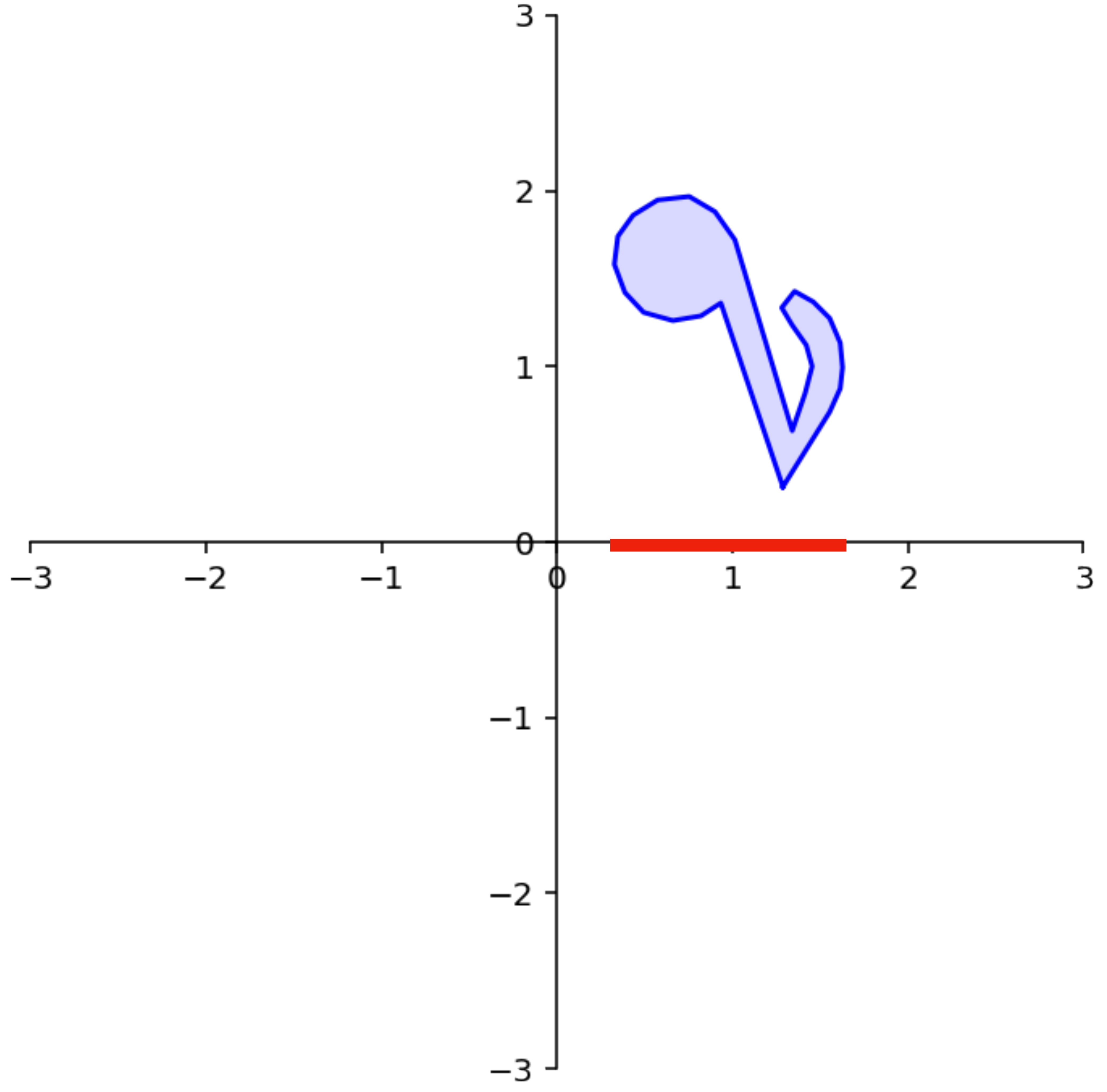
$$c_1 = \frac{\left\langle \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \right\rangle}{\left\| \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \right\|^2} = \frac{18 + 1 - 8}{9 + 1 + 1} = \frac{11}{11} = 1$$

$$c_2 = \frac{\left\langle \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\rangle}{\left\| \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\|^2} = \frac{-6 + 2 - 8}{1 + 4 + 1} = \frac{-12}{6} = -2$$

Orthogonal Projection

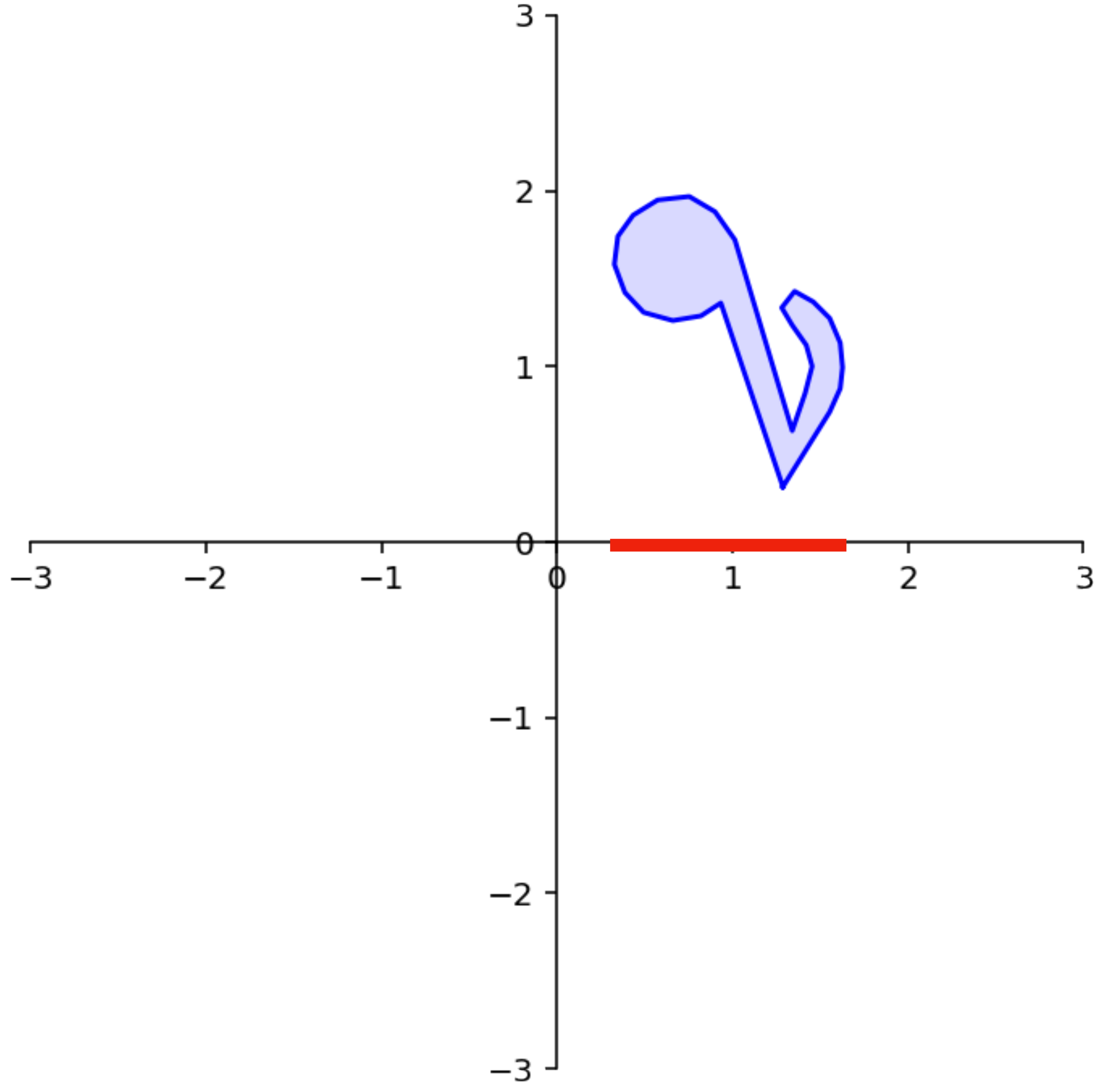
Why does that formula in the last example work?

Recall: Projection onto the x-axis



Recall: Projection onto the x-axis

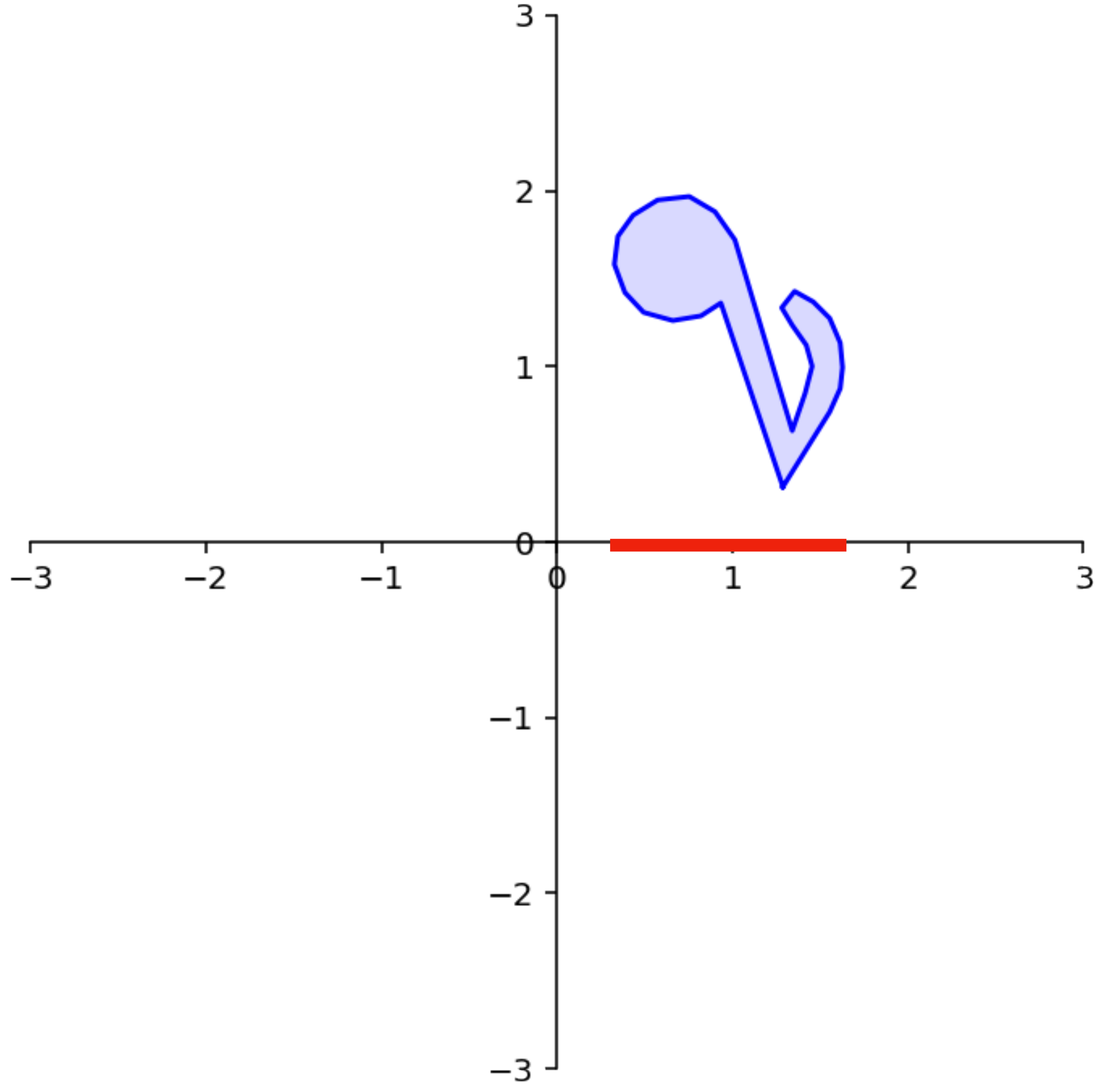
We've seen simple projections in R^2.



Recall: Projection onto the x-axis

We've seen simple projections in R^2.

We're going to generalize this idea.

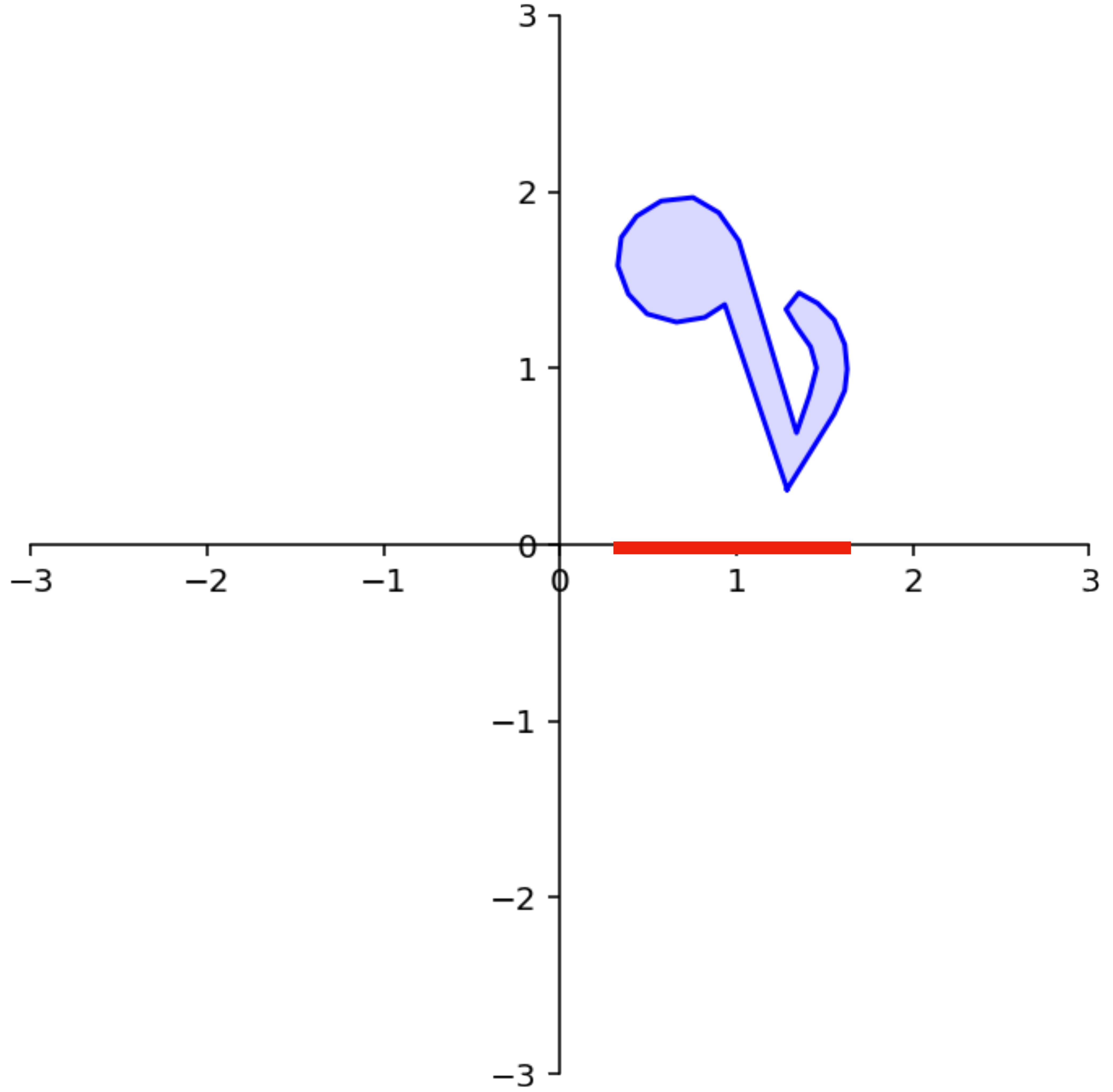


Recall: Projection onto the x-axis

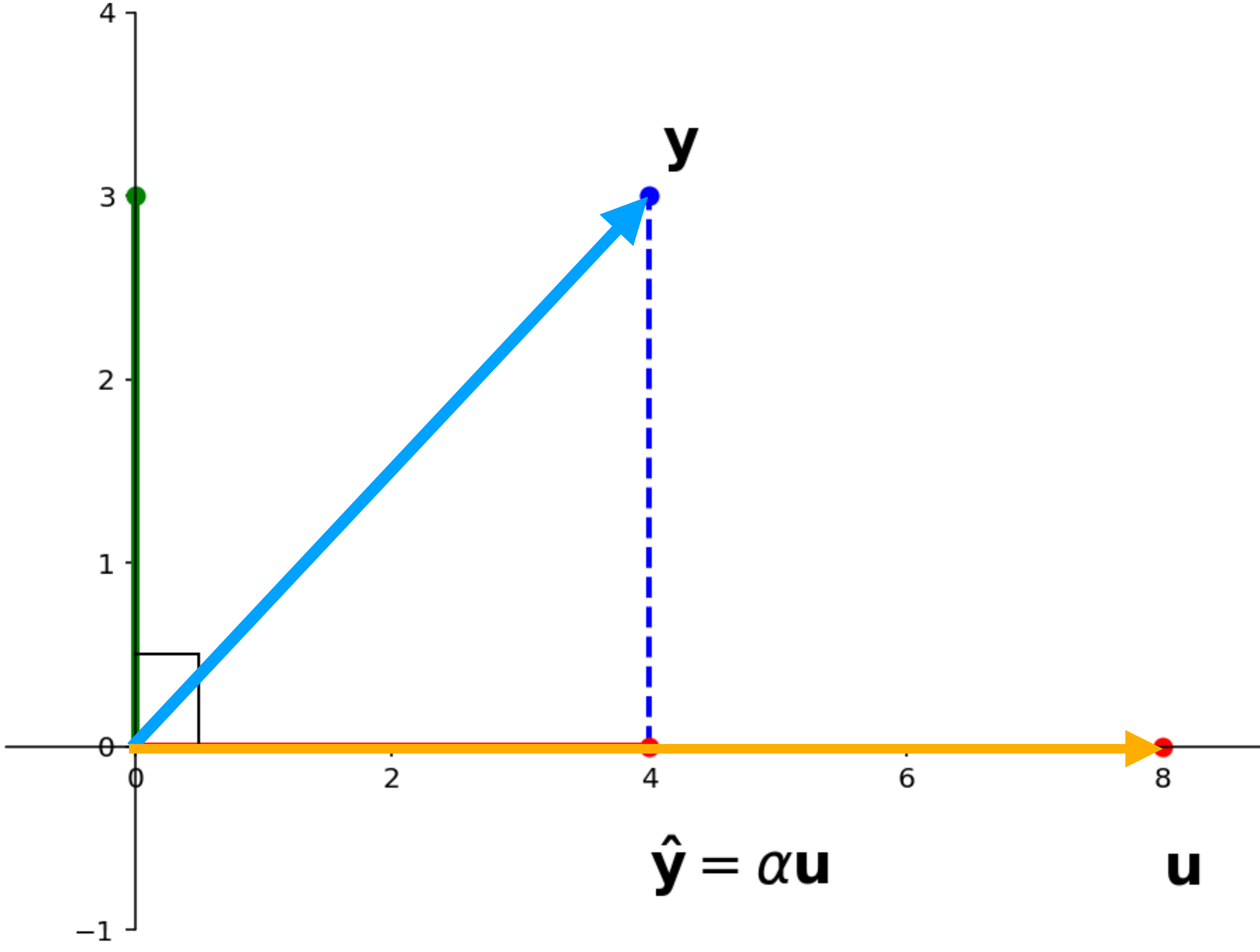
We've seen simple projections in R^2.

We're going to generalize this idea.

What we really did was a kind of projection onto the basis vectors.

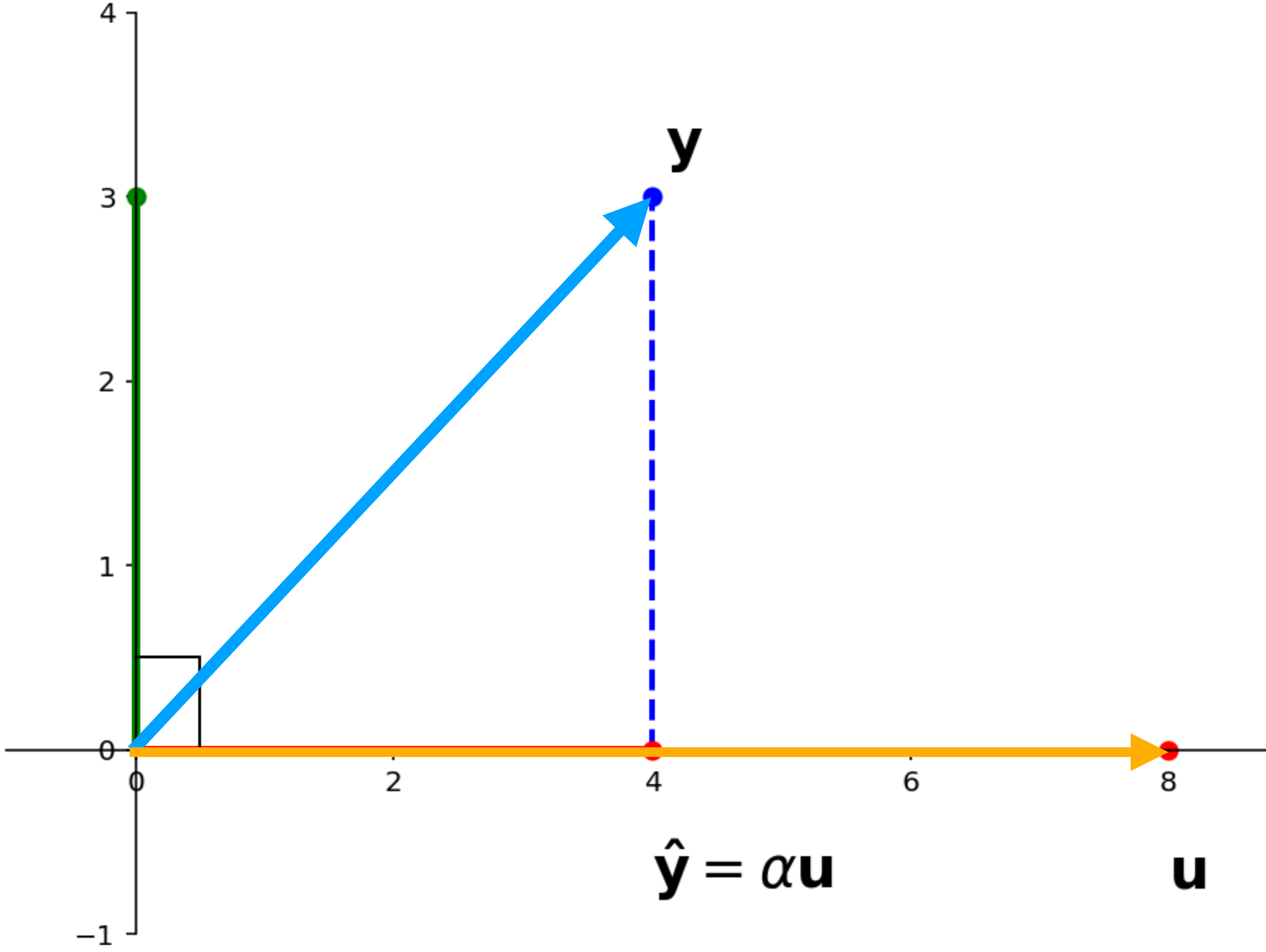


Orthogonal Projection



Orthogonal Projection

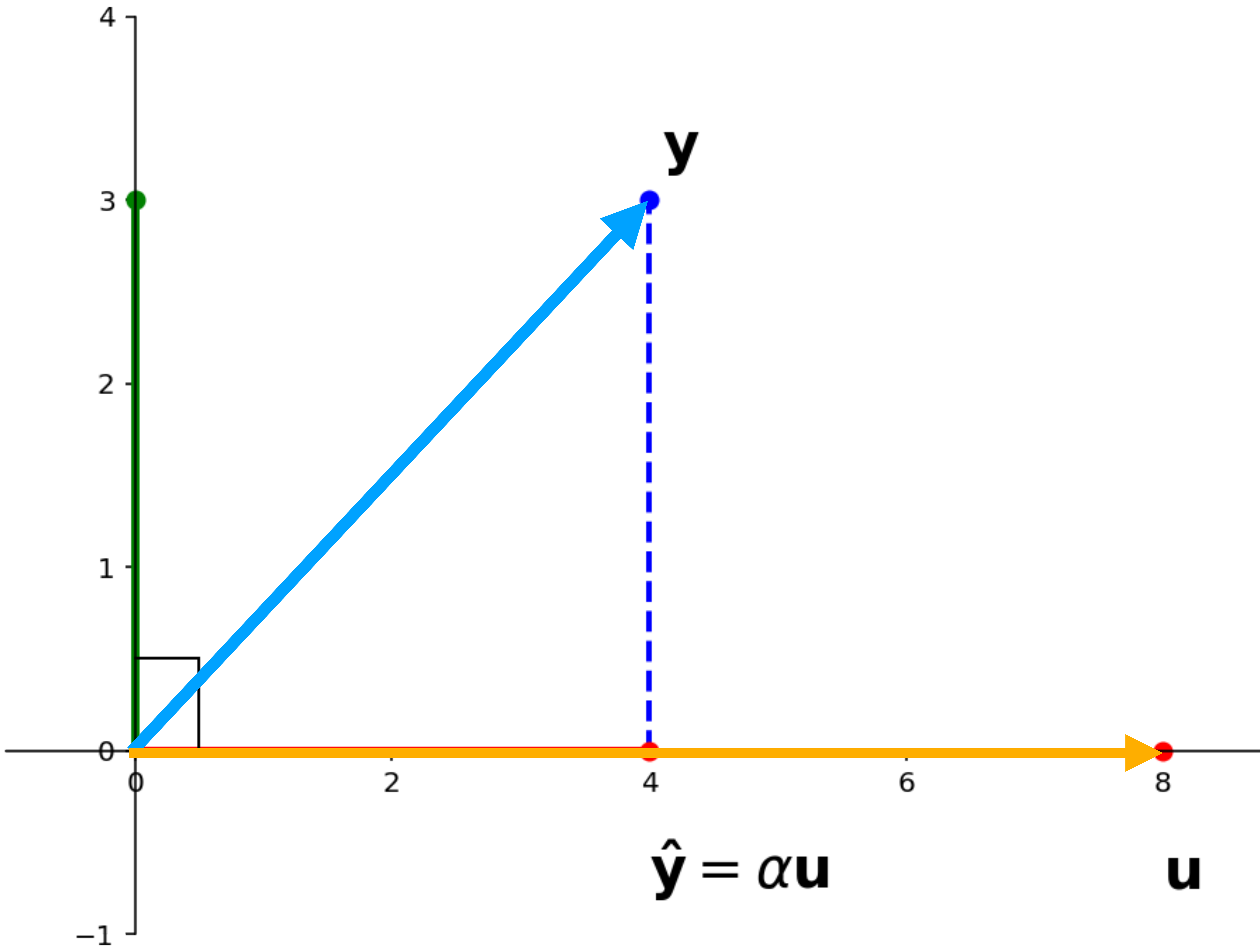
Question. Given vectors $\mathbf{y}$ and $\mathbf{u}$ in R^n, find vectors $\hat{\mathbf{y}}$ and $\mathbf{z}$ such that



Orthogonal Projection

Question. Given vectors $\mathbf{y}$ and $\mathbf{u}$ in R^n, find vectors $\hat{\mathbf{y}}$ and $\mathbf{z}$ such that

» $\mathbf{z}$ is orthogonal to $\mathbf{u}$ (i.e., $\mathbf{z} \cdot \mathbf{u} = 0$)

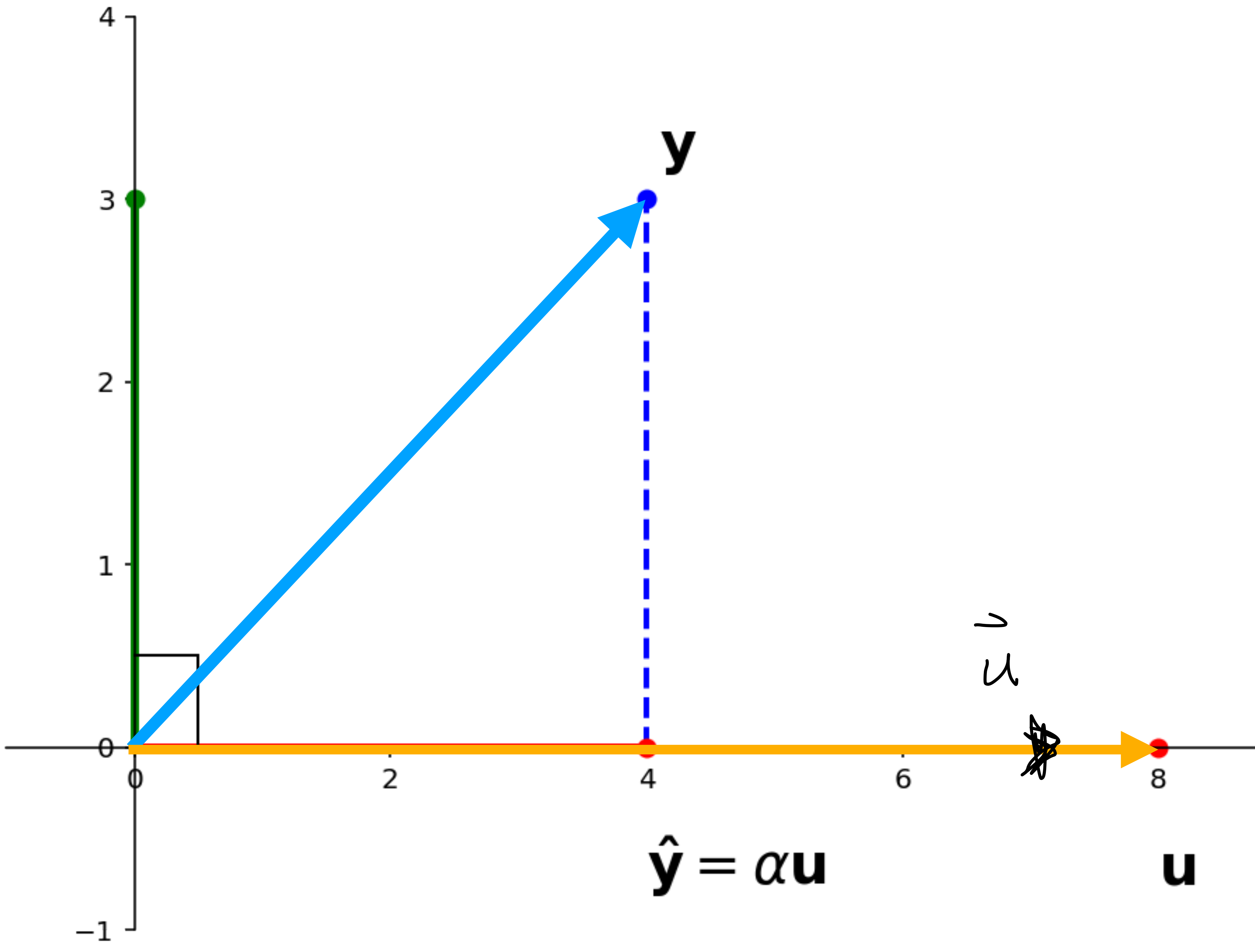


Orthogonal Projection

Question. Given vectors $\mathbf{y}$ and $\mathbf{u}$ in R^n, find vectors $\hat{\mathbf{y}}$ and $\mathbf{z}$ such that

» $\mathbf{z}$ is orthogonal to $\mathbf{u}$ (i.e., $\mathbf{z} \cdot \mathbf{u} = 0$)

» $\hat{\mathbf{y}} \in span\{\mathbf{u}\}$



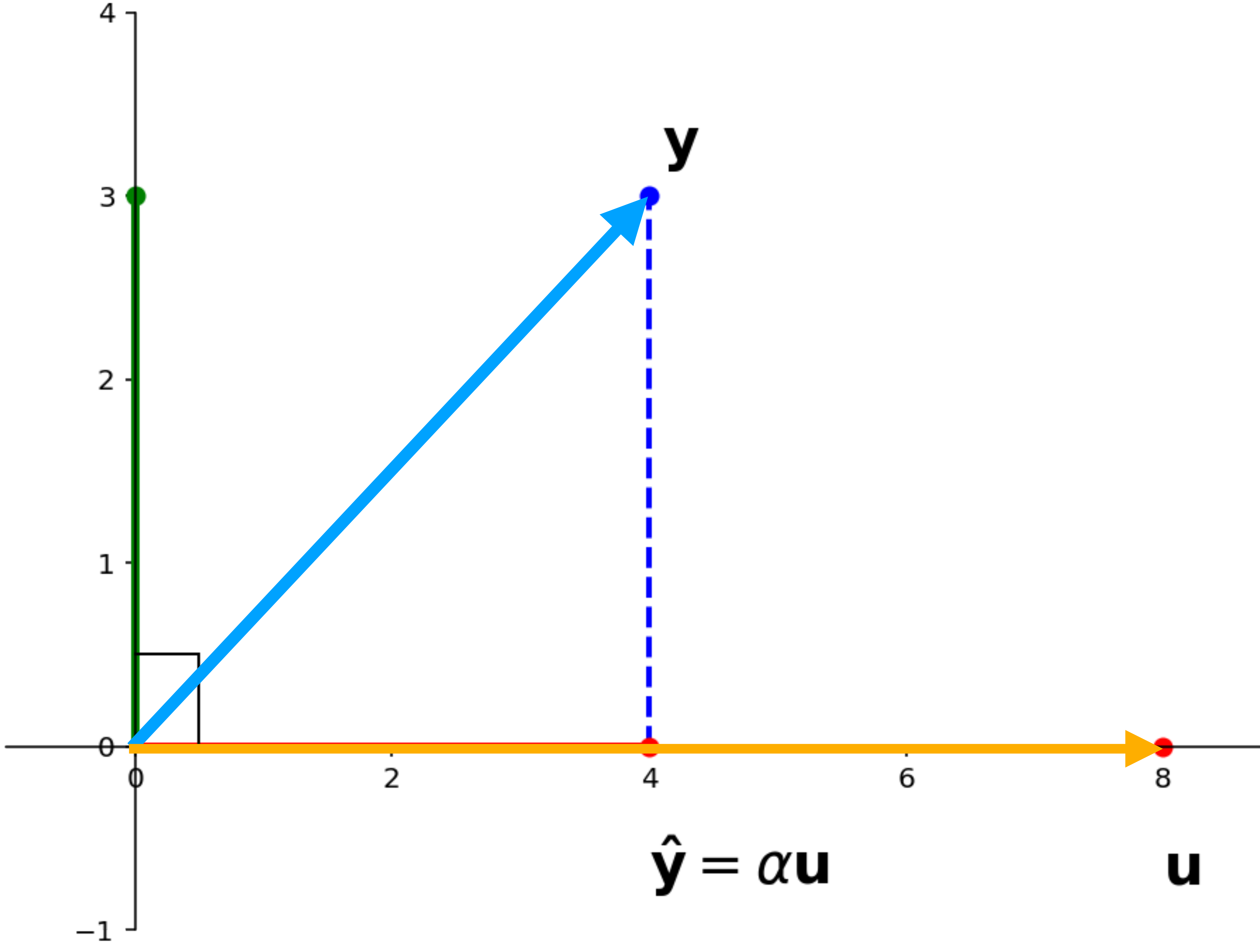
Orthogonal Projection

Question. Given vectors $\mathbf{y}$ and $\mathbf{u}$ in R^n, find vectors $\hat{\mathbf{y}}$ and $\mathbf{z}$ such that

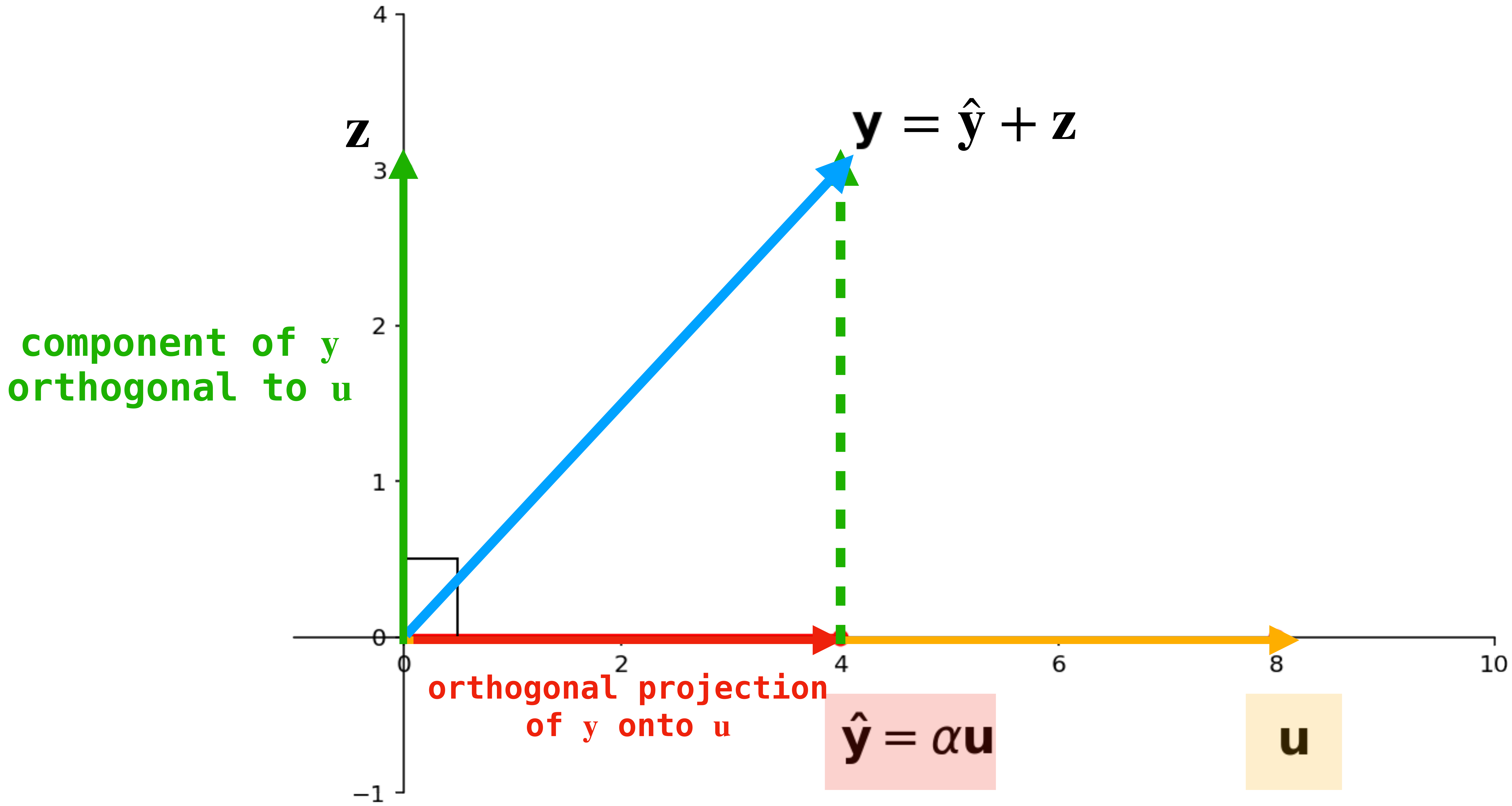
» $\mathbf{z}$ is orthogonal to $\mathbf{u}$ (i.e., $\mathbf{z} \cdot \mathbf{u} = 0$)

» $\hat{\mathbf{y}} \in span\{\mathbf{u}\}$

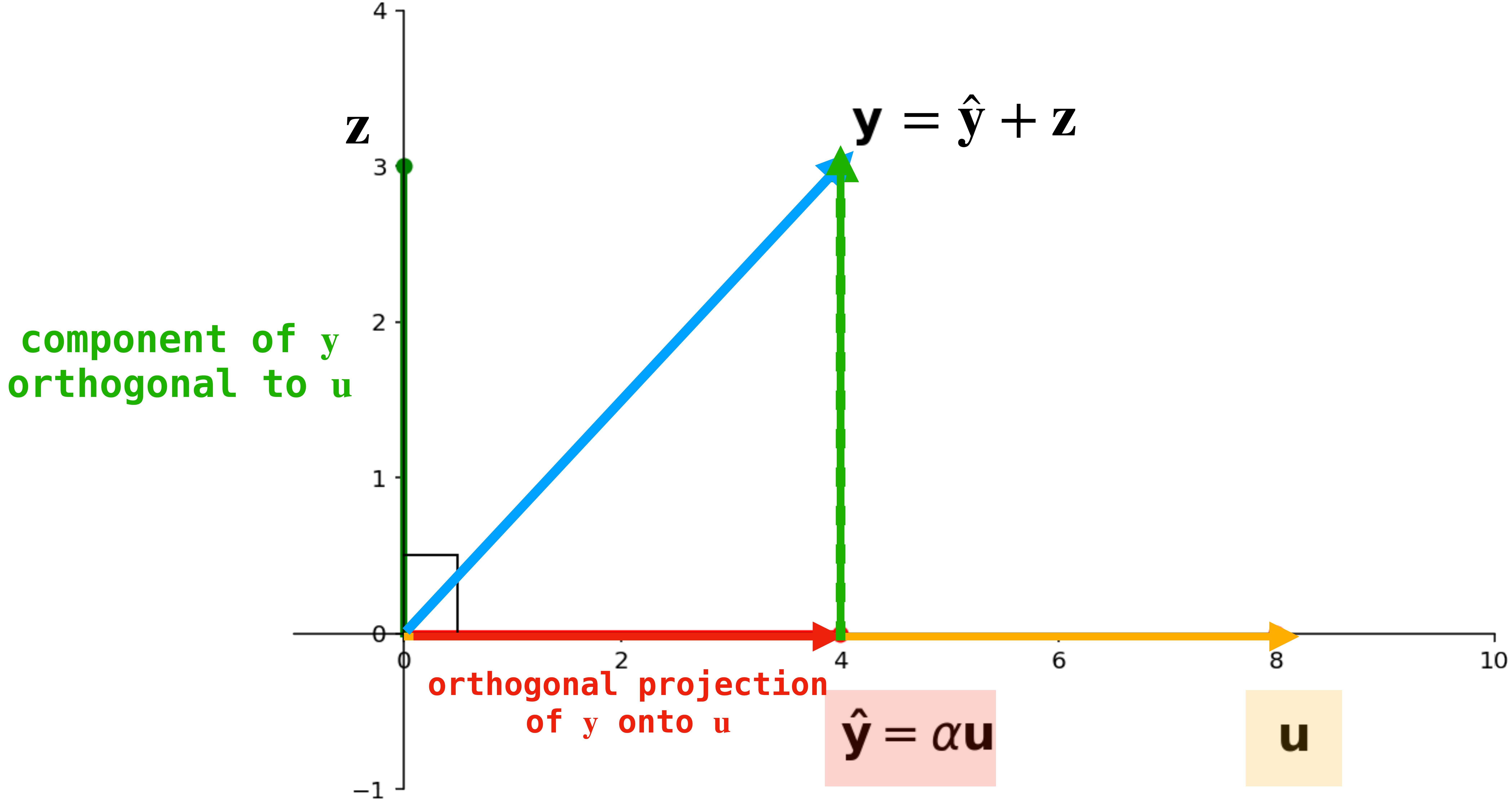
» $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$



Orthogonal Projection

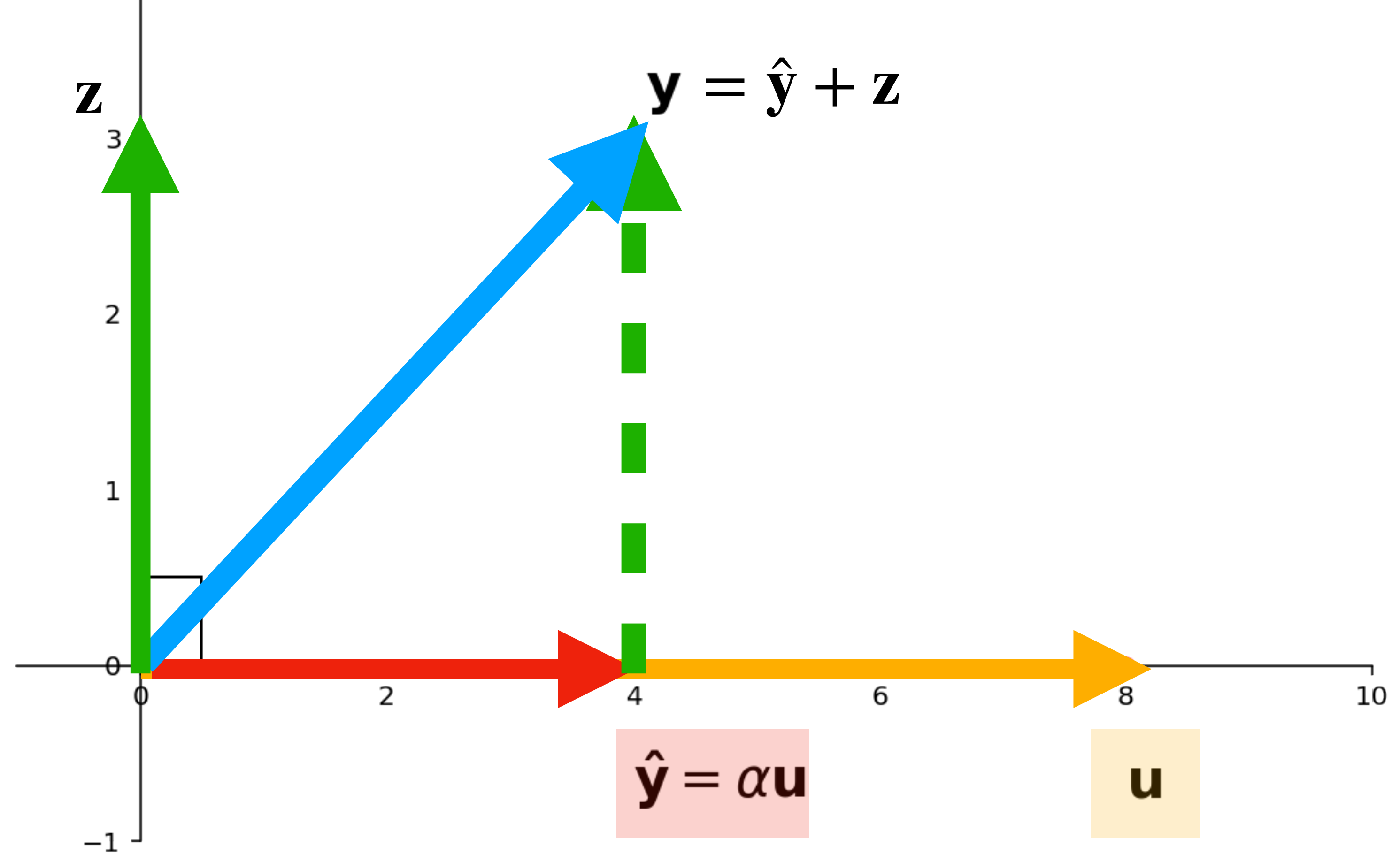


Orthogonal Projection



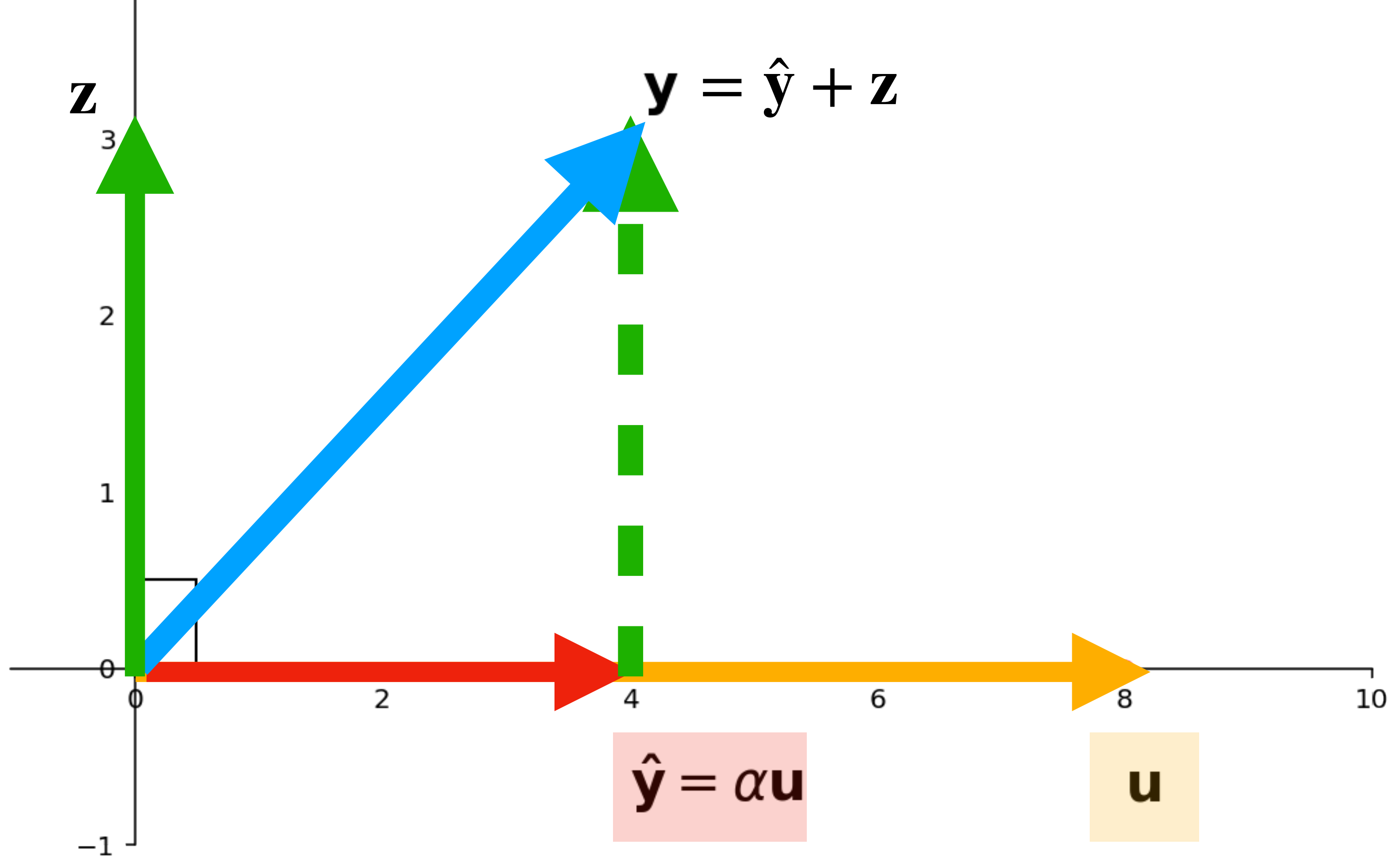
How do we find the orthogonal projection and orthogonal component?

What we know



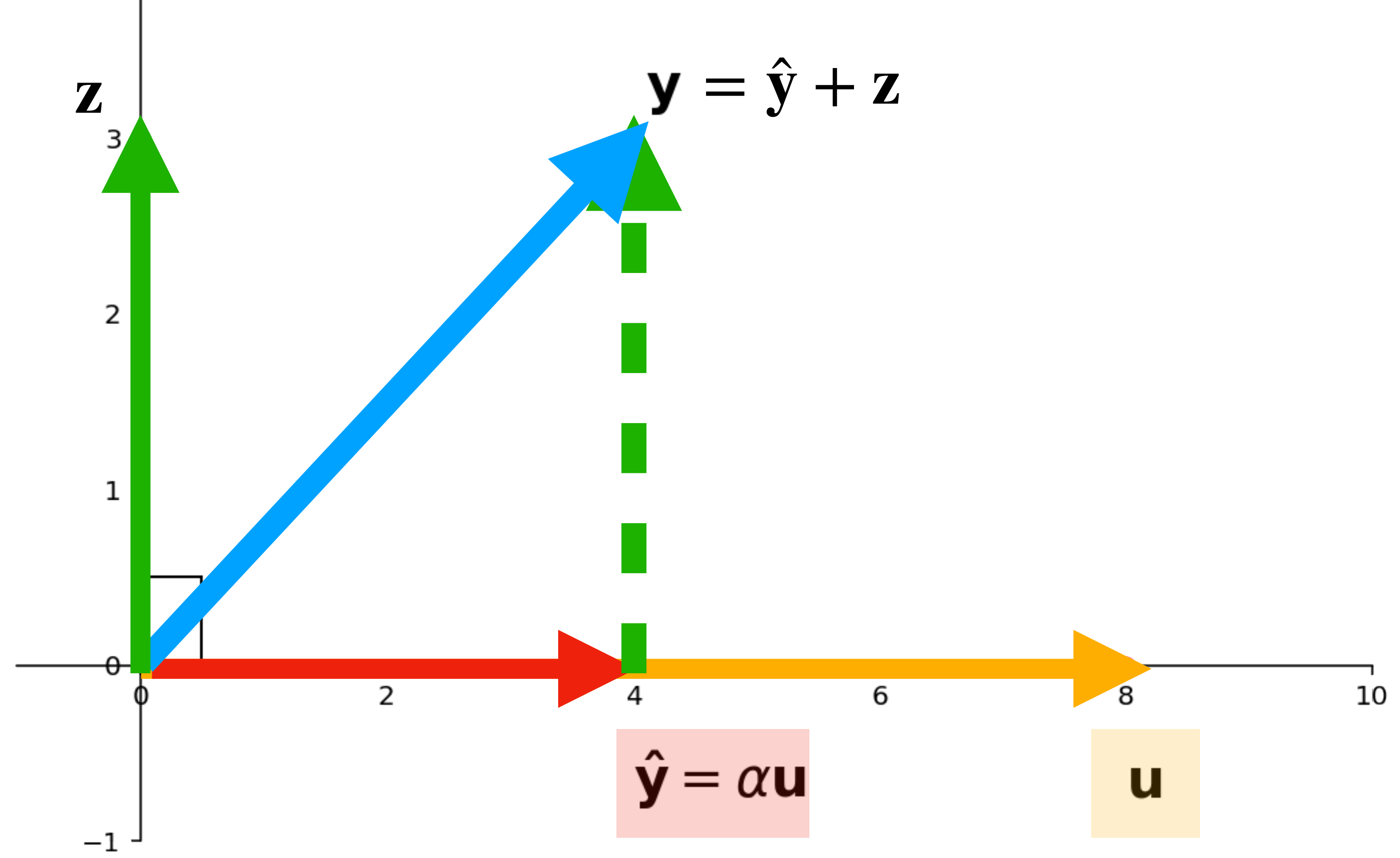
$\mathbf{z}$
$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$
$\hat{\mathbf{y}} = \alpha\mathbf{u}$
$\mathbf{u}$

What we know

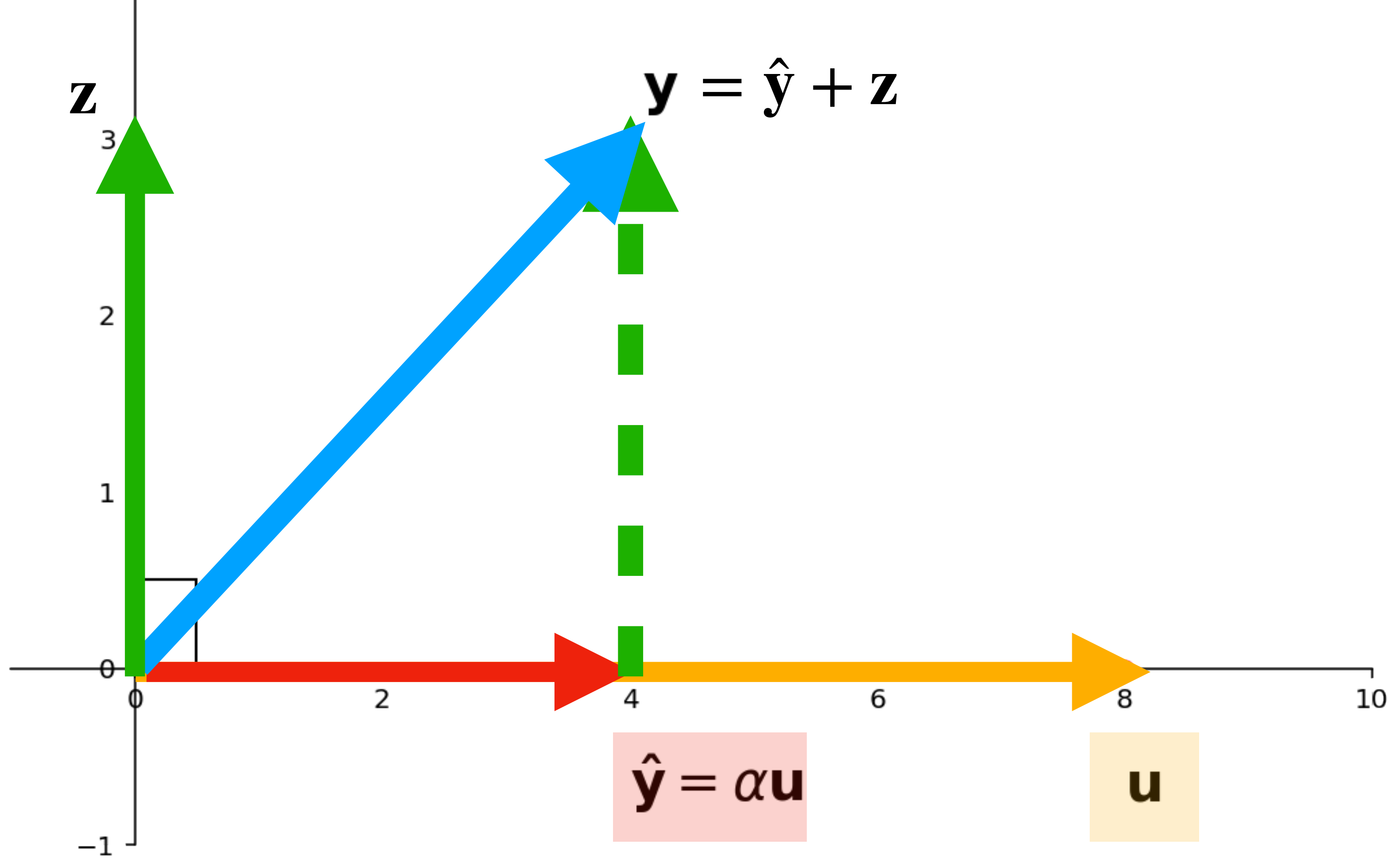


- $\hat{\mathbf{y}} = \alpha\mathbf{u}$ for some scalar α (since $\hat{\mathbf{y}} \in span\{\mathbf{u}\}$)

What we know



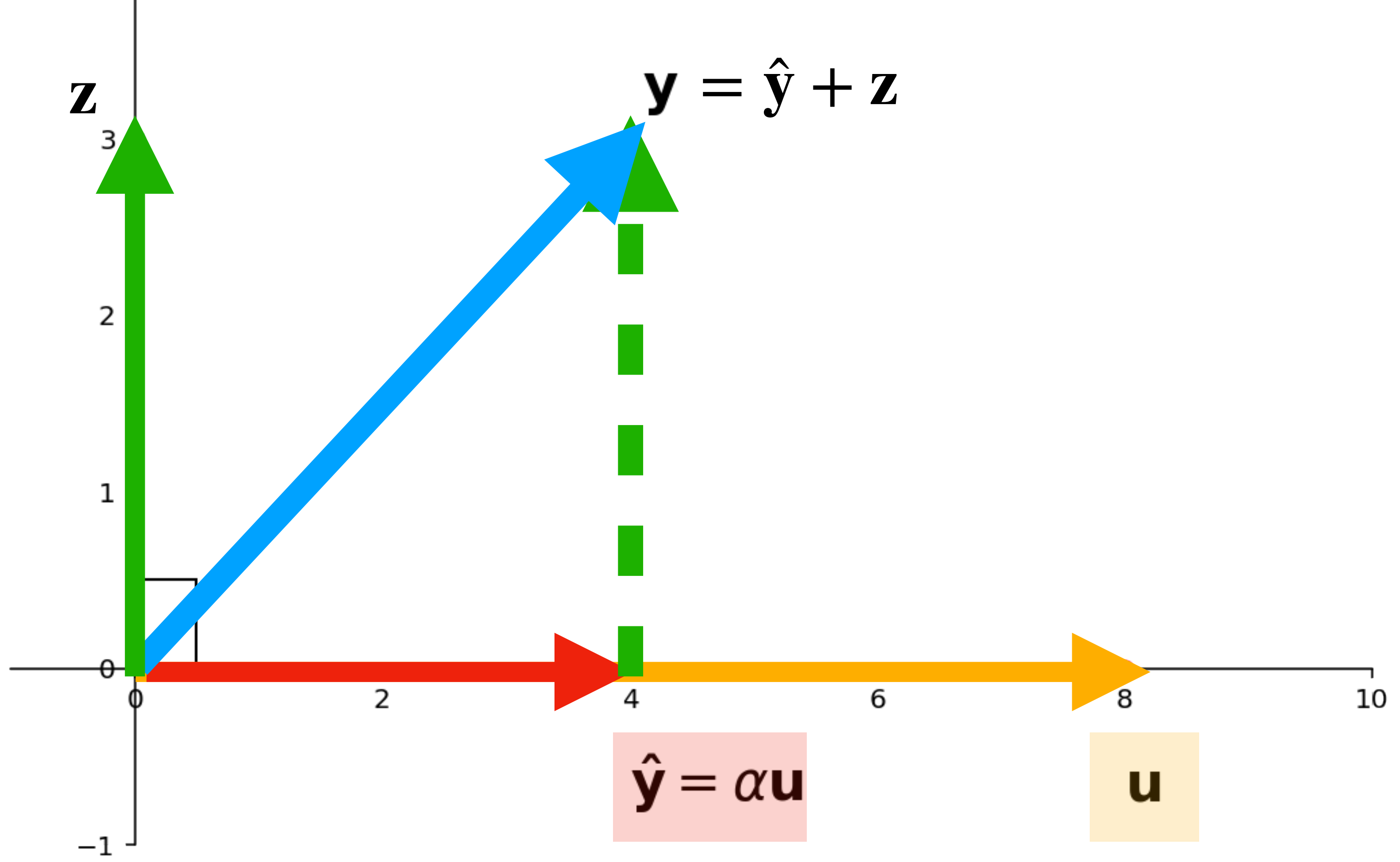
- $\hat{\mathbf{y}} = \alpha\mathbf{u}$ for some scalar α (since $\hat{\mathbf{y}} \in span\{\mathbf{u}\}$)
- $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}} = \mathbf{y} - \alpha\mathbf{u}$ (since $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$)



What we know

- $\hat{\mathbf{y}} = \alpha\mathbf{u}$ for some scalar α (since $\hat{\mathbf{y}} \in span\{\mathbf{u}\}$)
- $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}} = \mathbf{y} - \alpha\mathbf{u}$ (since $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$)
- $\langle \mathbf{z}, \mathbf{u} \rangle = 0$ (since $\mathbf{z}$ is orthogonal with $\mathbf{u}$)

What we know

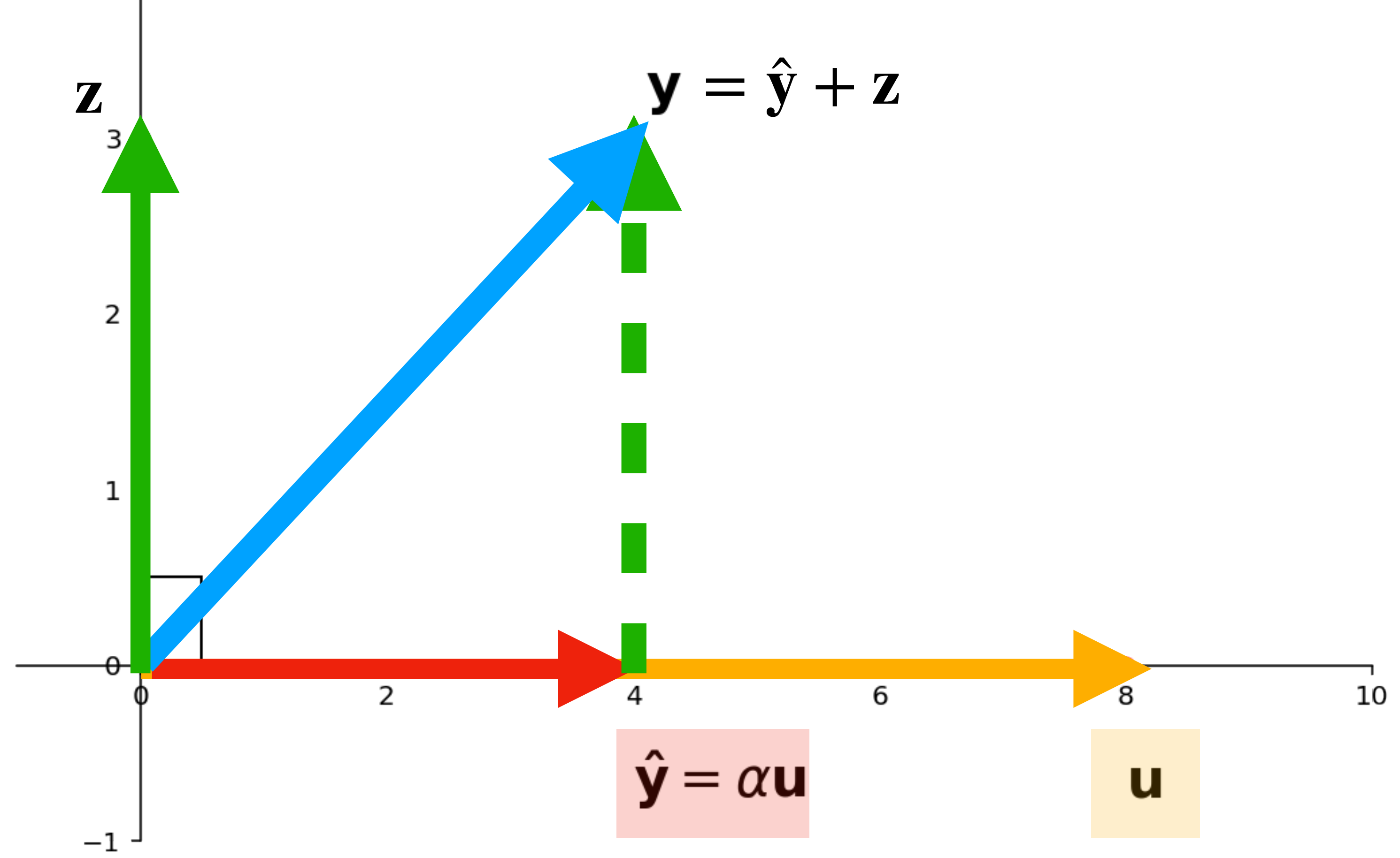


- $\hat{\mathbf{y}} = \alpha\mathbf{u}$ for some scalar α (since $\hat{\mathbf{y}} \in span\{\mathbf{u}\}$)
- $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}} = \mathbf{y} - \alpha\mathbf{u}$ (since $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$)
- $\langle \mathbf{z}, \mathbf{u} \rangle = 0$ (since $\mathbf{z}$ is orthogonal with $\mathbf{u}$)

Therefore:

$$\langle \mathbf{y} - \alpha\mathbf{u}, \mathbf{u} \rangle = 0$$

What we know



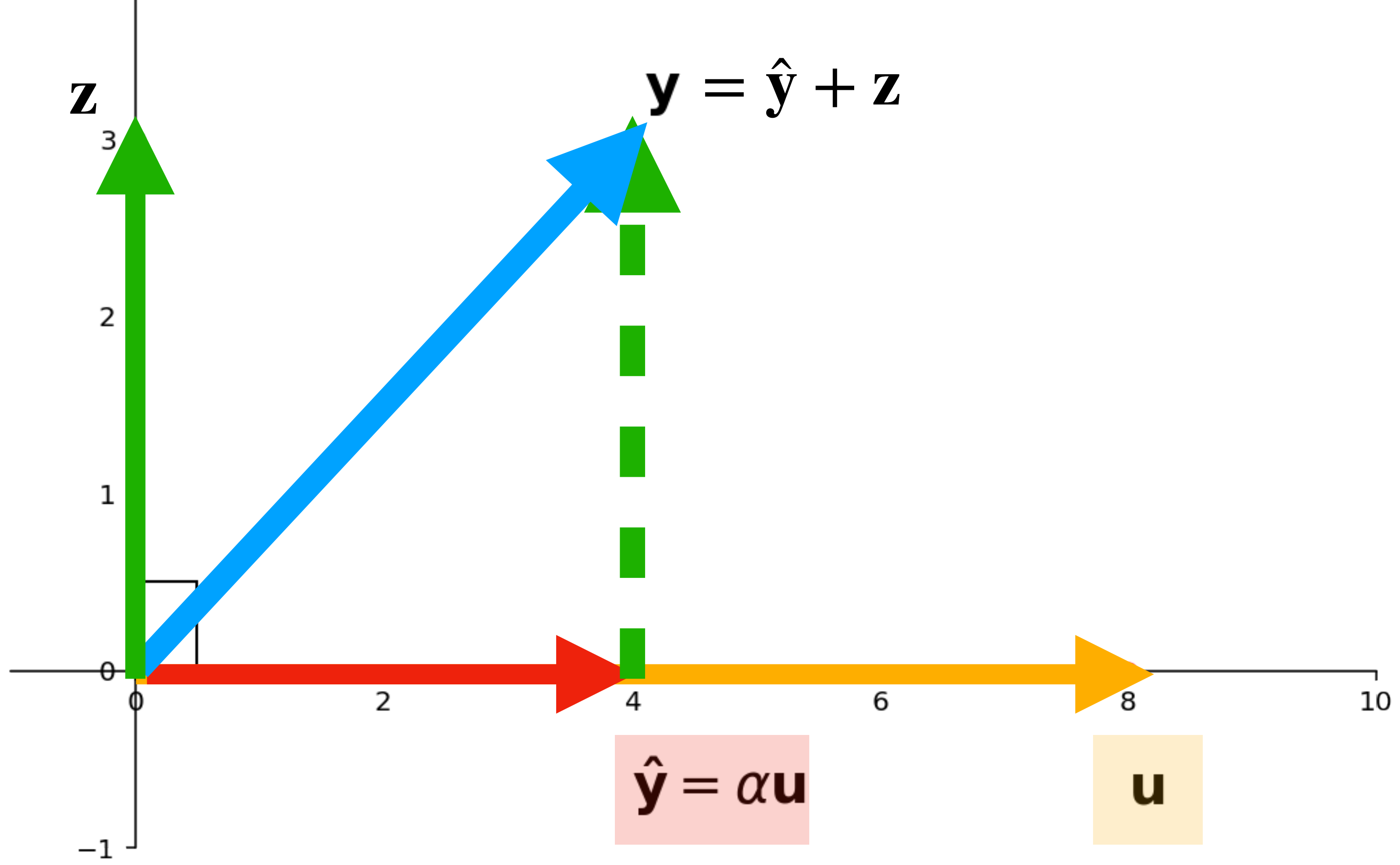
- $\hat{\mathbf{y}} = \alpha\mathbf{u}$ for some scalar α (since $\hat{\mathbf{y}} \in span\{\mathbf{u}\}$)
- $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}} = \mathbf{y} - \alpha\mathbf{u}$ (since $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$)
- $\langle \mathbf{z}, \mathbf{u} \rangle = 0$ (since $\mathbf{z}$ is orthogonal with $\mathbf{u}$)

Therefore:

$$\langle \mathbf{y} - \alpha\mathbf{u}, \mathbf{u} \rangle = 0$$

Once we have α, we can compute both $\hat{y}$ and z

Step 1: Finding α



$$\langle \mathbf{y} - \alpha\mathbf{u}, \mathbf{u} \rangle = 0$$

Let's solve for α, $\hat{\mathbf{y}}$ and $\mathbf{z}$:

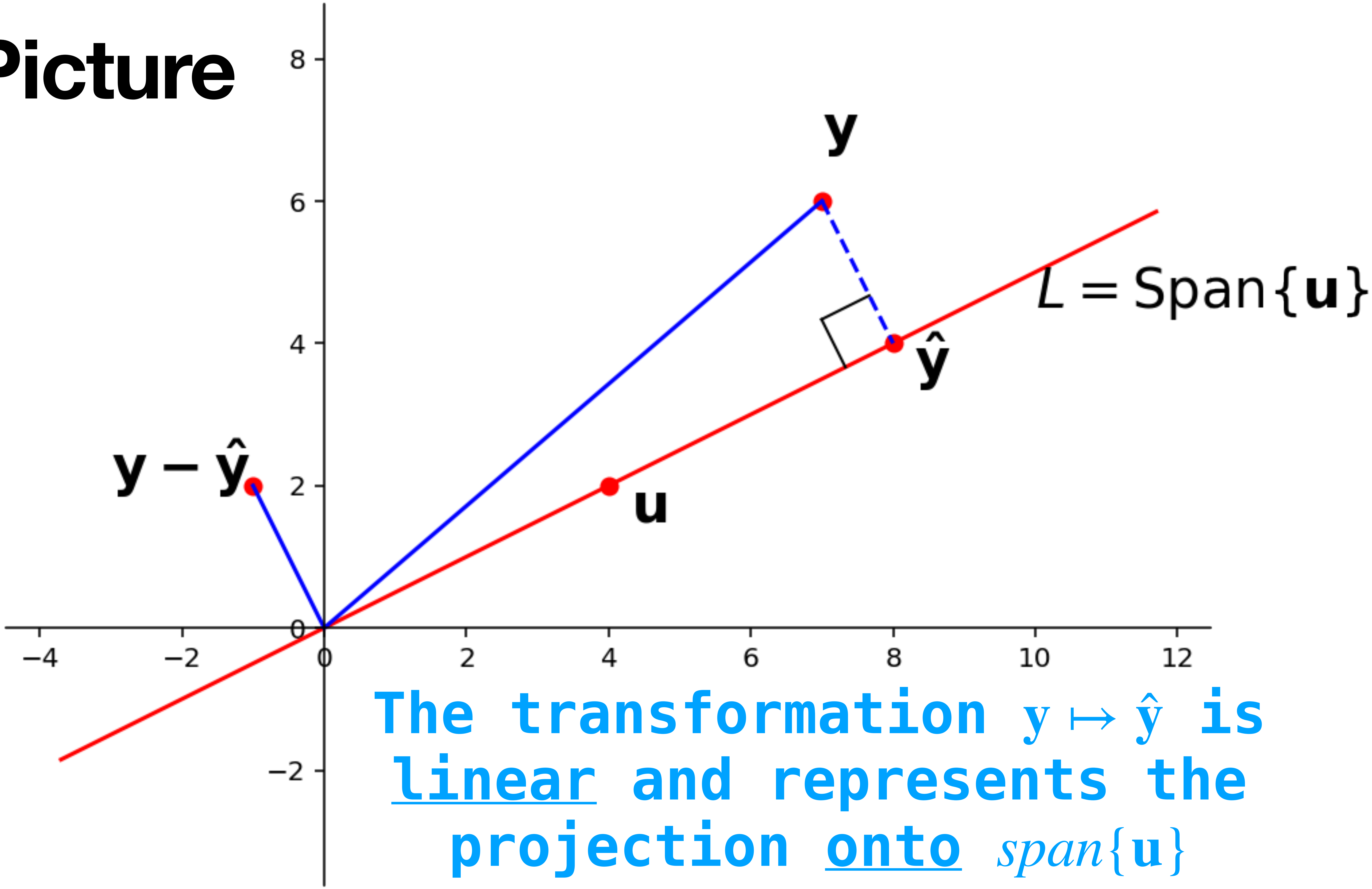
$$\langle y, u \rangle - \alpha \langle u, u \rangle = 0$$

$$\alpha \langle u, u \rangle = \langle y, u \rangle$$

$$\alpha = \frac{\langle y, u \rangle}{\langle u, u \rangle}$$

$$\hat{y} = \frac{\langle y, u \rangle}{\langle u, u \rangle} \vec{u}$$

The Picture



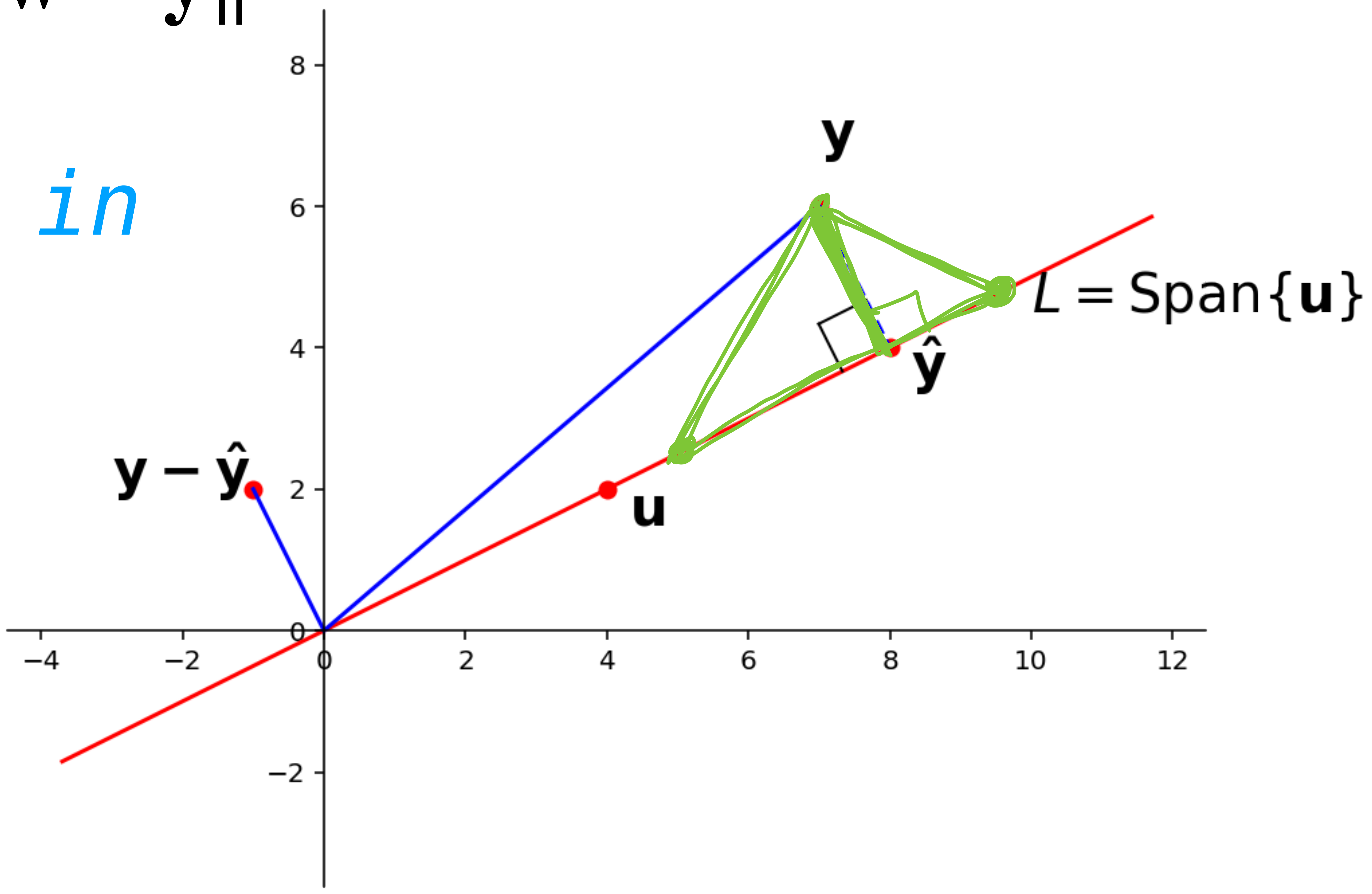
The transformation $\mathbf{y} \mapsto \hat{\mathbf{y}}$ is <u>linear</u> and represents the projection <u>onto</u> $span\{\mathbf{u}\}$

$\hat{\mathbf{y}}$ and Distance

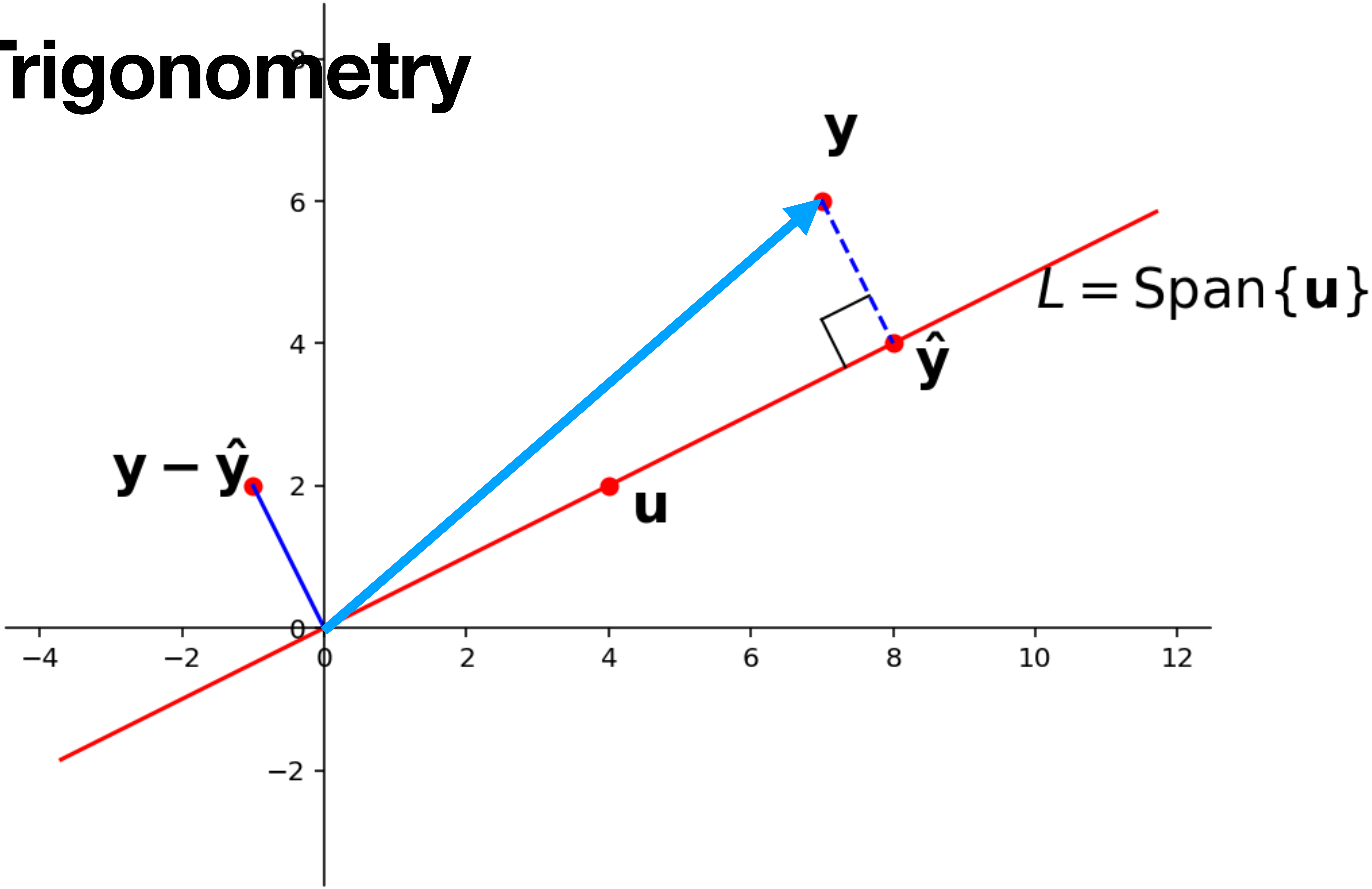
Theorem. $\|\hat{\mathbf{y}} - \mathbf{y}\| = \min_{\mathbf{w} \in span\{\mathbf{u}\}} \|\mathbf{w} - \mathbf{y}\|$

$\hat{\mathbf{y}}$ is the closest vector in $span\{\mathbf{u}\}$ to $\mathbf{y}$.

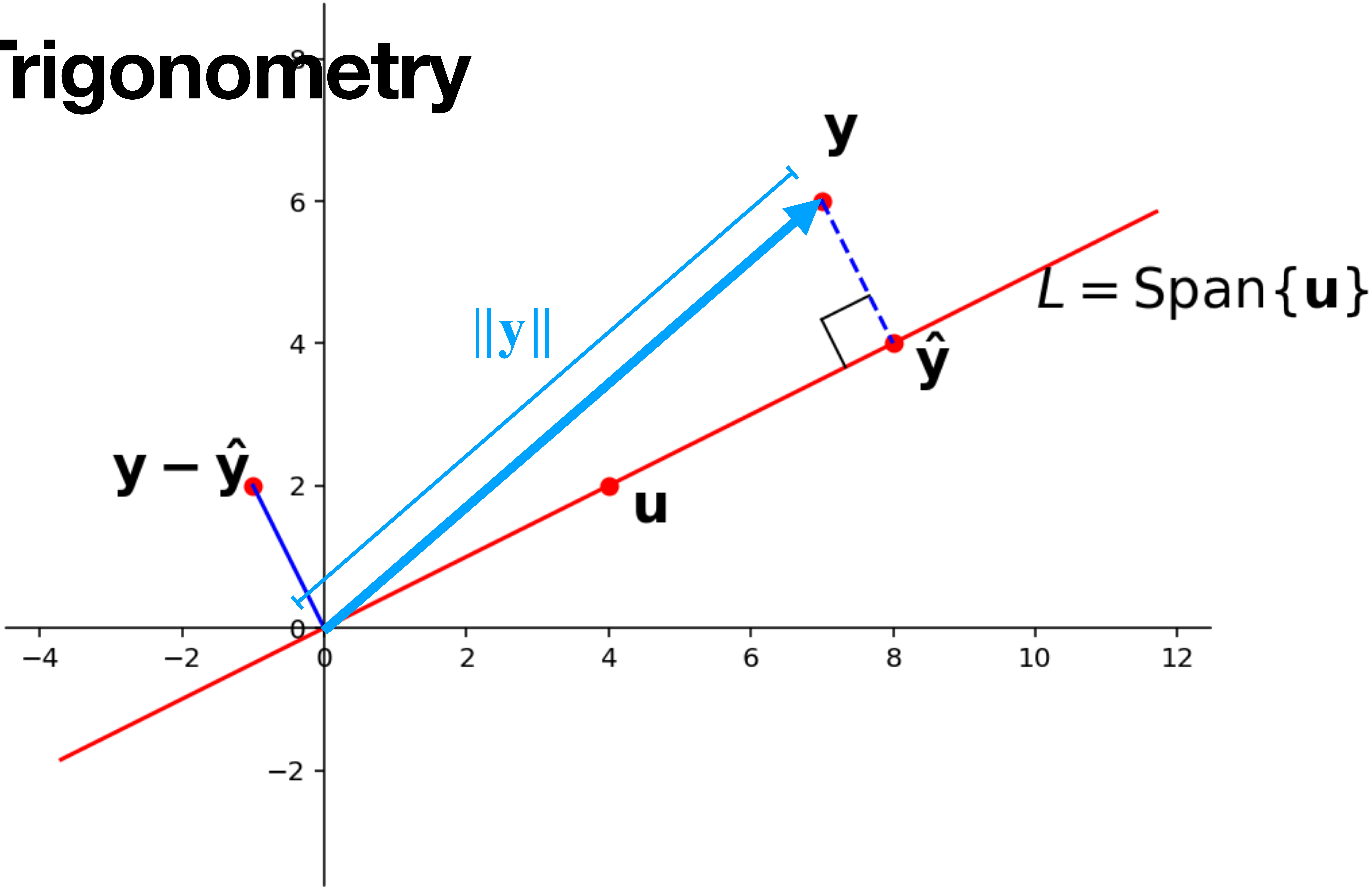
"Proof" by inspection:



The Trigonometry

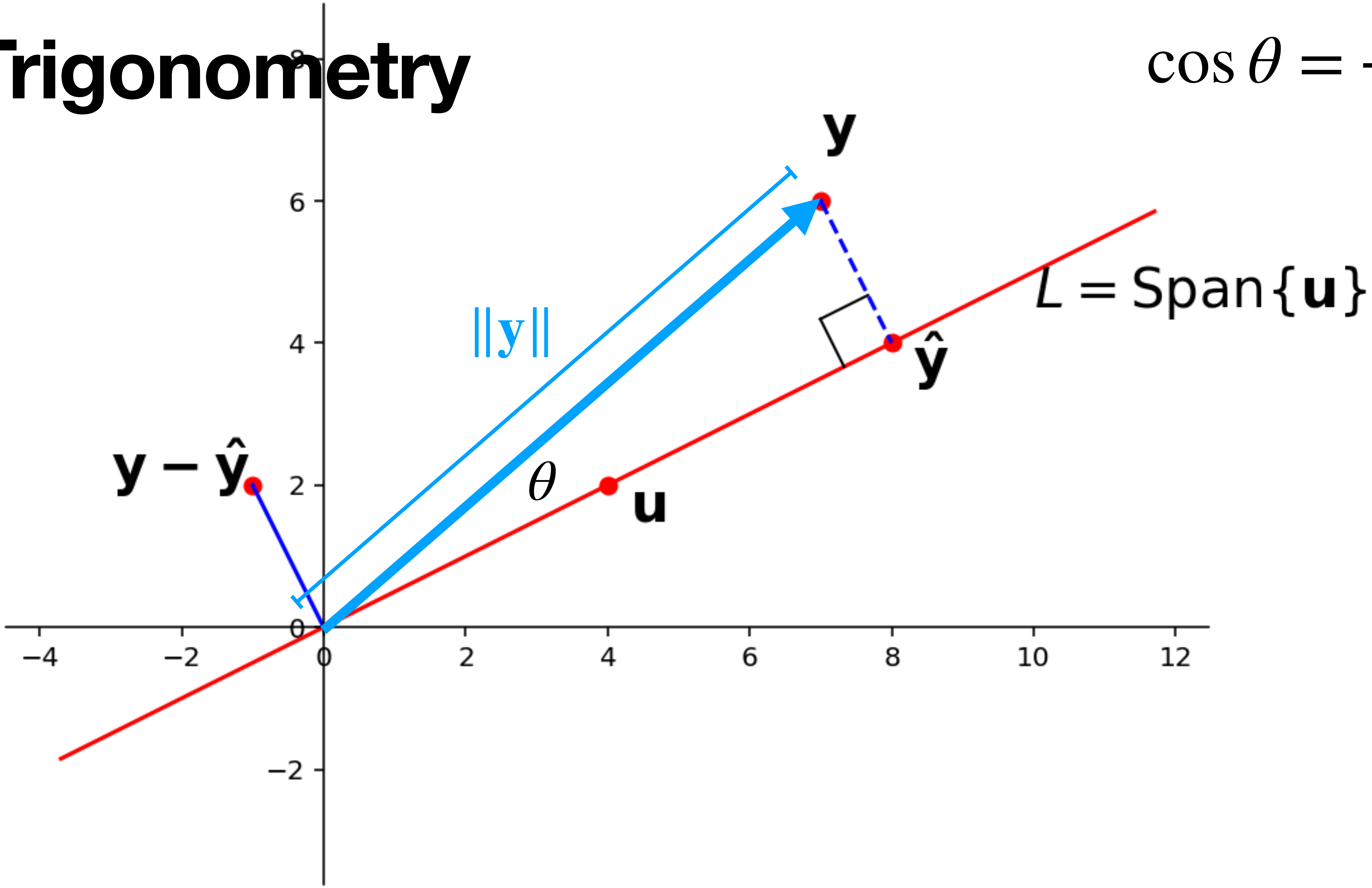


The Trigonometry



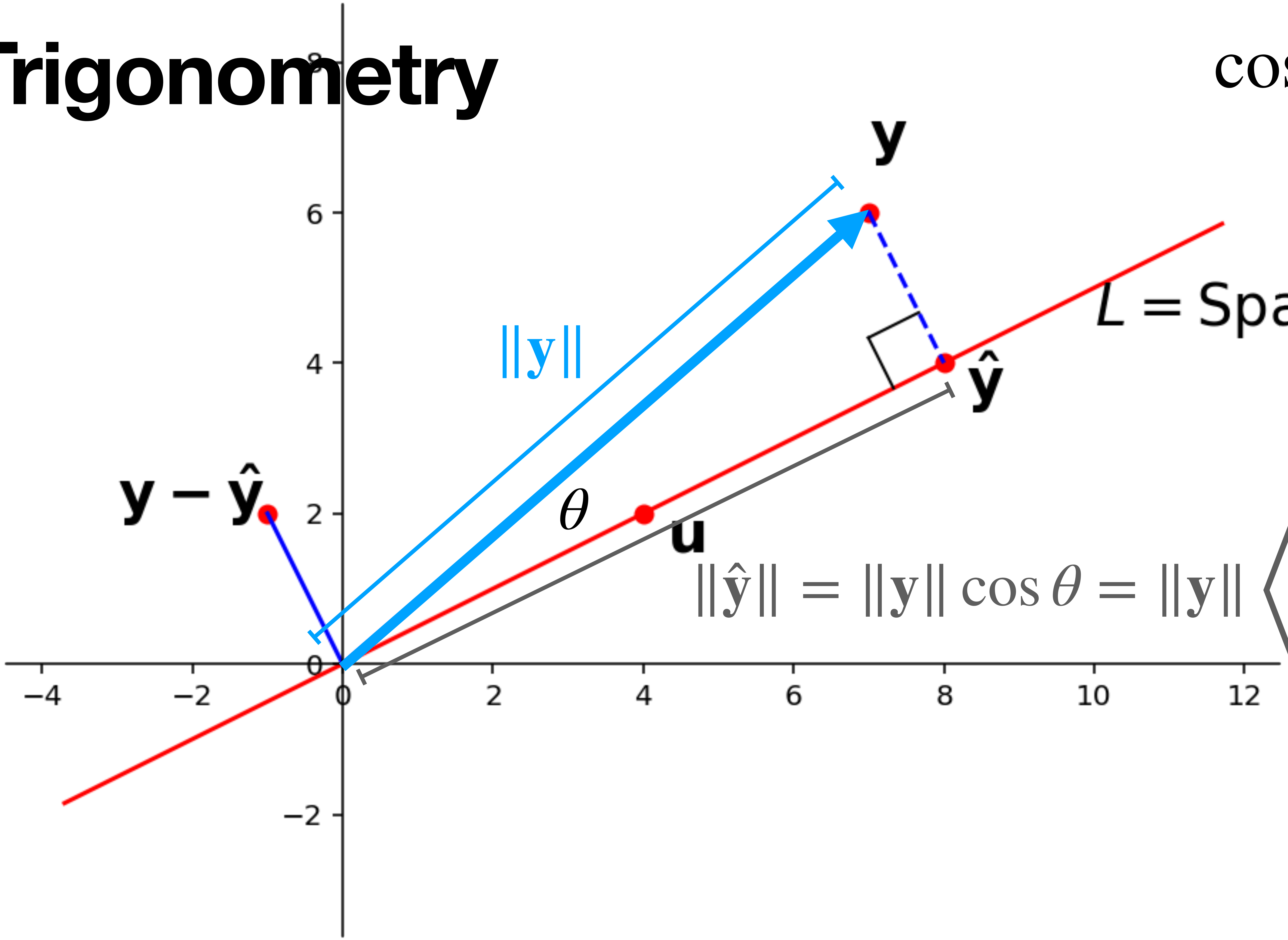
The Trigonometry

$$\cos\theta = \frac{\|\hat{\mathbf{y}}\|}{\|\mathbf{y}\|}$$



The Trigonometry

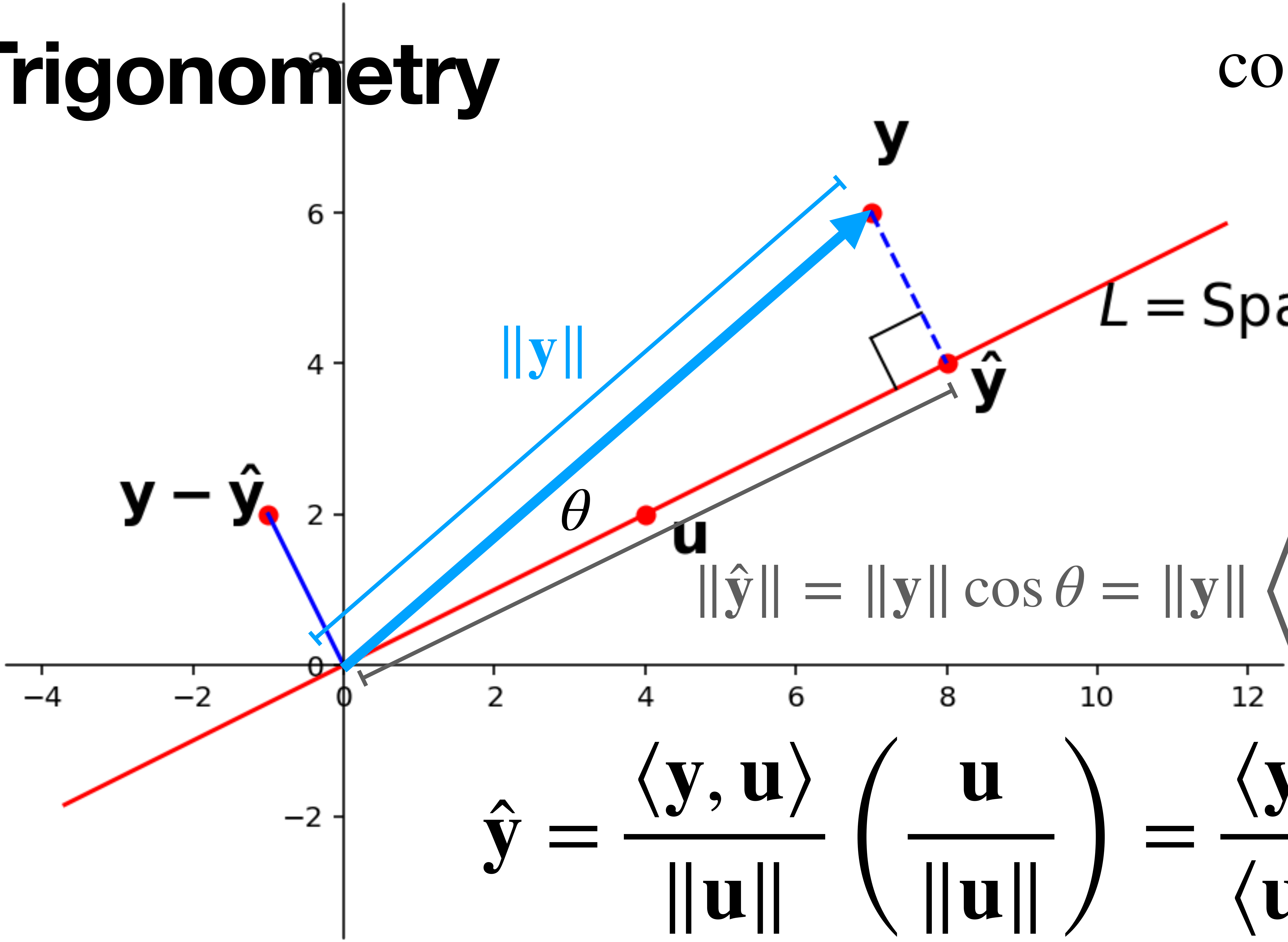
$$\cos\theta = \frac{\|\hat{\mathbf{y}}\|}{\|\mathbf{y}\|}$$



$$\|\hat{\mathbf{y}}\| = \|\mathbf{y}\|\cos\theta = \|\mathbf{y}\|\left\langle \frac{\mathbf{y}}{\|\mathbf{y}\|}, \frac{\mathbf{u}}{\|\mathbf{u}\|} \right\rangle$$

The Trigonometry

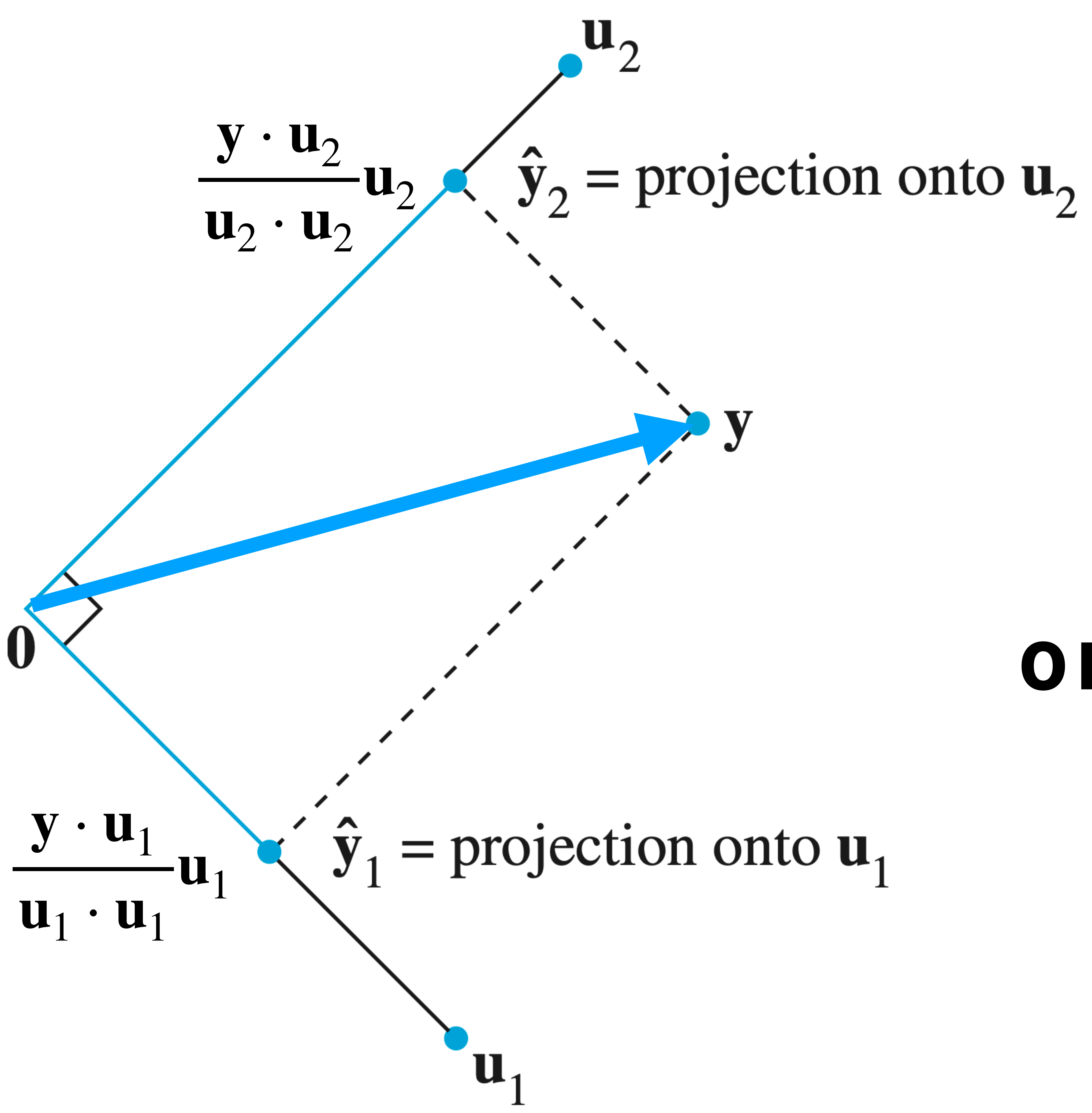
$$\cos\theta = \frac{\|\hat{\mathbf{y}}\|}{\|\mathbf{y}\|}$$



$$\|\hat{\mathbf{y}}\| = \|\mathbf{y}\|\cos\theta = \|\mathbf{y}\|\left\langle \frac{\mathbf{y}}{\|\mathbf{y}\|}, \frac{\mathbf{u}}{\|\mathbf{u}\|} \right\rangle$$

$$\hat{\mathbf{y}} = \frac{\langle \mathbf{y}, \mathbf{u} \rangle}{\|\mathbf{u}\|}\left(\frac{\mathbf{u}}{\|\mathbf{u}\|}\right) = \frac{\langle \mathbf{y}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle}\mathbf{u}$$

Orthogonal Projections and Orthogonal Bases



Each component of $\mathbf{y}$ written in terms of an *orthogonal* basis is an **orthogonal projection onto to a basis vector**

Linear Algebra and its Applications, Lay, Lay, McDonald

How To:

Question. Find the projection of $\mathbf{y}$ onto the span of $\mathbf{u}$.

Solution. Calculate $\alpha = \dfrac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}$, then the solution is $\alpha\mathbf{u}$.

Question

Find the matrix which implements orthogonal projection onto the span of $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$.

Answer

$$\frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{bmatrix}$$

Orthonormal Sets

Orthogonal sets would be easier to work with if every vector was a unit vector

Orthonormality

Orthonormality

Definition. A set $\{u_1, u_2, \ldots, u_p\}$ is an **orthonormal set** if of it an orthogonal set of unit vectors.

Orthonormality

Definition. A set $\{u_1, u_2, \ldots, u_p\}$ is an **orthonormal set** if of it an orthogonal set of unit vectors.

Definition. An **orthonormal basis** of the subspace W is a basis of W which is an orthonormal set.

Orthonormality

Definition. A set $\{u_1, u_2, \ldots, u_p\}$ is an **orthonormal set** if of it an orthogonal set of <u>unit</u> vectors.

Definition. An **orthonormal basis** of the subspace W is a basis of W which is an orthonormal set.

ortho·normal

Orthonormality

Definition. A set $\{u_1, u_2, \ldots, u_p\}$ is an **orthonormal set** if of it an orthogonal set of unit vectors.

Definition. An **orthonormal basis** of the subspace W is a basis of W which is an orthonormal set.

ortho·normal

orthogonal/perpendicular

Orthonormality

Definition. A set $\{u_1, u_2, \ldots, u_p\}$ is an **orthonormal set** if of it an orthogonal set of unit vectors.

Definition. An **orthonormal basis** of the subspace W is a basis of W which is an orthonormal set.

ortho·normal

orthogonal/perpendicular

normalized/made unit vectors

Orthonormal Matrices

Definition. A matrix is **orthonormal** if its columns form an orthonormal set.

The notes call a square orthonormal matrix an **orthogonal** matrix.

Orthonormal Matrices

Definition. A matrix is **orthonormal** if its columns form an orthonormal set.

The notes call a square orthonormal matrix an **orthogonal** matrix.

This is incredibly confusing, but we'll try to be consistent and clear.

Orthonormal Matrices and Transposition

Theorem. For an $m \times n$ orthonormal matrix U

$$U^T U = I_m$$

Verify: $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \begin{bmatrix} u_1 & u_2 \end{bmatrix} = \begin{bmatrix} \langle u_1, u_1 \rangle & \langle u_1, u_2 \rangle \\ \langle u_2, u_1 \rangle & \langle u_2, u_2 \rangle \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Inverses of Orthogonal Matrices

Theorem. If an $n \times n$ matrix U is orthogonal (square orthonormal) then it is invertible and

$$U^{-1} = U^T$$

Verify: ✓

$$U^T U = I$$

$$\therefore U^T = U^{-1}$$

Orthonormal Matrices and Inner Products

Theorem. For a $m \times n$ orthonormal matrix U, and any vectors x and y in R^n

$$\langle Ux, Uy \rangle = \langle x, y \rangle$$

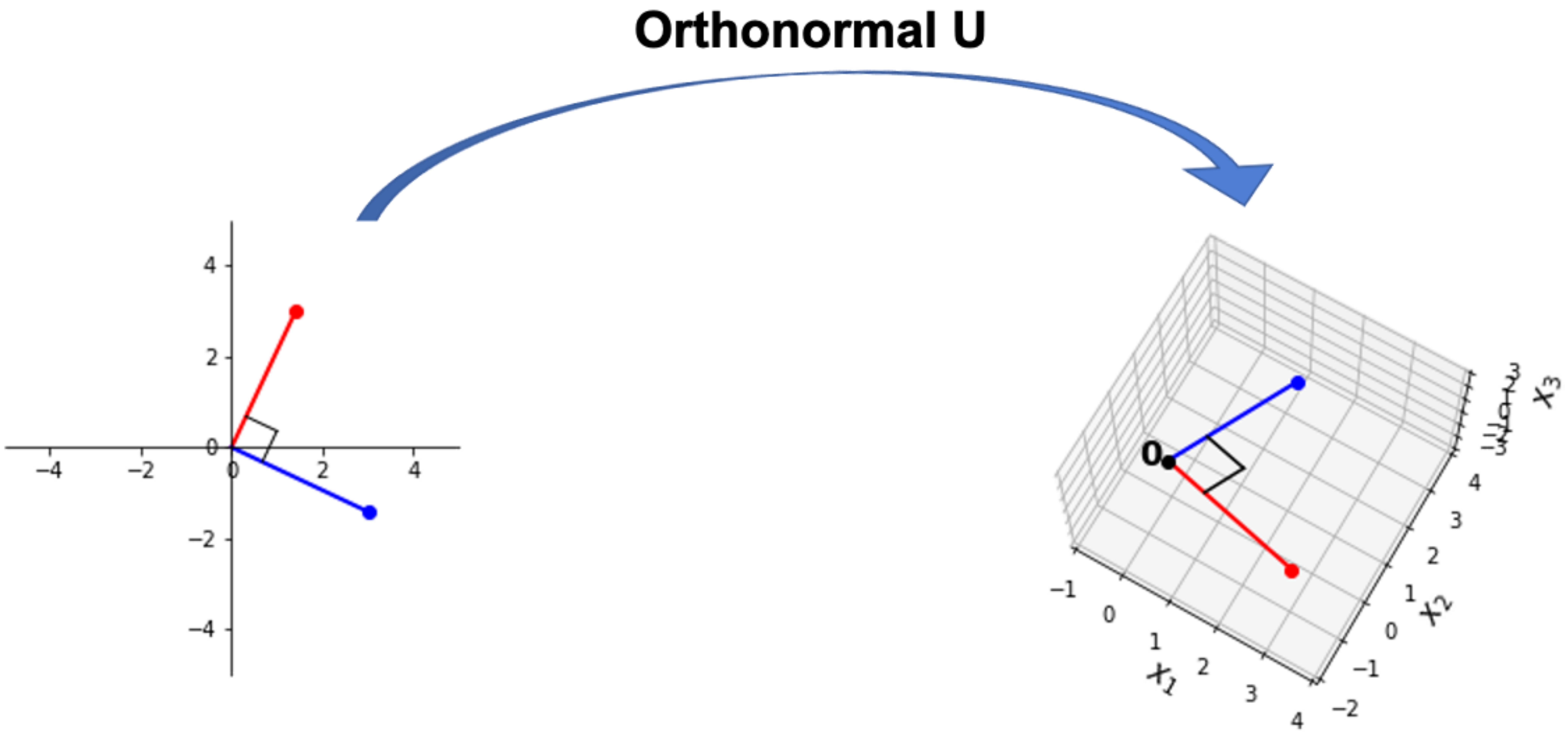
Orthonormal matrices preserve inner products.

Verify:

Length, Angle, Orthogonality Preservation

Since lengths and angles are defined in terms of inner products, they are also preserved by orthonormal matrices:

The Picture



Example

$$U = \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix} \qquad x = \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix}$$

Question (Conceptual)

Suppose A is an $m \times n$ matrix with orthogonal but ***not*** *orthonormal columns. What is $A^T A$?*

Answer

If $A = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \dots \quad \mathbf{a}_n]$ then $A^T A$ is a diagonal matrix D where

$$D_{ii} = \|\mathbf{a}_i\|^2$$

Summary

Orthogonal sets allow for simpler calculations of coordinates.

Finding these coordinates is a really about find the orthogonal projections onto each vector in the orthogonal set.

We can apply these ideas to matrices and describe a class of very well behaved transformations via orthonormal matrices.