Singular Value Decomposition

Geometric Algorithms
Lecture 26

CAS CS 132

Introduction

Recap Problem (+ Course Evaluations)

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

Find an orthogonal diagonalization of $\cancel{\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}}$.

https://www.bu.edu/courseeval

Answer

$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ $\boxed{D = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}}$ $A = PDP^{-1} = PDP^T$

① Find eigenvalues

$A - \lambda I = \begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}$

$(3-\lambda)^2 - 1$

$\lambda^2 - 6\lambda + 8 =$

$(\lambda - 4)(\lambda - 2)$

$\lambda = 4, 2$

② Find eigenvectors

(i) $A - 4I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$

$(A - 4I)\vec{x} = \vec{0}$

$\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(ii) $A - 2I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$\vec{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(3.0) Normalize eigenvectors

$\vec{x}_1' = \frac{1}{\|\vec{x}_1\|}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\vec{x}_2' = \frac{1}{\|\vec{x}_2\|}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\boxed{P = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}$

$P^{-1} = P^T = P$

Objectives

1. Finish up our discussion of quadratic forms.
2. Introduce the singular value decomposition (probably the most important matrix decomposition for computer science).
3. Talk very briefly about what to do after this course if you want (or have to) to see more linear algebra.

Quadratic Forms (Finishing Up)

Quadratic Forms

which is a sum

Definition. A **quadratic form** is an function of variables $x_1, \ldots, x_n$ in which every term has degree two.

Examples: $Q(x_1, x_2, x_3) = x_1^2 + 2x_2^2 - 3x_3^2 + 5x_1x_3$

Non-examples:

$$Q(x_1, x_2) = x_1^3 - x_1x_2$$

$$Q(x_1, x_2) = x_1^2 - x_1$$

Quadratic Forms and Symmetric Matrices

Fact. Every quadratic form can be represented as

$$\mathbf{x}^T A \mathbf{x} \qquad \langle x, Ax \rangle$$

where A is <u>symmetric</u>.

Example: $\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2x_1 \\ -x_2 \end{bmatrix} =$

$$\boxed{2x_1^2 - x_2^2}$$

Example: Computing the Quadratic Form for a Matrix

$$A = \begin{bmatrix} 3 & \boxed{-2} \\ \boxed{-2} & 7 \end{bmatrix}$$

This means, given a symmetric matrix A, we can compute its corresponding quadratic form:

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3x_1 - 2x_2 \\ -2x_1 + 7x_2 \end{bmatrix} =$$

$$x_1(3x_1 - 2x_2) + x_2(-2x_1 + 7x_2) = 3x_1^2 \boxed{-2} x_1x_2 + \boxed{(-2)} x_2x_1 + 7x_2^2$$

$$= \boxed{3x_1^2 \boxed{-4} x_1x_2 + 7x_2^2}$$

Quadratic forms and Symmetric Matrices (Again)

diagonal entries

off diag entries

Furthermore, we can generally say

$$\mathbf{x}^T A \mathbf{x} = \boxed{\sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j} = \boxed{\sum_{i=1}^{n} A_{ii} x_i^2 + \sum_{i \neq j} \boxed{(A_{ij} + A_{ji})} x_i x_j}$$

Verify:

$$x^T(Ax) = \sum_{i=1}^{n} \vec{x}_i (A\vec{x})_i = \sum_{i=1}^{n} \vec{x}_i \left(\sum_{j=1}^{n} A_{ij} \vec{x}_j \right)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j$$

A Slightly more Complicated Example

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 0 \\ -1 & 0 & 5 \end{bmatrix}$$

Let's expand $\mathbf{x}^T A \mathbf{x}$:

$$Q(x_1, x_2, x_3) = 1x_1^2 + 3x_2^2 + 5x_3^2 + 4x_1x_2 - 2x_1x_3$$

Matrices from Quadratic Forms

$$Q(\mathbf{x}) = \boxed{5}x_1^2 + \boxed{3}x_2^2 + \boxed{2}x_3^2 - x_1x_2 + 8x_2x_3$$

We can also go in the other direction. Let's express this as $\mathbf{x}^T A\mathbf{x}$:

$$A = \begin{bmatrix} 5 & -1/2 & 0 \\ -1/2 & 3 & 4 \\ 0 & 4 & 2 \end{bmatrix}$$

How To: Matrices of Quadratic Forms

Problem. Given a quadratic form $Q(\mathbf{x})$, find the symmetric matrix A such that $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$.

Solution.

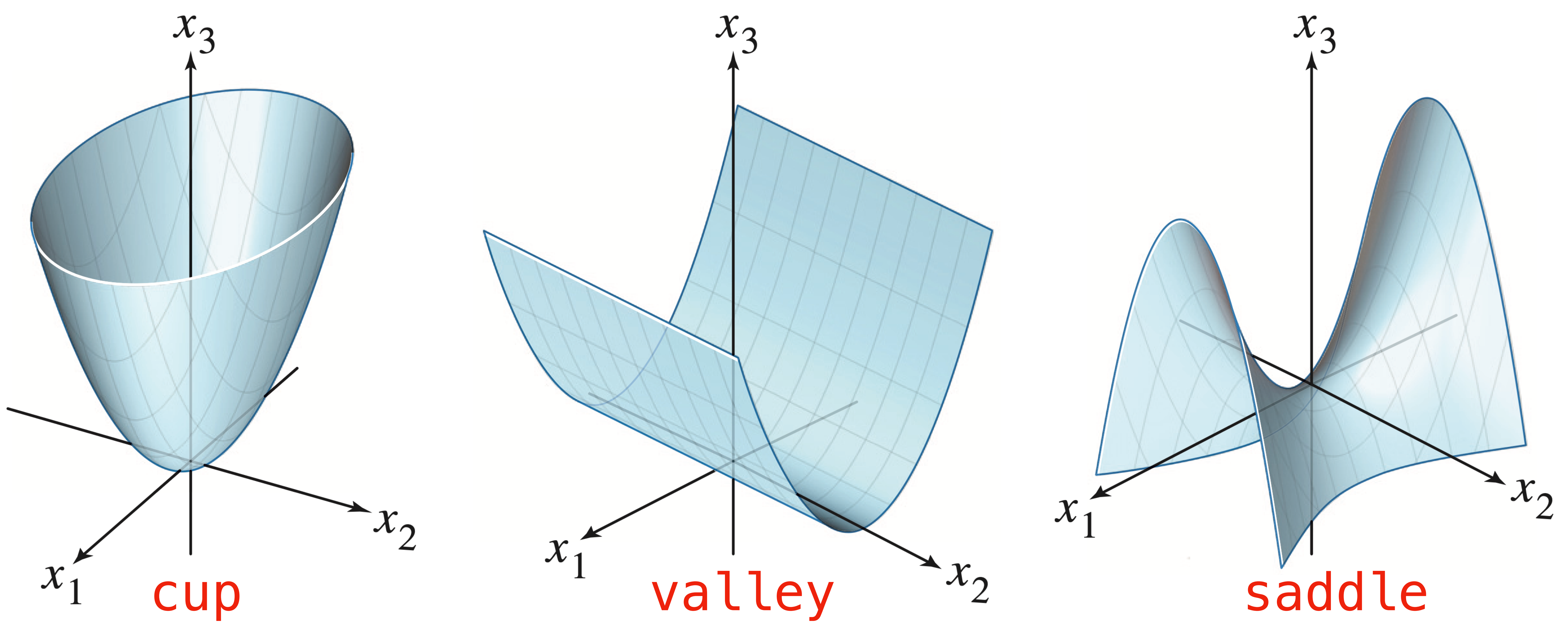
- if $Q(\mathbf{x})$ has the term αx_i^2 then $A_{ii} = \alpha$
- if $Q(\mathbf{x})$ has the term $\alpha x_i x_j$, then $A_{ij} = A_{ji} = \dfrac{\alpha}{2}$

Example

$$Q(x_1, x_2, x_3, x_4) = x_1^2 + 3x_2^2 - 2x_3x_4 - 6x_4^2 + 7x_1x_3$$

Find the symmetric matrix A *such that* $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$*.*

Shapes of of Quadratic Forms

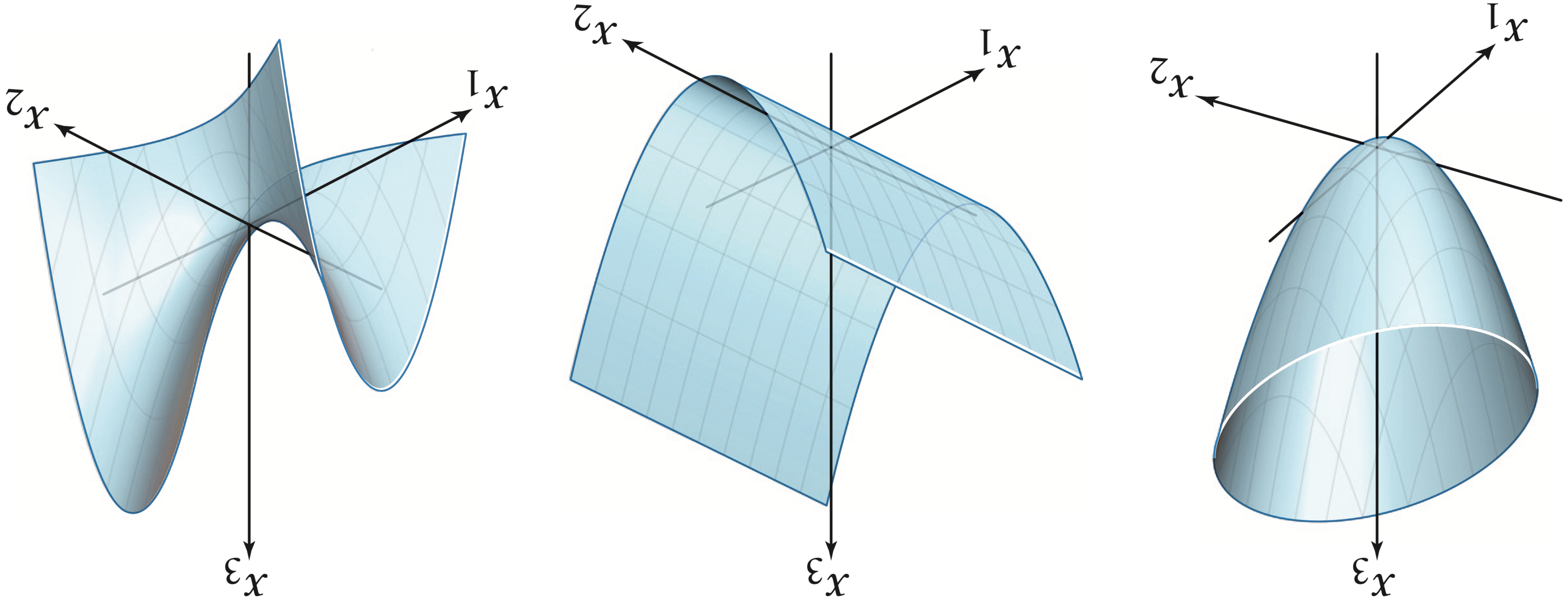


There are essentially three possible shapes (six if you include the negations).

Can we determine what shape it will be mathematically?

Linear Algebra and its Applications, Lay, Lay, McDonald

Shapes of of Quadratic Forms



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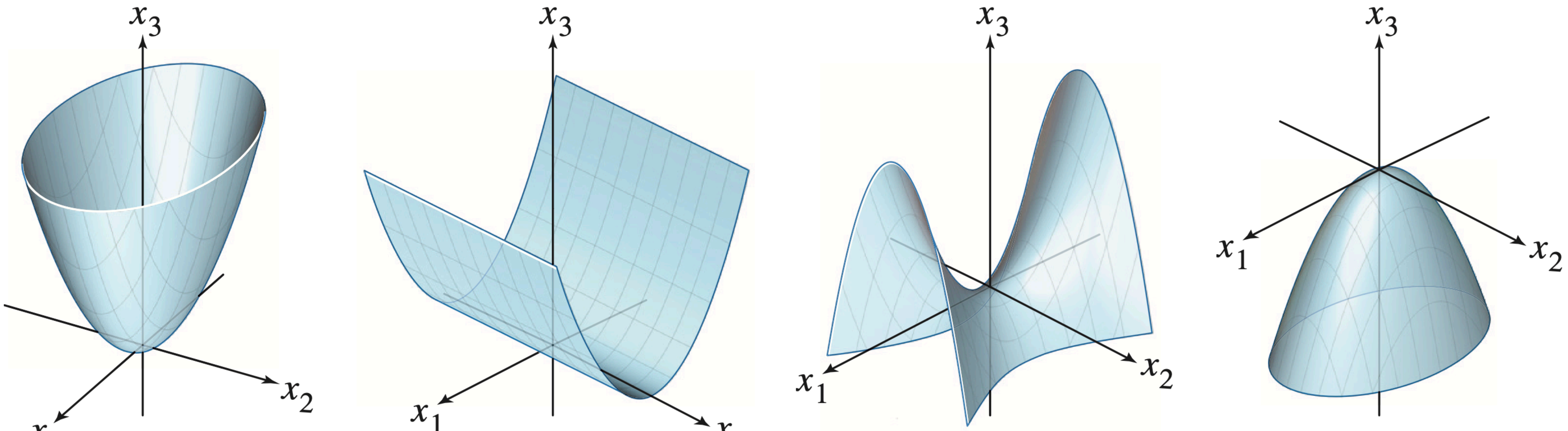
Definiteness

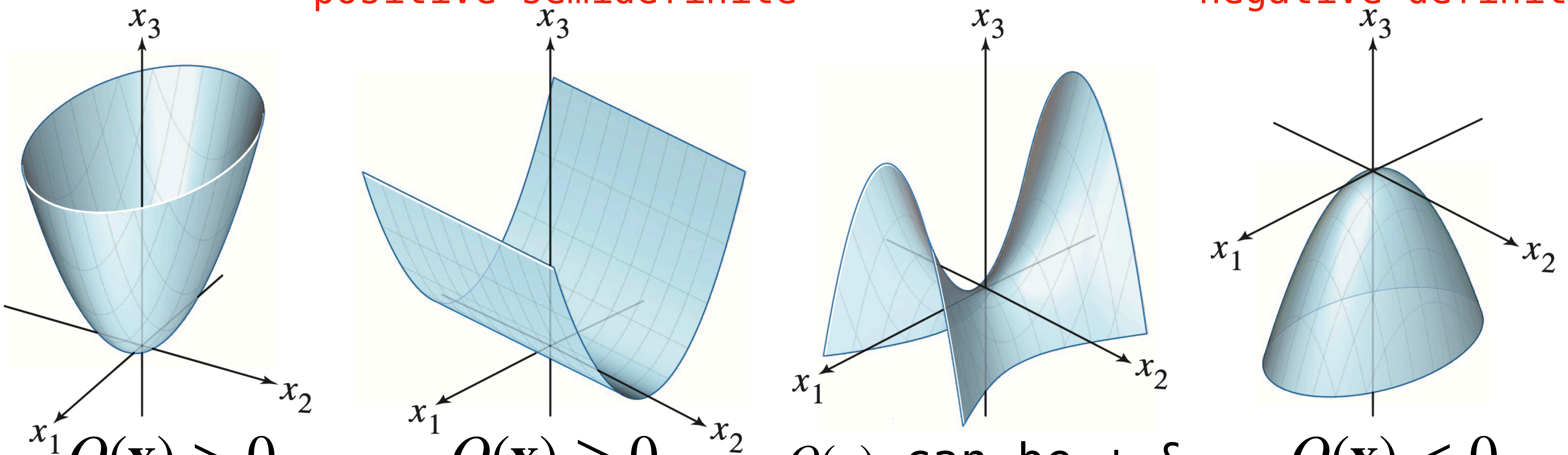


$Q(\mathbf{x}) > 0$ $\quad$ $Q(\mathbf{x}) \geq 0$ $\quad$ $Q(\mathbf{x})$ can be + & – $\quad$ $Q(\mathbf{x}) < 0$

For $\mathbf{x} \neq \mathbf{0}$, each of the above graphs satisfy the associated properties.

Definiteness

positive semidefinite negative definite



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positive definite indefinite

For $\mathbf{x} \neq \mathbf{0}$, each of the above graphs satisfy the associated properties.

Definiteness and Eigenvectors

Theorem. For a symmetric matrix A, the quadratic form $\mathbf{x}^T A \mathbf{x}$

- **positive definite** ≡ all positive eigenvalues
- **positive semidefinite** ≡ all nonnegative eigenvalues
- **indefinite** ≡ positive and negative eigenvalues
- **negative definite** ≡ all negative eigenvalues

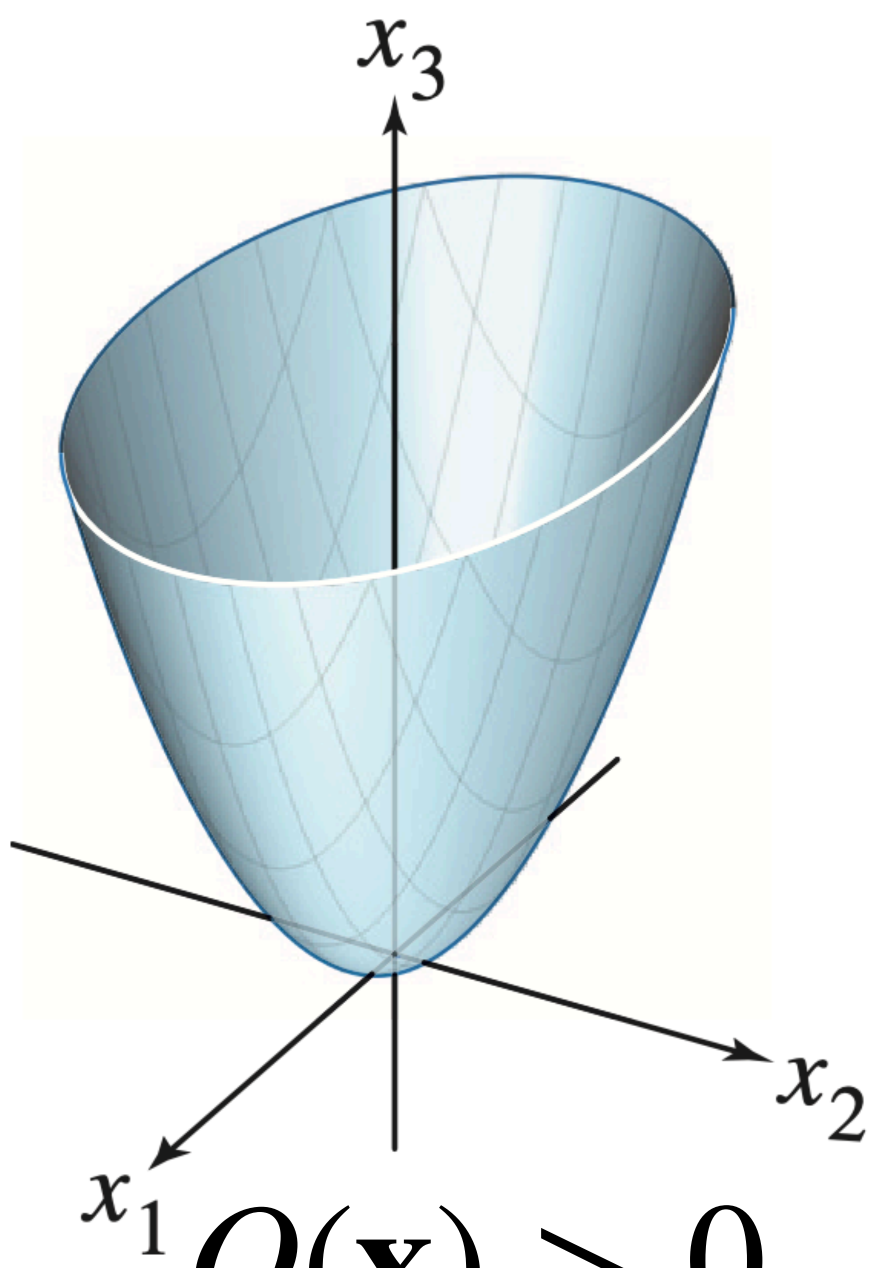
Definiteness

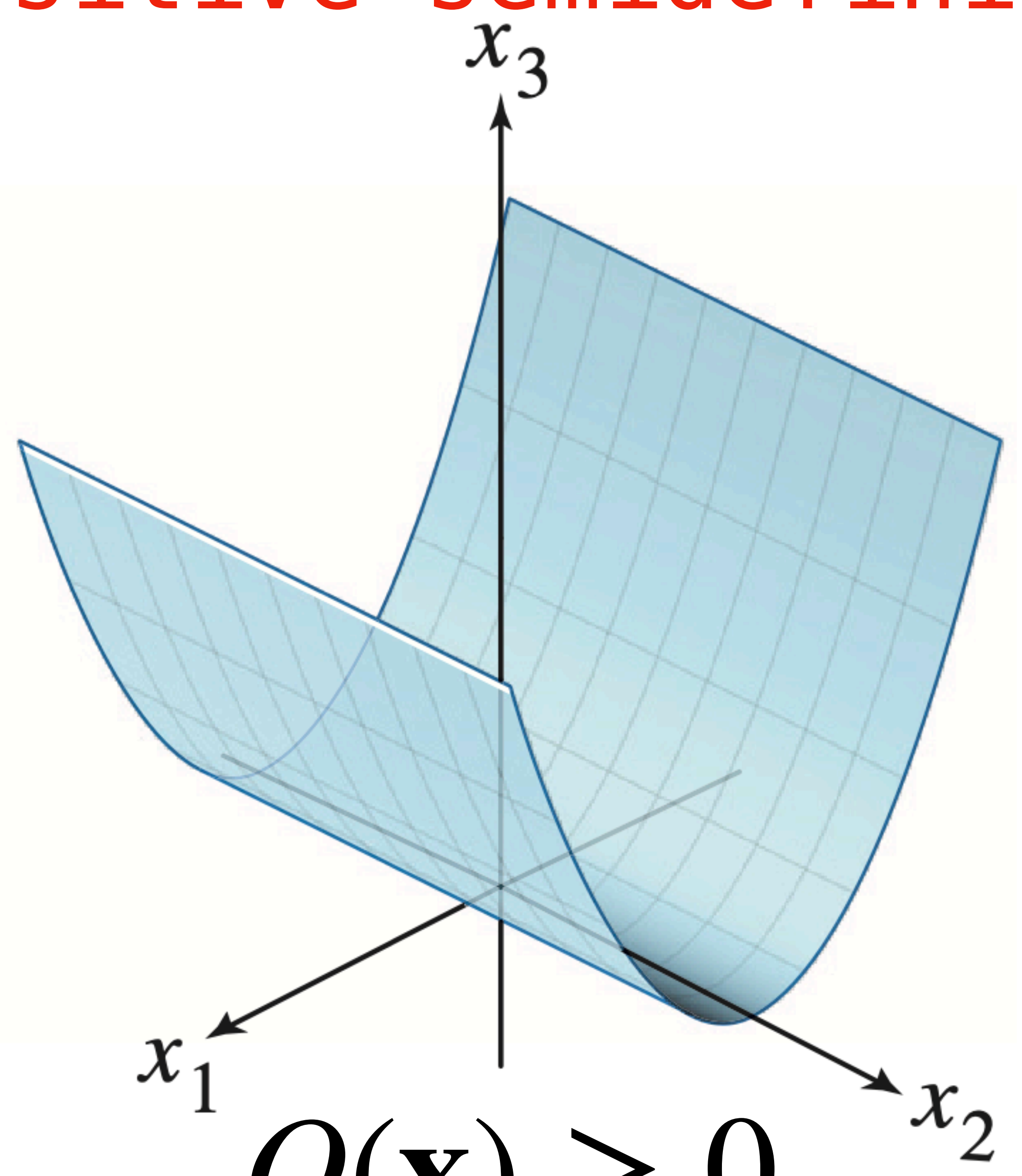
all nonneg. eigenvals

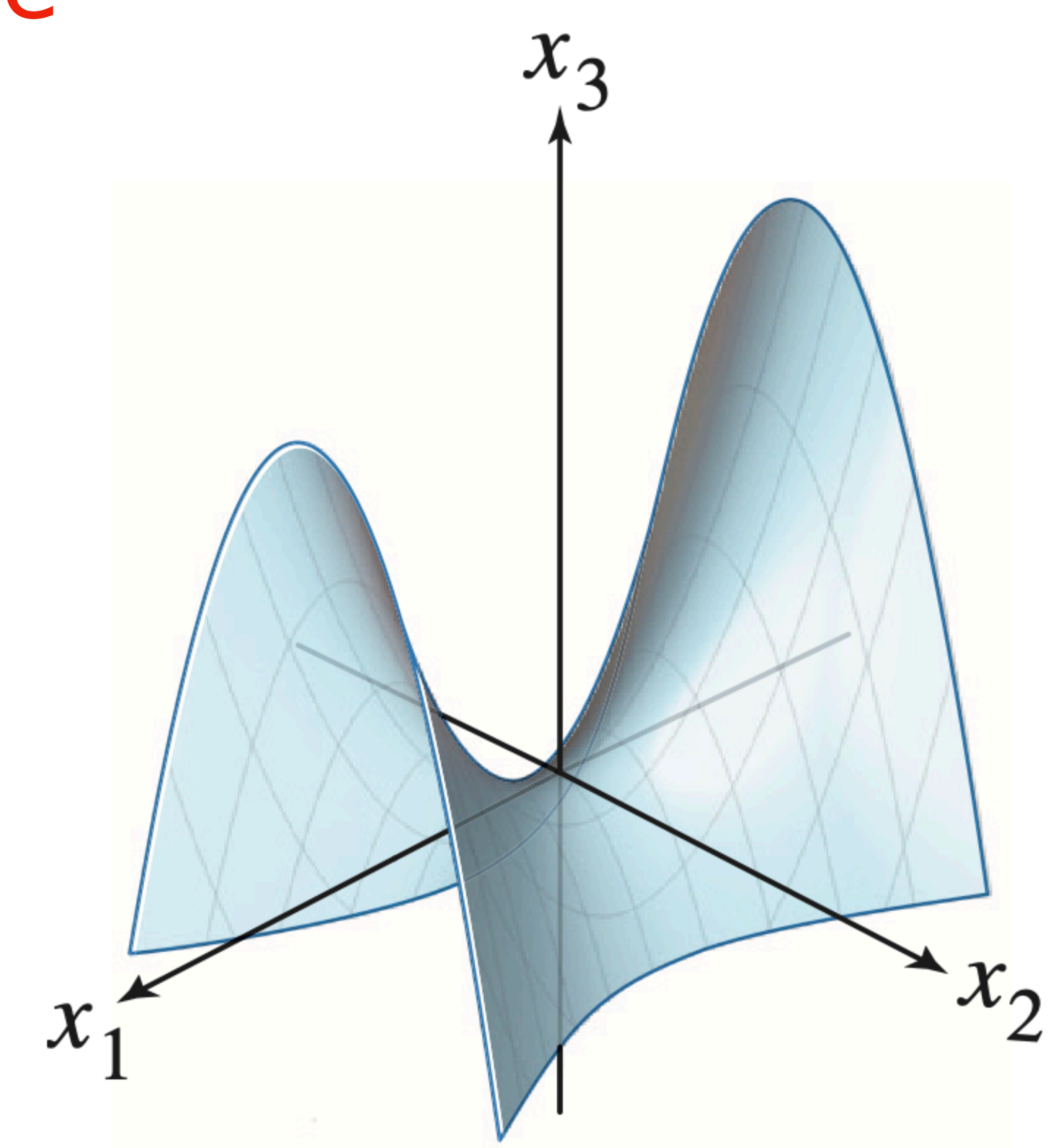
positive semidefinite

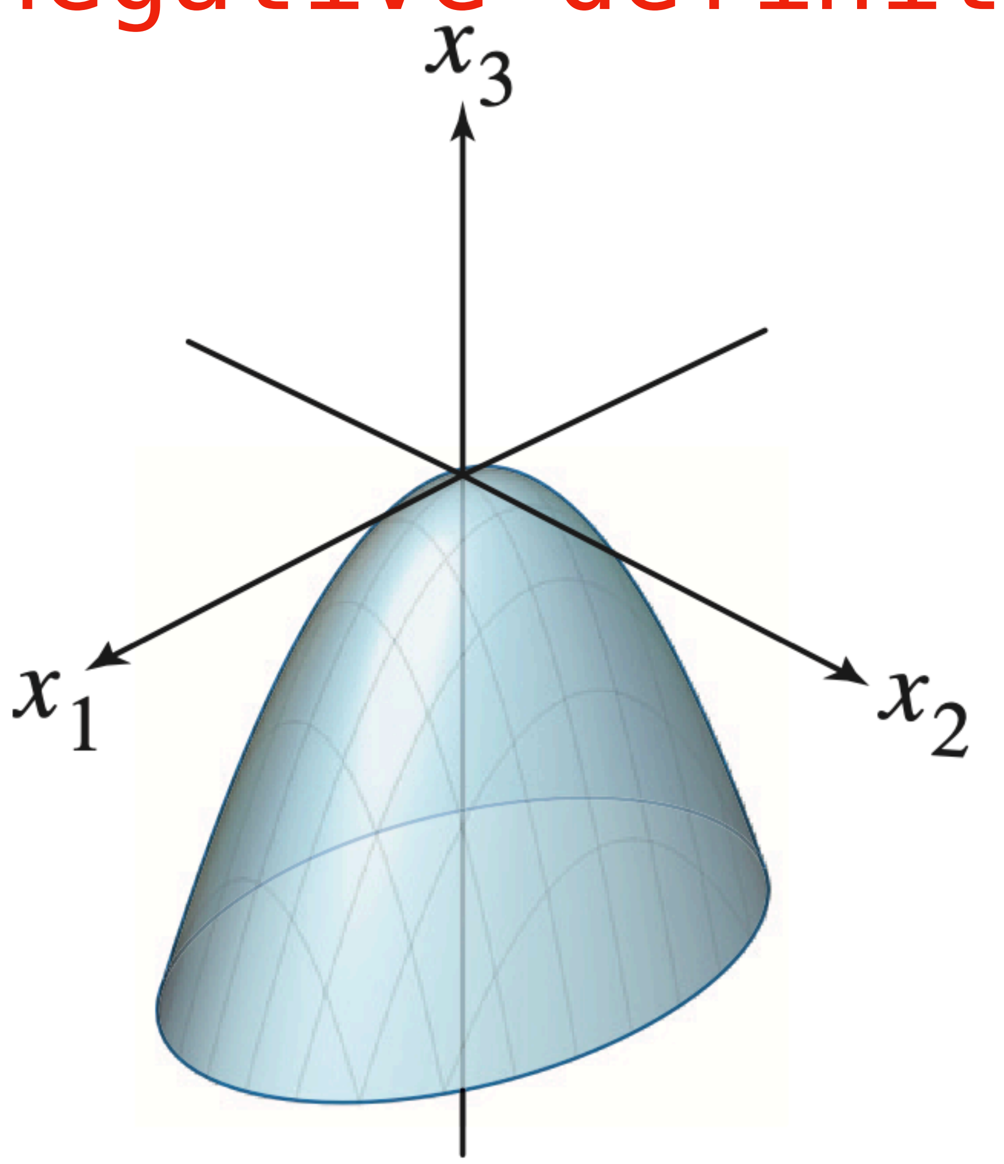
all neg. eigenvals

negative definite









$Q(\mathbf{x}) > 0$ $\quad$ $Q(\mathbf{x}) \geq 0$ $\quad$ $Q(\mathbf{x})$ can be + & − $\quad$ $Q(\mathbf{x}) < 0$

positive definite

all pos. eigenvals

indefinite

pos. and neg. eigenvals

Example

$$\det(A-\lambda I) = \frac{1}{\cancel{(1-\lambda)}}(3-\lambda)\cancel{(1-\lambda)}\left((1-\lambda)^2-4\right)$$

$$(3-\lambda)(\lambda^2-2\lambda-3) = -(3-\lambda)^2(\lambda+1)$$

$$Q(x_1,x_2,x_3) = 3x_1^2 + x_2^2 + 4x_2x_3 + x_3^2$$

$\boxed{\lambda = 3, -1}$

Let's determine which case this is:

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \qquad A-\lambda I = \begin{bmatrix} 3-\lambda & 0 & 0 \\ 0 & 1-\lambda & 2 \\ 0 & 2 & 1-\lambda \end{bmatrix} \sim$$

$$\begin{bmatrix} 3-\lambda & 0 & 0 \\ 0 & 1-\lambda & 2 \\ 0 & 2(1-\lambda) & (1-\lambda)^2 \end{bmatrix} \sim \begin{bmatrix} 3-\lambda & 0 & 0 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & (1-\lambda)^2-4 \end{bmatrix}$$

Constrained Optimization

In General

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Given a function $f : \mathbb{R}^n \to \mathbb{R}$ and a set of vectors X from $\mathbb{R}^n$ the **constrained minimization problem** for f over X is the problem of determining

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Given a function $f: \mathbb{R}^n \to \mathbb{R}$ and a set of vectors X from $\mathbb{R}^n$ the **constrained minimization problem** for f over X is the problem of determining

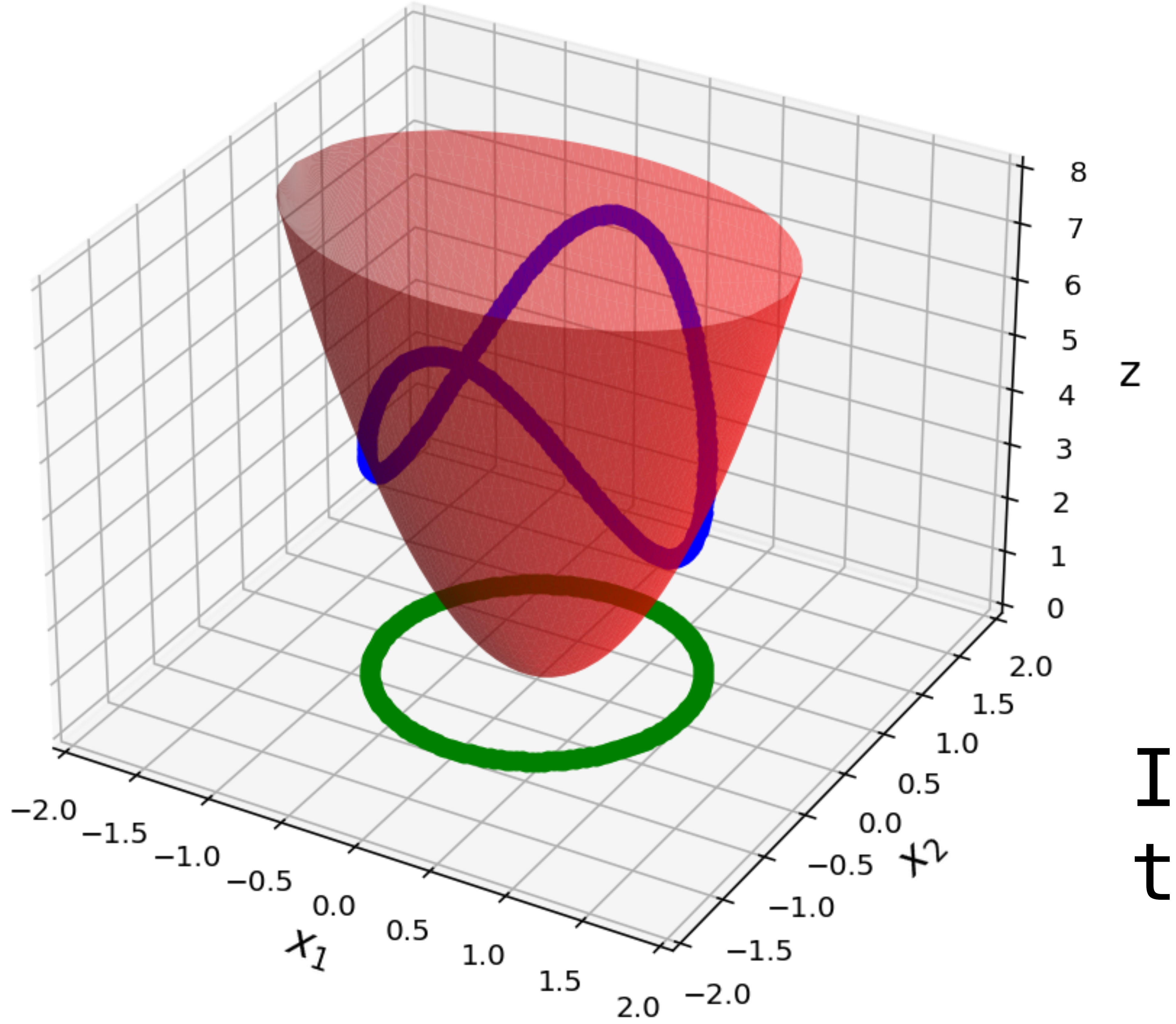
$$\min_{\mathbf{v} \in X} f(\mathbf{v})$$

(analogously for maximization)

Find the smallest value of $f(\mathbf{v})$ subject to choosing a vector in X

Constrained Optimization for Quadratic Forms and Unit Vectors

$$\text{mini/maximize } \mathbf{x}^T A \mathbf{x} \text{ subject to } \|\mathbf{x}\| = 1$$



It's common to constraint to unit vectors.

3

Example: $3x_1^2 + 7x_2^2 \le 7$

$3(0)^2 + 7(-1)^2 = 7$ (max)

$3(-1)^2 + 7(0)^2 = 3$ (min)

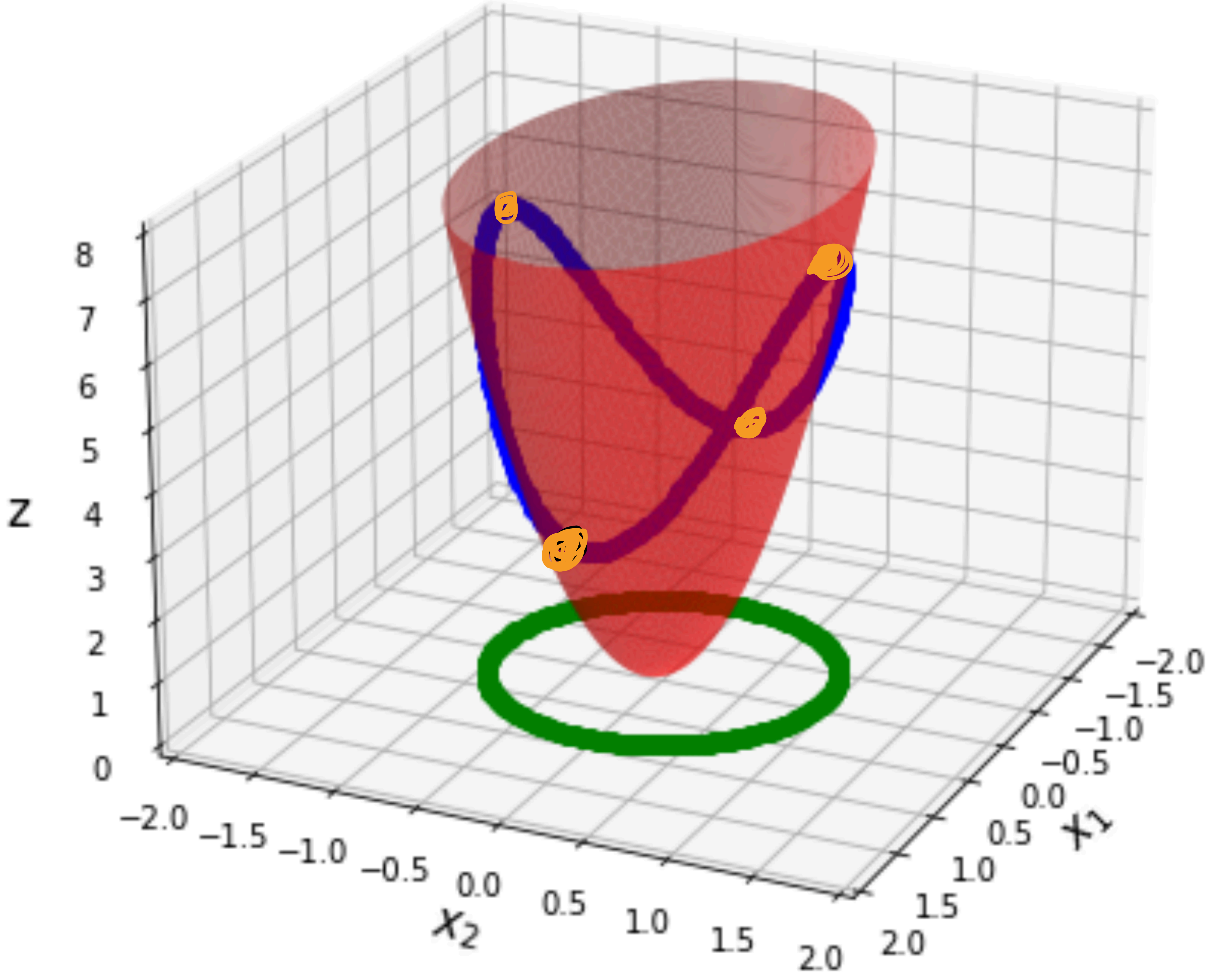
What are the min/max values?:

$$3x_1^2 + 7x_2^2 \le 7x_1^2 + 7x_2^2$$

$$= 7(\underbrace{x_1^2 + x_2^2}_{1})$$

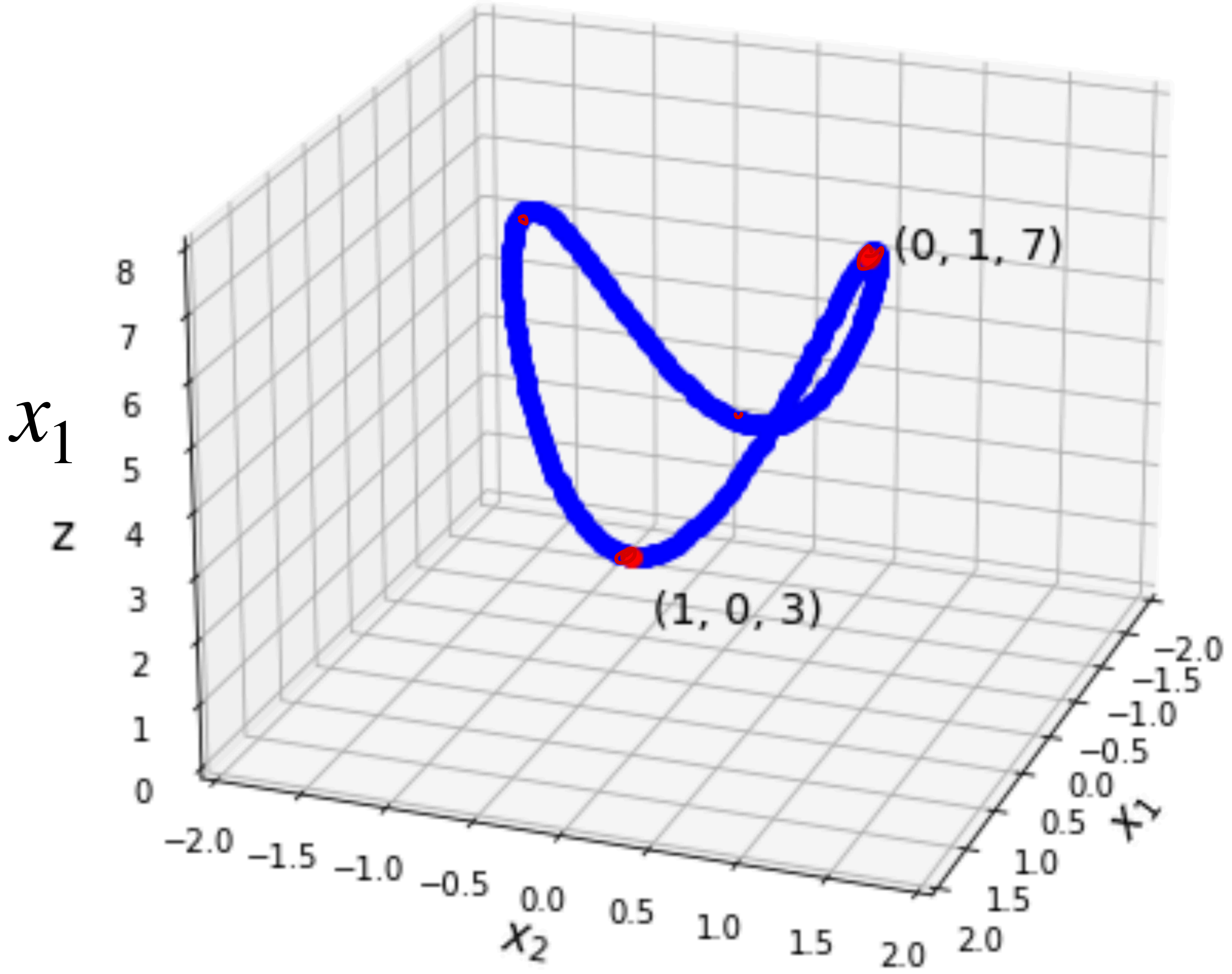
$$= 7$$

$$3x_1^2 + 7x_2^2 \ge 3$$



Example: $3x_1^2 + 7x_2^2$

The minimum and maximum values are attained when the "weight" of the vector is distributed all on x_1 or x_2.



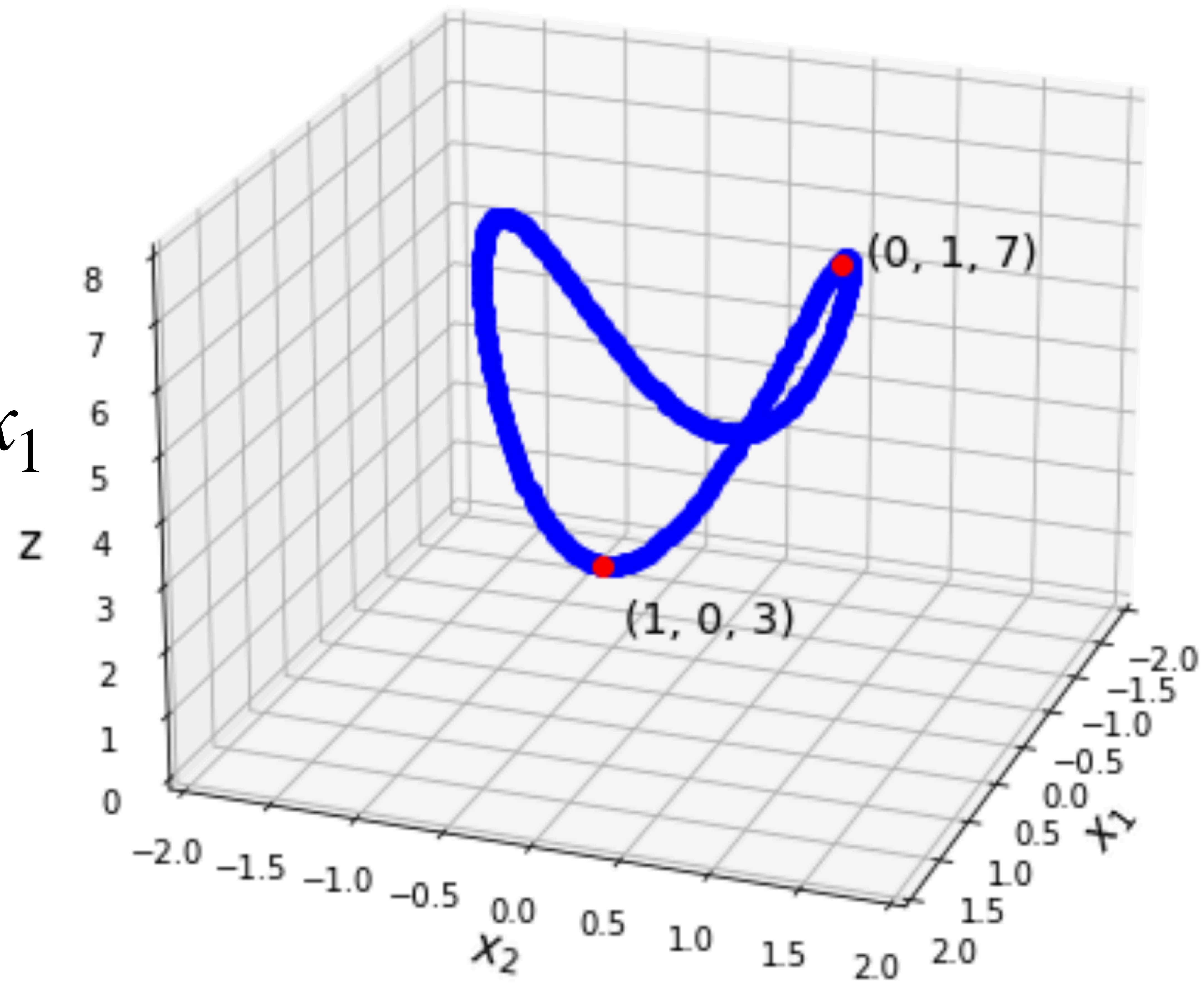
Example: $3x_1^2 + 7x_2^2$

What is the matrix?:

$$\begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix}$$

eigenvalues : 3 , 7

x_1



Constrained Optimization and Eigenvalues

Theorem. For a symmetric matrix A, with *largest* eigenvalue λ_1 and *smallest* eigenvalue λ_n

$$\max_{\|\mathbf{x}\|=1} \mathbf{x}^T A \mathbf{x} = \lambda_1 \qquad \min_{\|\mathbf{x}\|=1} \mathbf{x}^T A \mathbf{x} = \lambda_n$$

No matter the shape of A, this will hold.

How To: Constrained Optimization

How To: Constrained Optimization

Problem. Find the maximum value of $\mathbf{x}^T A \mathbf{x}$ subject to $\|\mathbf{x}\| = 1$.

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Solution. Find the largest eigenvalue of A, this will be the maximum value.

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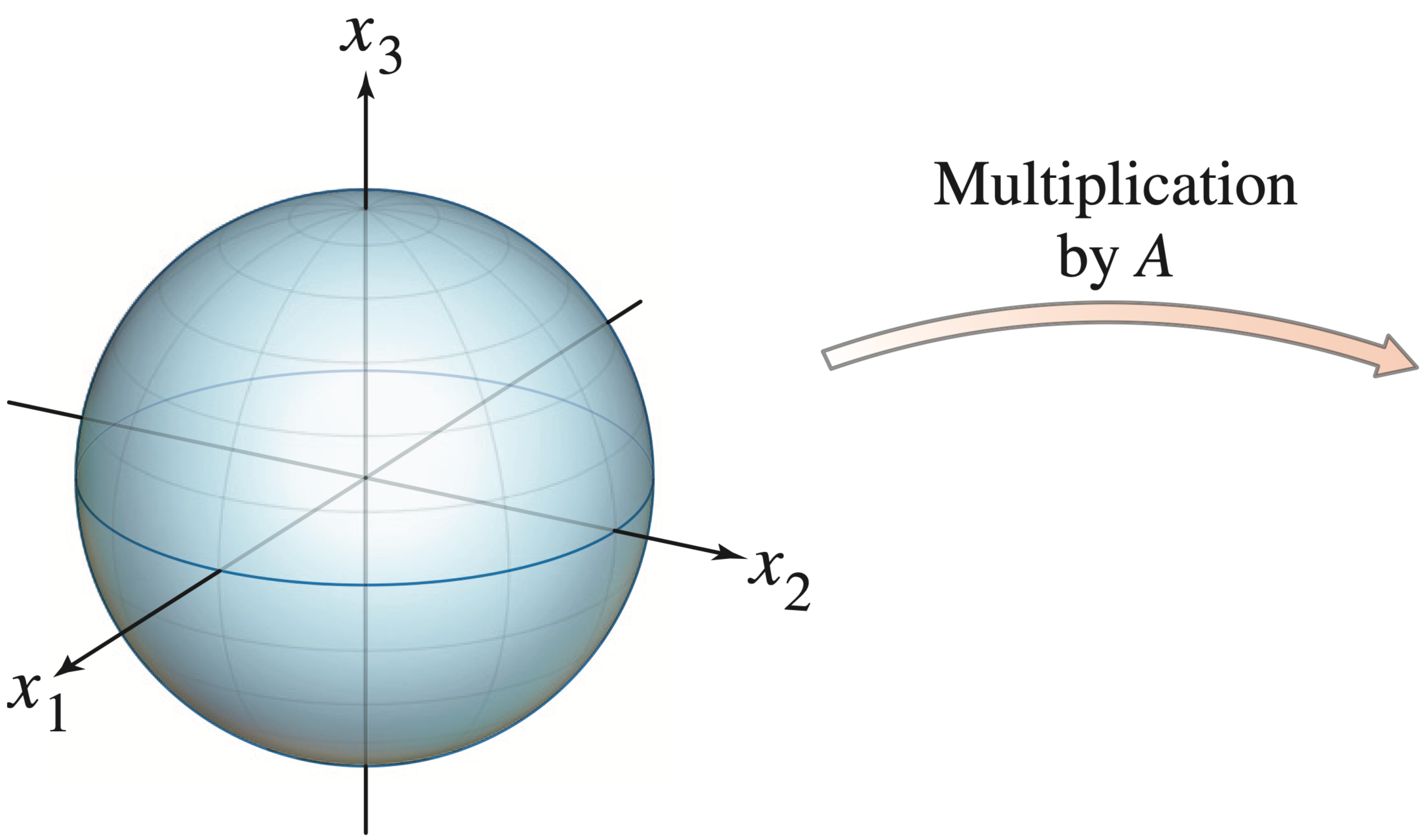
(Use NumPy)

Singular Value Decomposition

Question

What shape is a the unit sphere after a linear transformation?

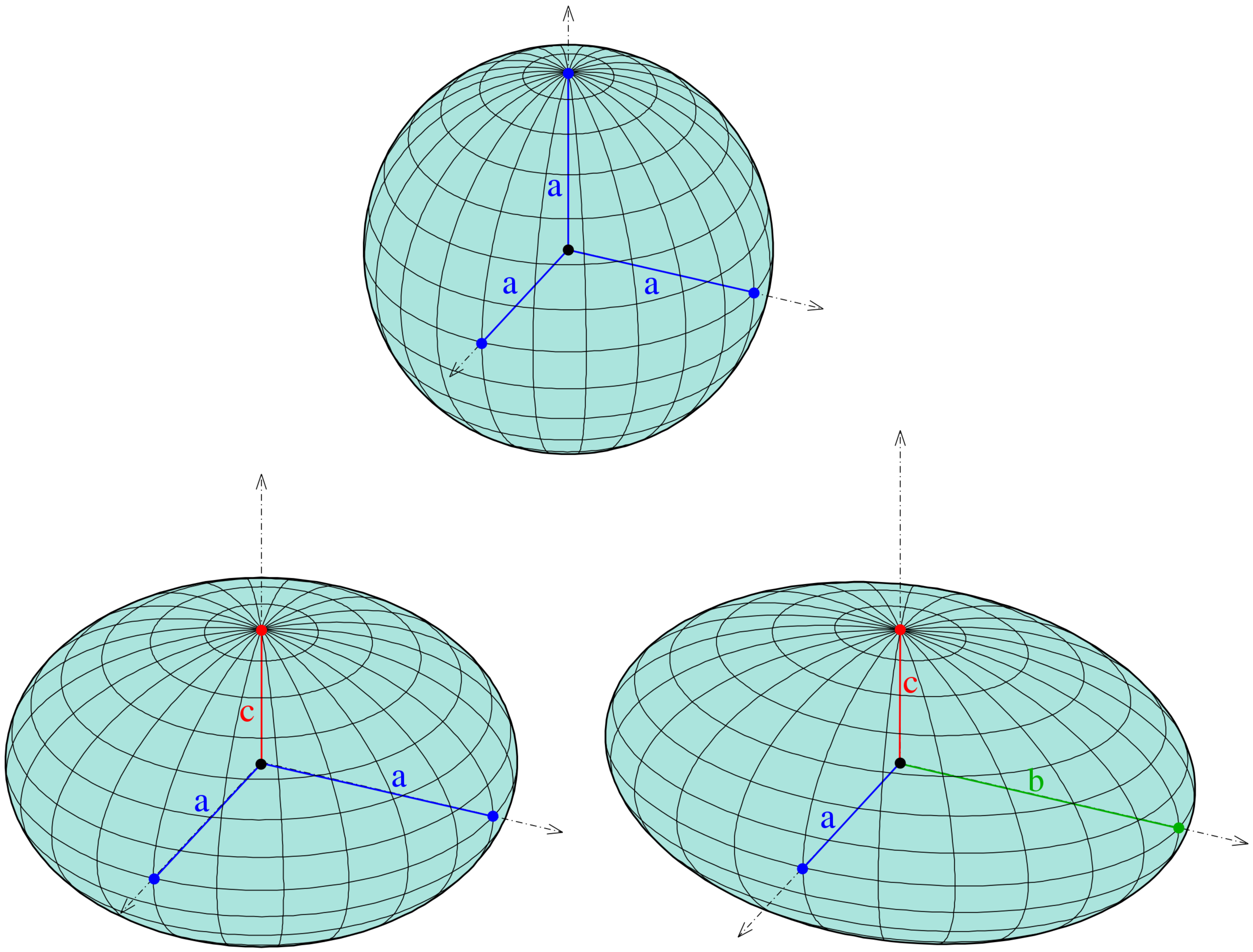


???

Ellipsoids

Ellipsoids are spheres "stretched" in orthogonal directions called the **axes of symmetry** or the **principle axes.**

Linear transformations maps <u>spheres</u> to <u>ellipsoids</u>.

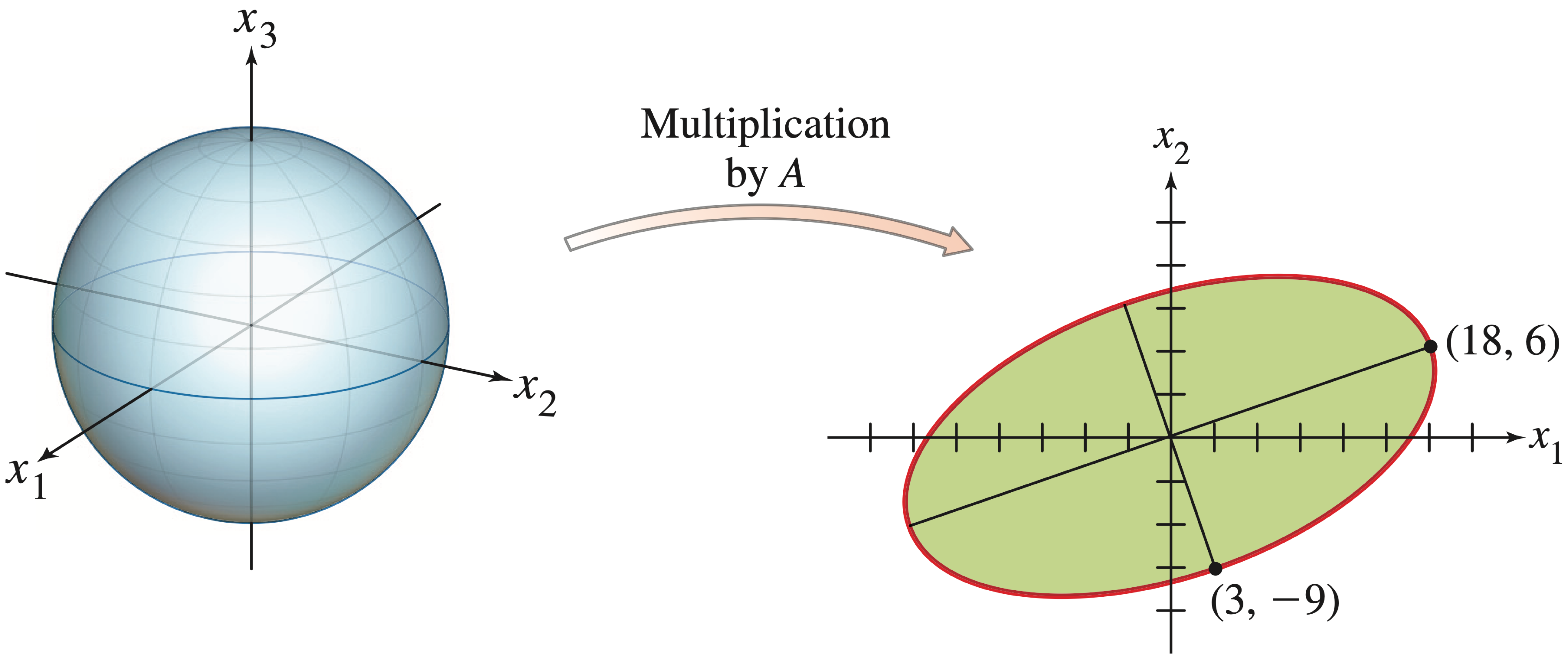


https://commons.wikimedia.org/wiki/File:Ellipsoide.svg

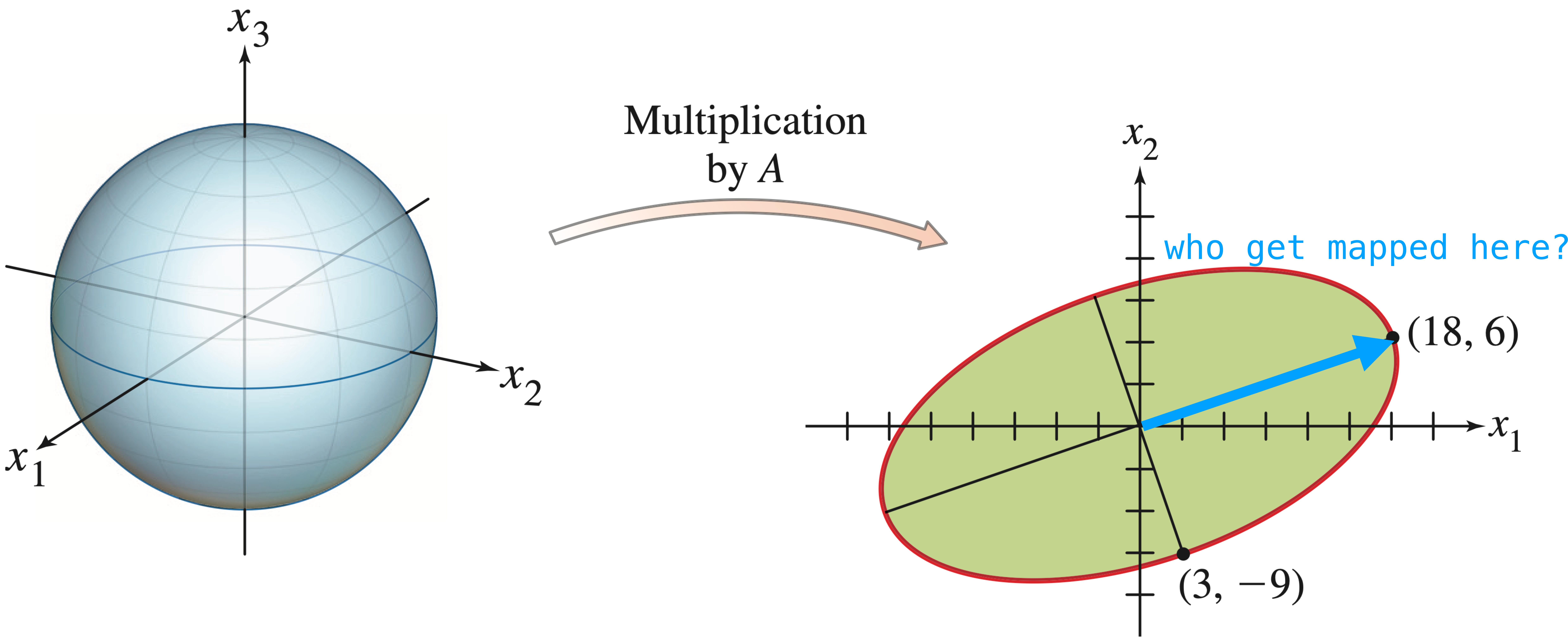
Simple Example : Scaling Matrices

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

The Picture

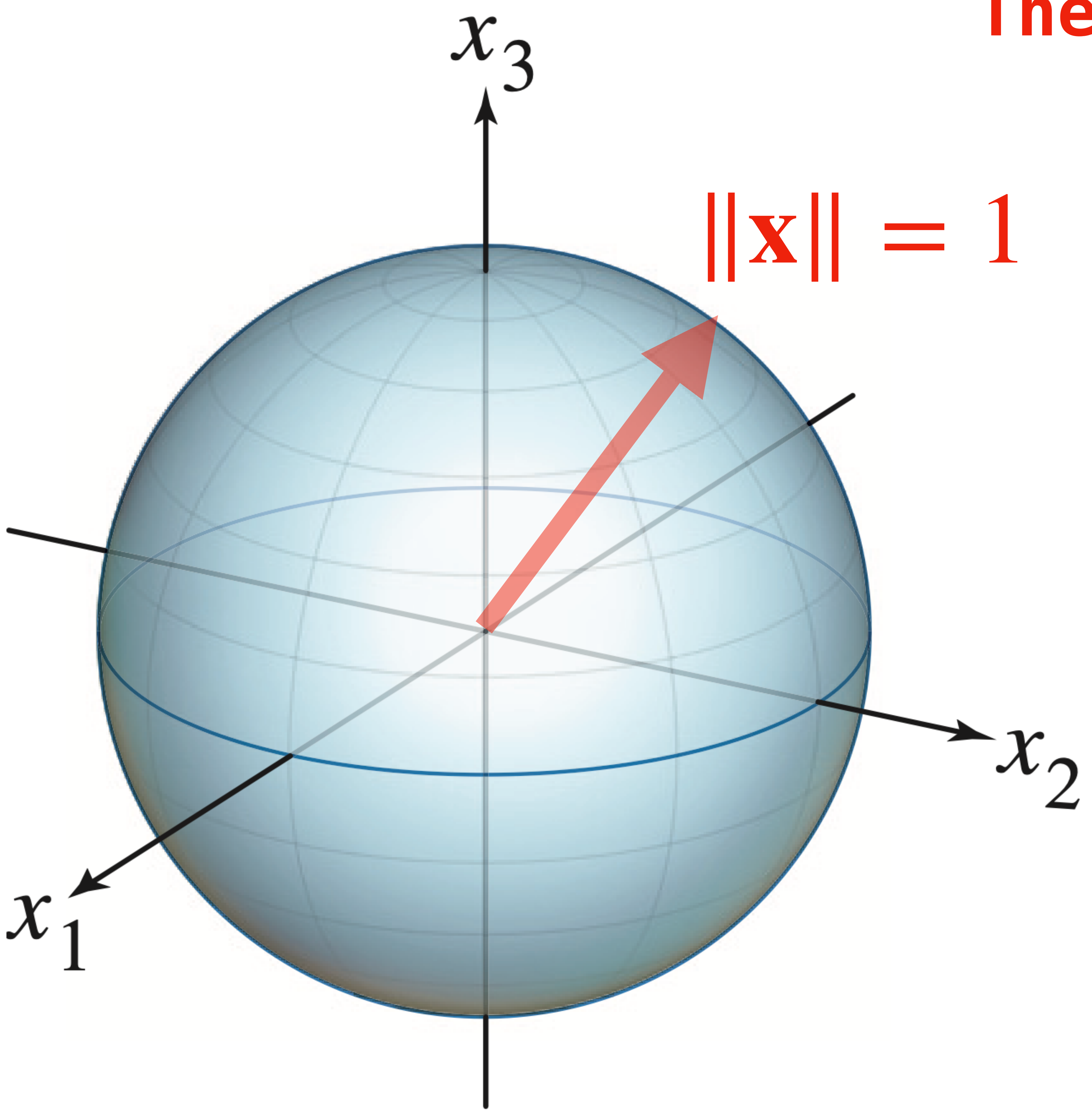


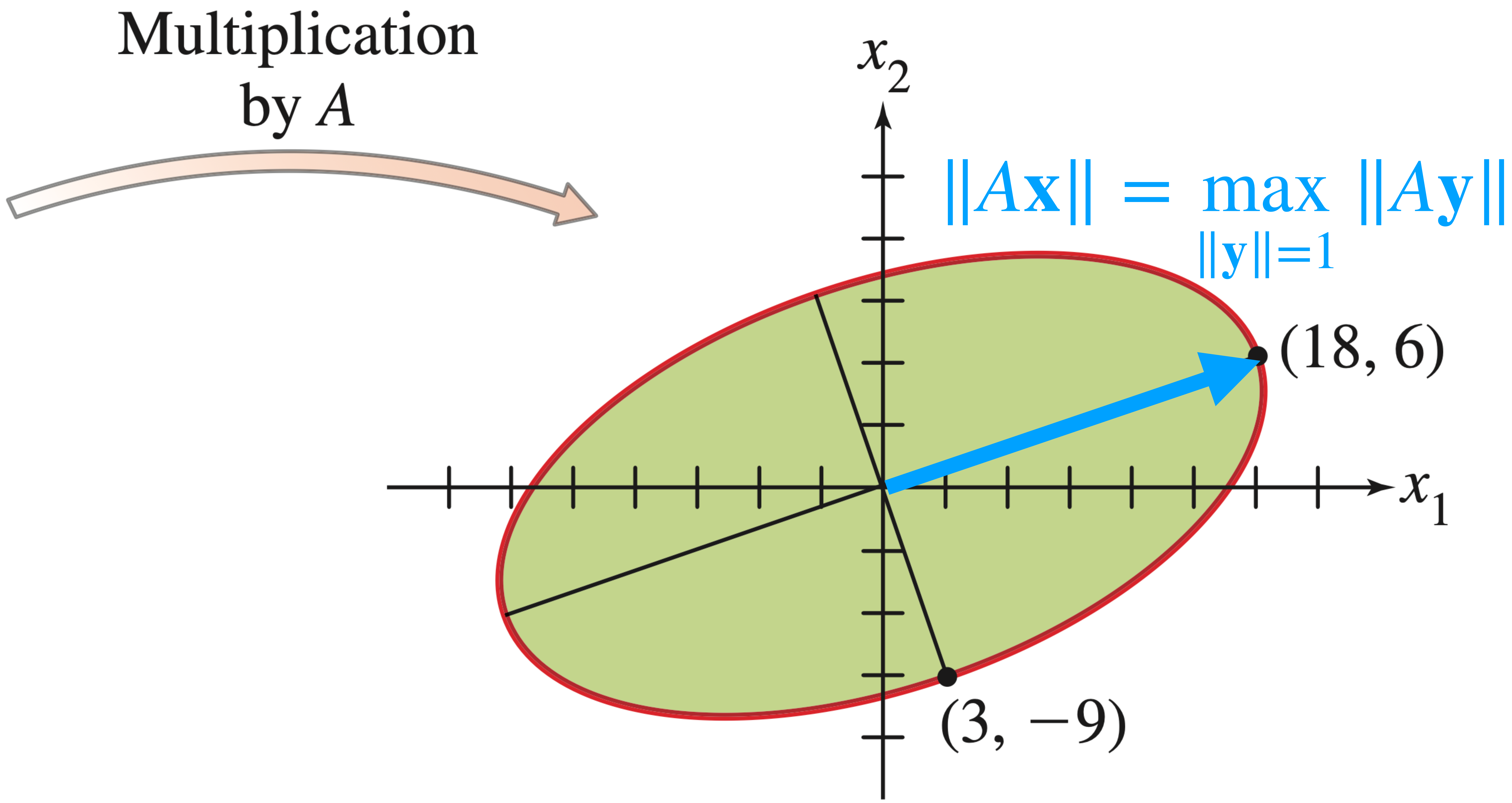
The Picture



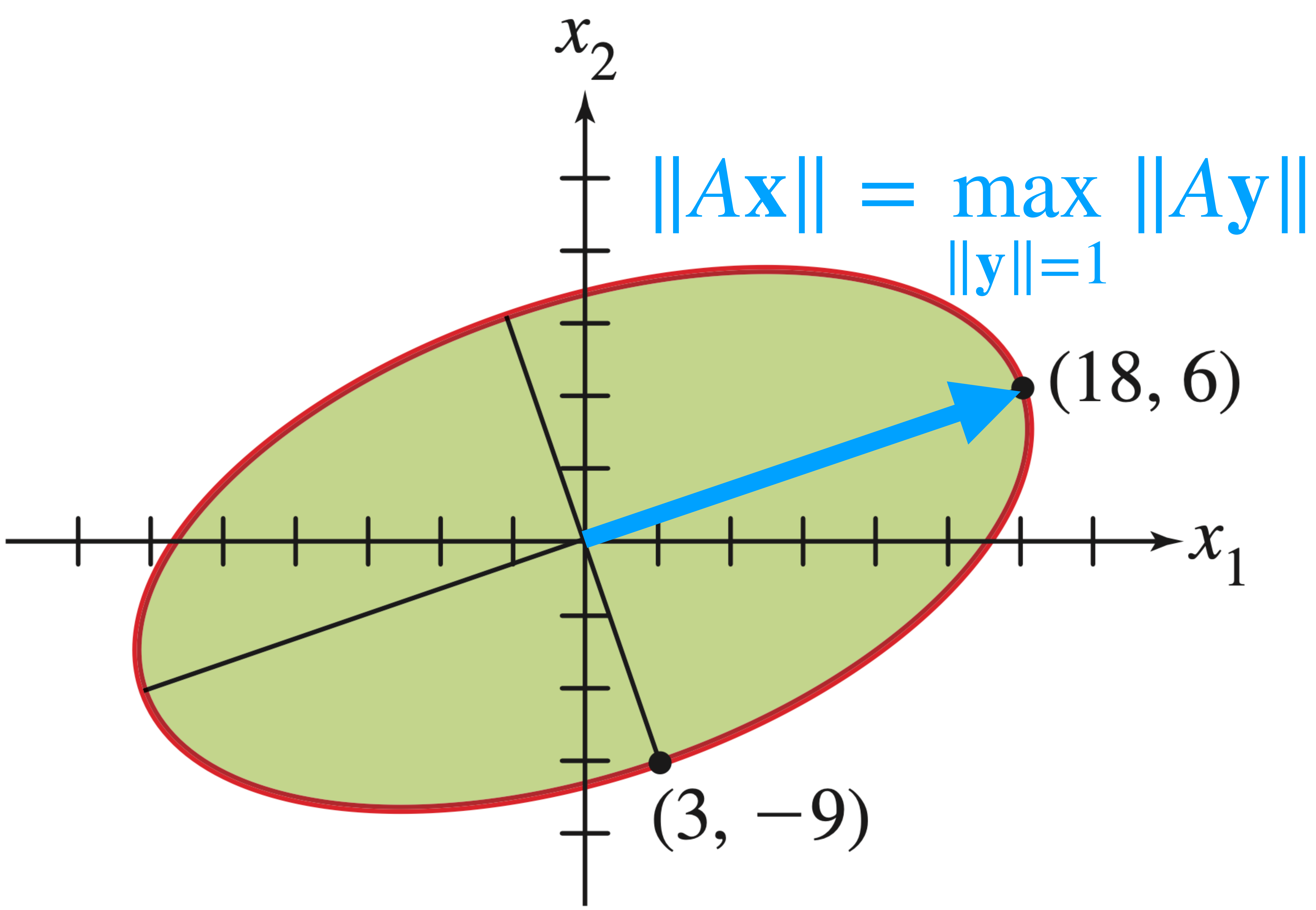
The Picture

The longest end of the ellipse is the solution to a constrained optimization problem



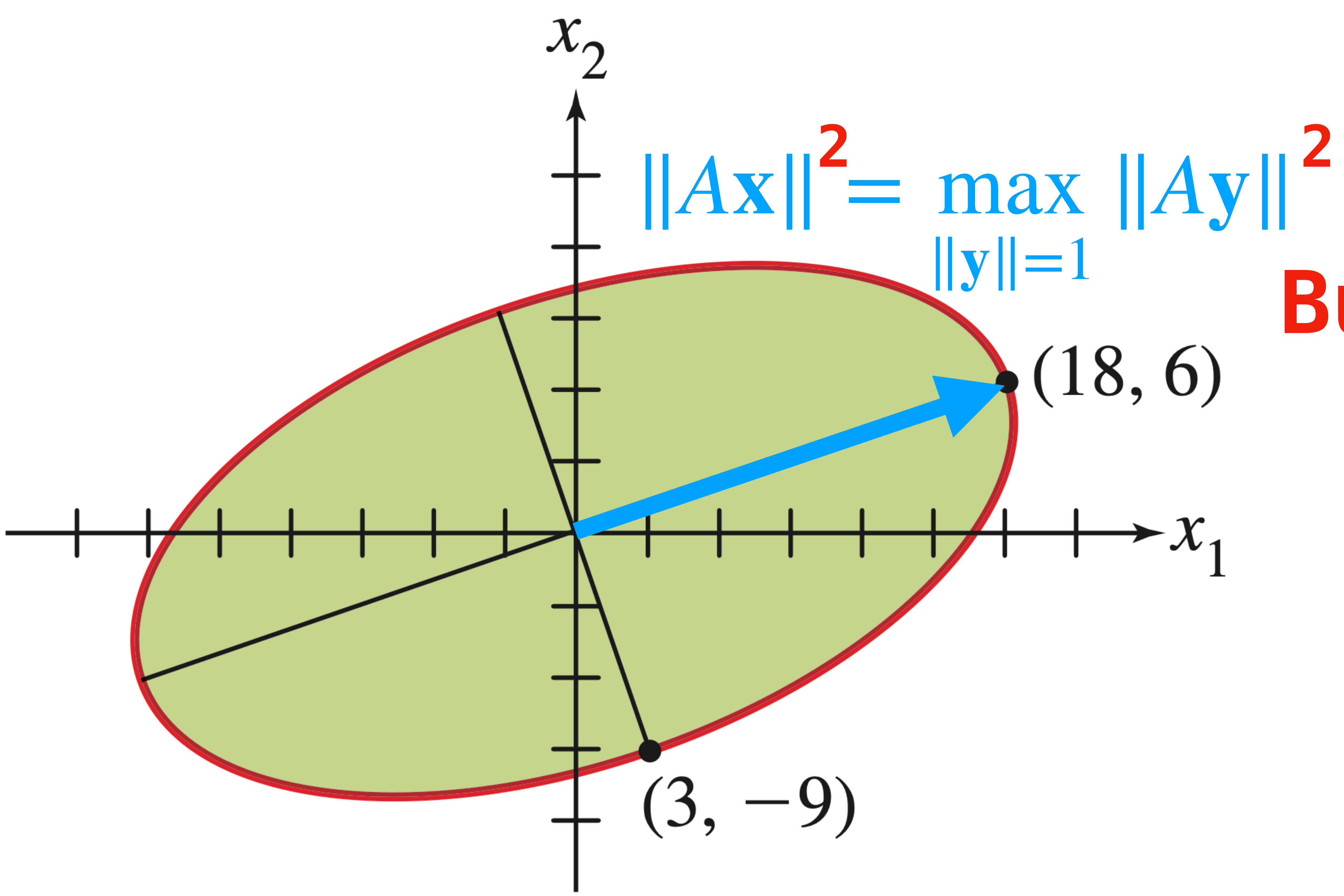


The Picture



`This is not a quadratic form...`

The Picture



But this is.

This is not a quadratic form...

A Quadratic Form

What does $\|A\mathbf{x}\|^2$ look like?:

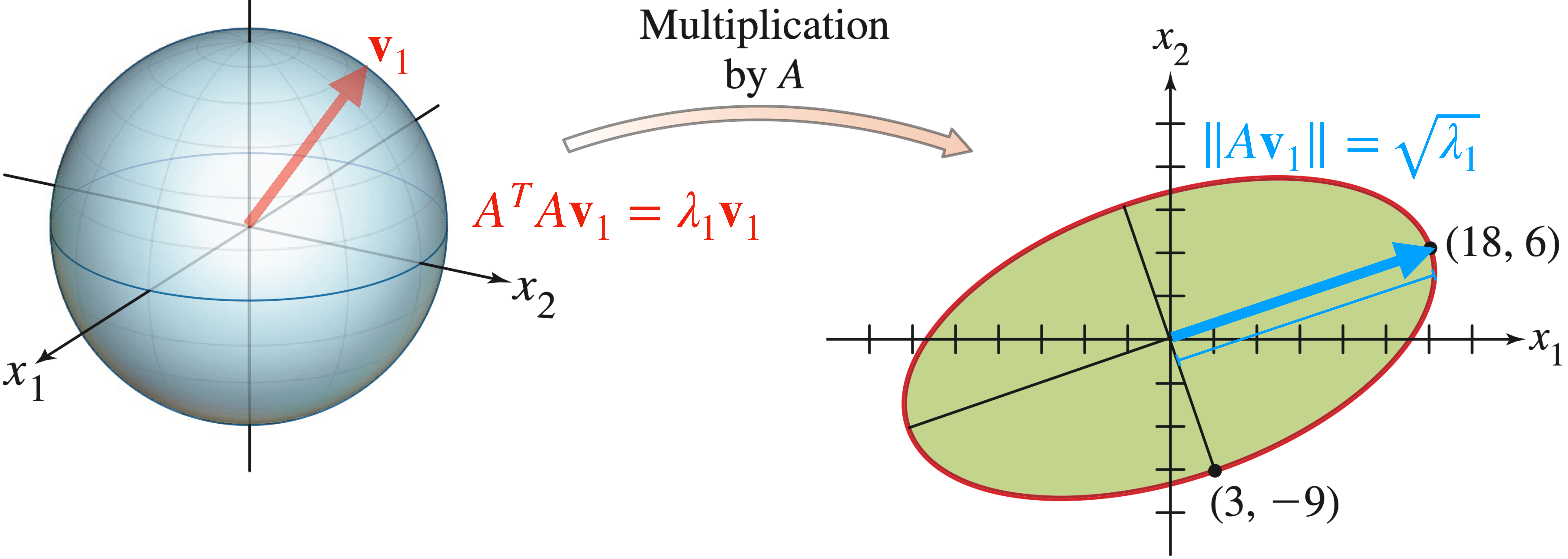
$$\langle A\vec{x}, A\vec{x}\rangle = (A\vec{x})^T A\vec{x}$$

$$\vec{x}^T \boxed{A^T A} \vec{x}$$

The Picture

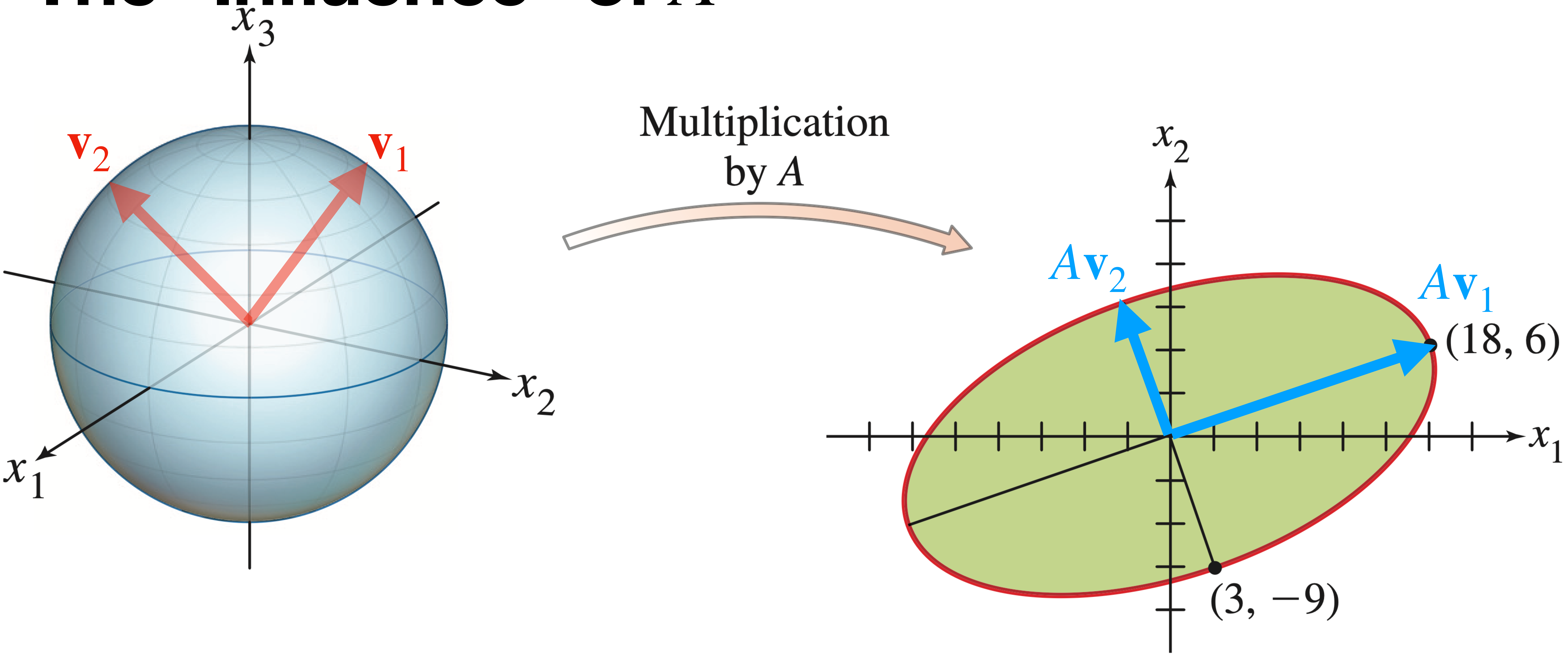
The ~~largest~~ eigenvector of A^TA

for largest eigenvalue



$\mathbf{v}_1$ solves the constrained optimization problem.

The "Influence" of A



$\mathbf{v}_1$ is "most affected" by A and $\mathbf{v}_2$ is "least affected"

Properties of $A^T A$

Properties of $A^T A$

» It's symmetric. $(A^T A)^T = A^T A^{TT} = A^T A$

Properties of $A^T A$

- It's symmetric.
- So its <u>orthogonally diagonalizable</u>.

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- **There is an orthogonal basis of eigenvectors.**

Properties of $A^T A$

- It's symmetric.
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- It's eigenvalues are nonnegative.

Properties of A^TA

- It's symmetric.
- So its orthogonally diagonalizable.
- **There is an orthogonal basis of eigenvectors.**
- It's eigenvalues are nonnegative.
- **It's positive semidefinite.**

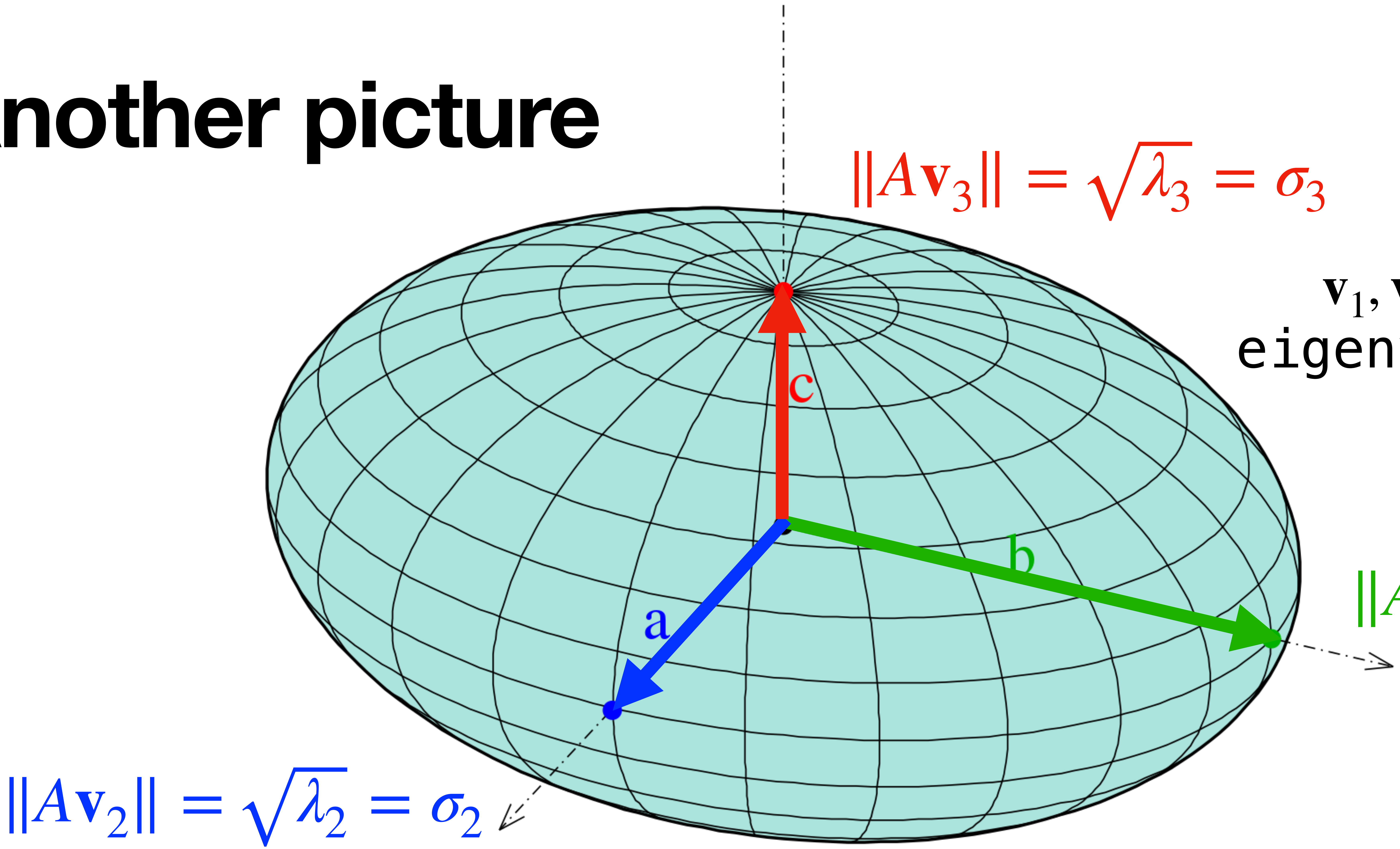
Singular Values

Definition. For an $m \times n$ matrix A, the **singular values** of A are the n values

$$\sigma_1 \geq \sigma_2 \ldots \geq \sigma_n \geq 0$$

where $\sigma_i = \sqrt{\lambda_i}$ and λ_i is an eigenvalue of $A^T A$.

Another picture



$\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are the eigenvectors of A^TA

The **singular values** are the <u>lengths</u> of the *axes of symmetry* of the ellipsoid after transforming the unit sphere.

https://commons.wikimedia.org/wiki/File:Ellipsoide.svg

Every $m \times n$ matrix transforms the unit m–sphere into an n–ellipsoid.

So <u>every</u> $m \times n$ matrix has n singular values.

What else can we say?

Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be an **orthogonal** eigenbasis of $\mathbb{R}^n$ for A^TA and suppose A has r <u>nonzero</u> singular values.

Theorem. $A\mathbf{v}_1, \ldots, A\mathbf{v}_r$ is an orthogonal basis of $\text{Col}(A)$.

What else can we say?

Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be an **orthogonal** eigenbasis of $\mathbb{R}^n$ for A^TA and suppose A has r nonzero singular values.

Theorem. $A\mathbf{v}_1, \ldots, A\mathbf{v}_r$ is an orthogonal basis of $\text{Col}(A)$.

This is the most important theorem for SVD.

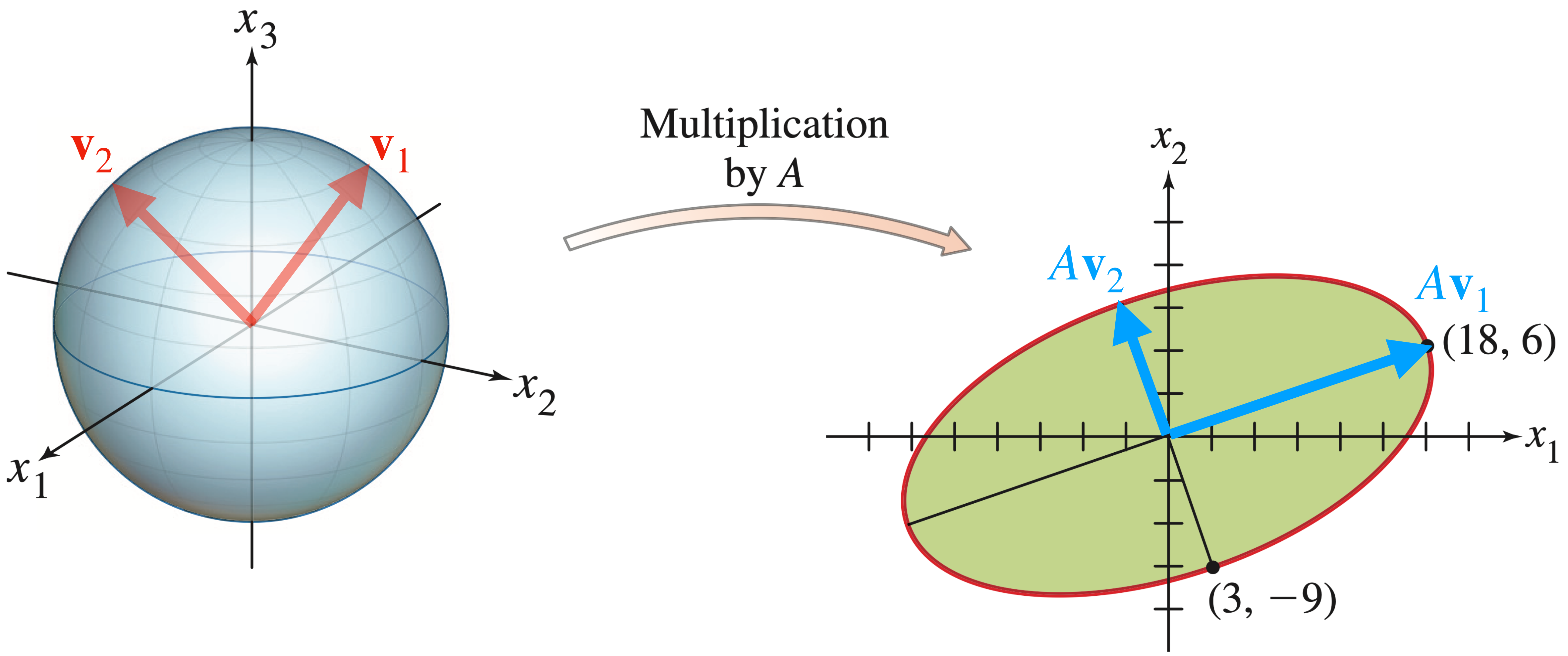
Verifying it

Let's show $A\mathbf{v}_1, \ldots, A\mathbf{v}_r$ are linearly independent:

Verifying it

Let's show $A\mathbf{v}_1, \ldots, A\mathbf{v}_r$ span $\text{Col}(A)$:

Putting it all together



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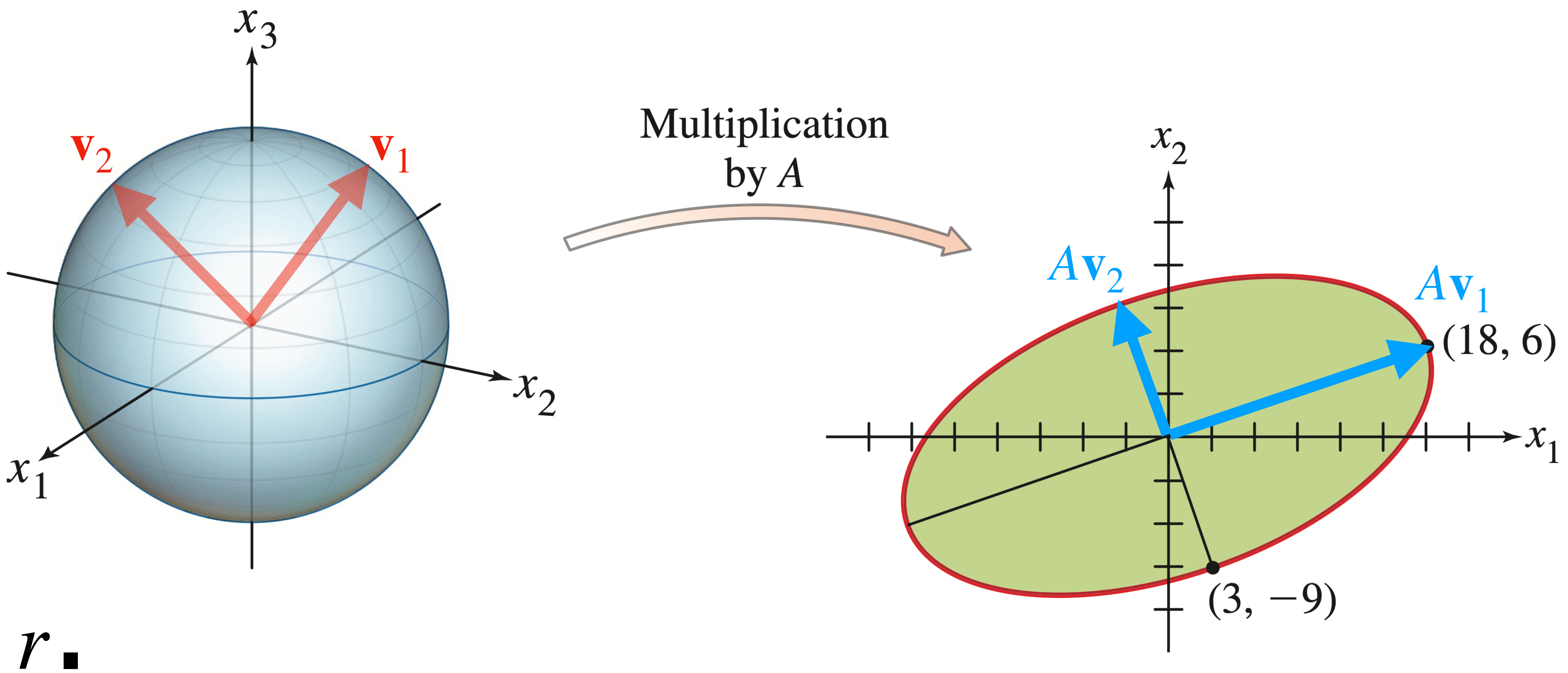
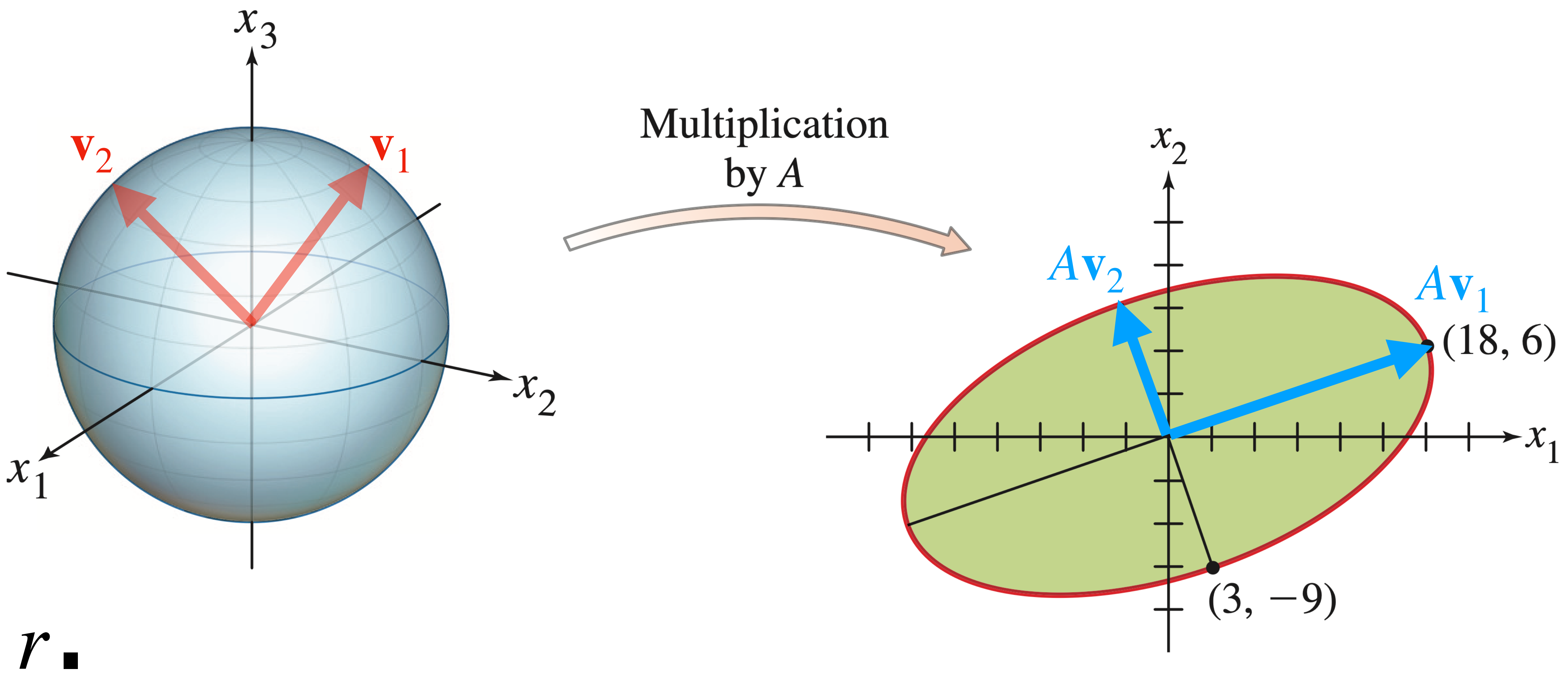


Let A be an $m \times n$ matrix of rank r.

Putting it all together



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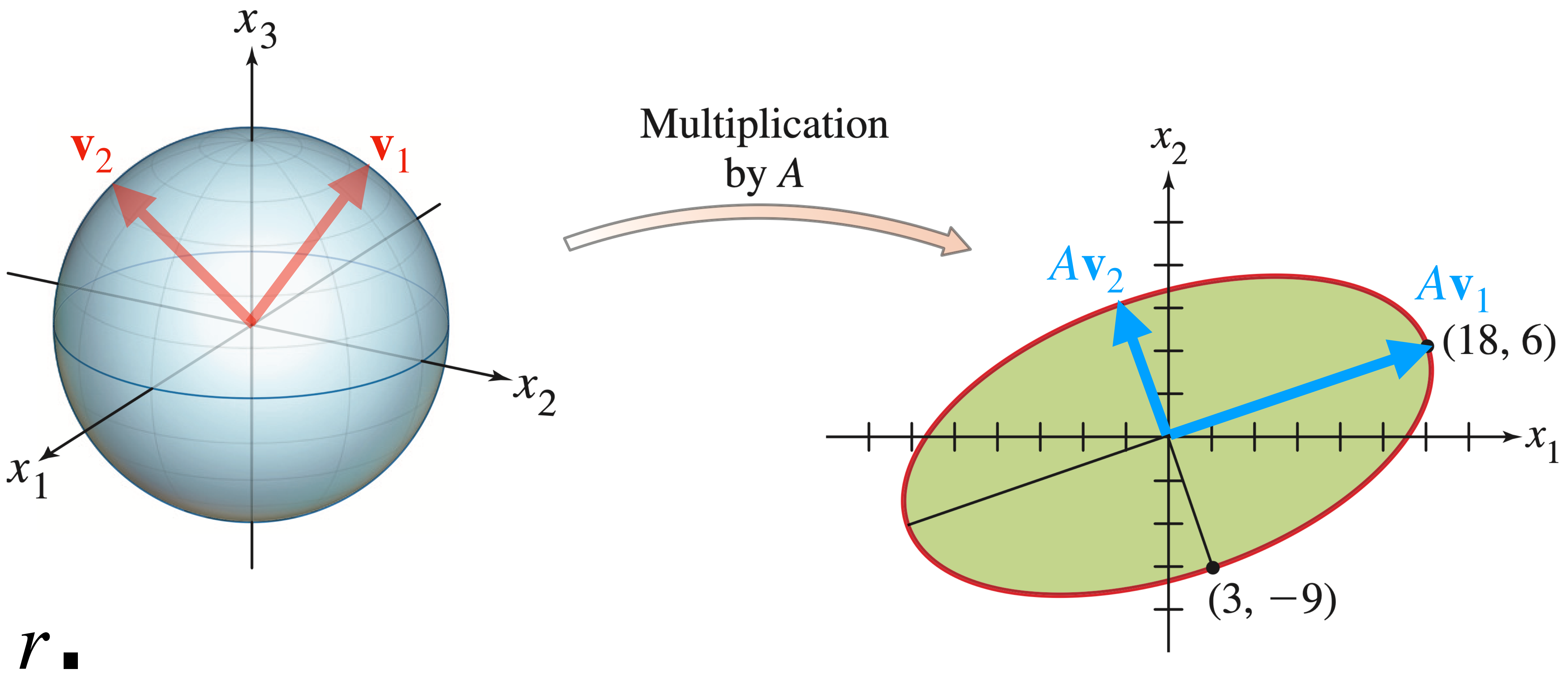
<u>What we know:</u>

Putting it all together

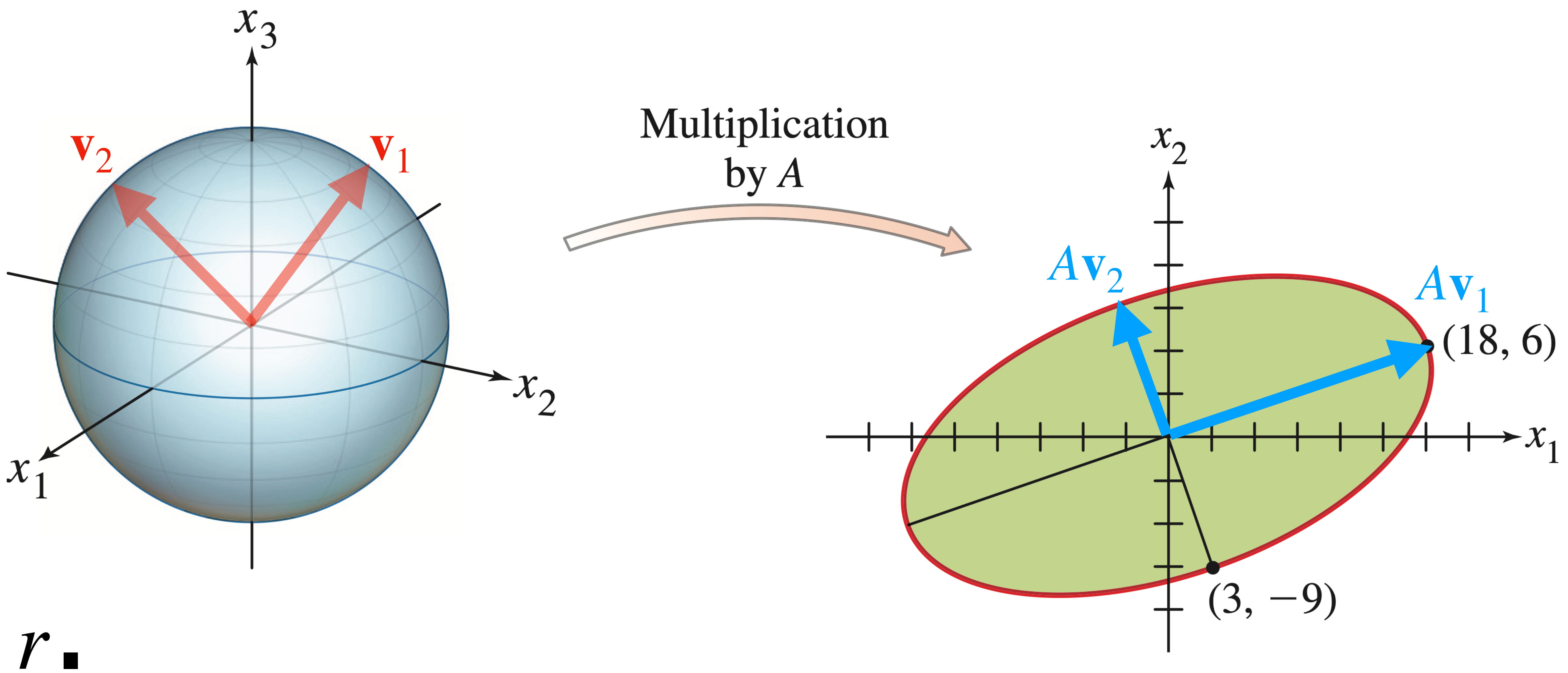


Let A be an $m \times n$ matrix of rank r.

<u>What we know:</u>

» We can find orthonormal vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ in $\mathbb{R}^n$ such that $A\mathbf{v}_1, \ldots, A\mathbf{v}_r$ in $\mathbb{R}^m$ form an orthogonal basis for $\mathrm{Col}(A)$.

Putting it all together



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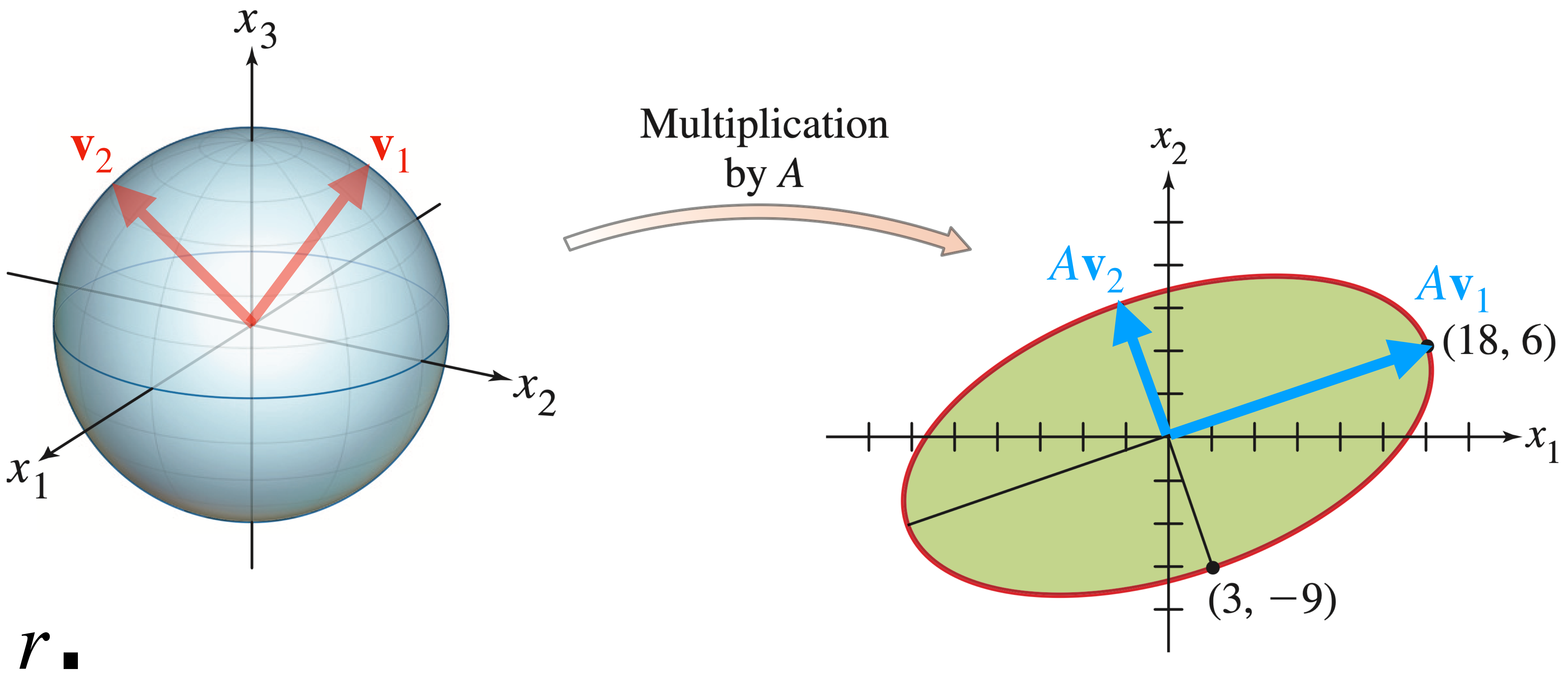
» So if we take $\mathbf{u}_i = \dfrac{A\mathbf{v}_i}{\|A\mathbf{v}_i\|}$, we get an **orthonormal** basis of Col(A)

Putting it all together



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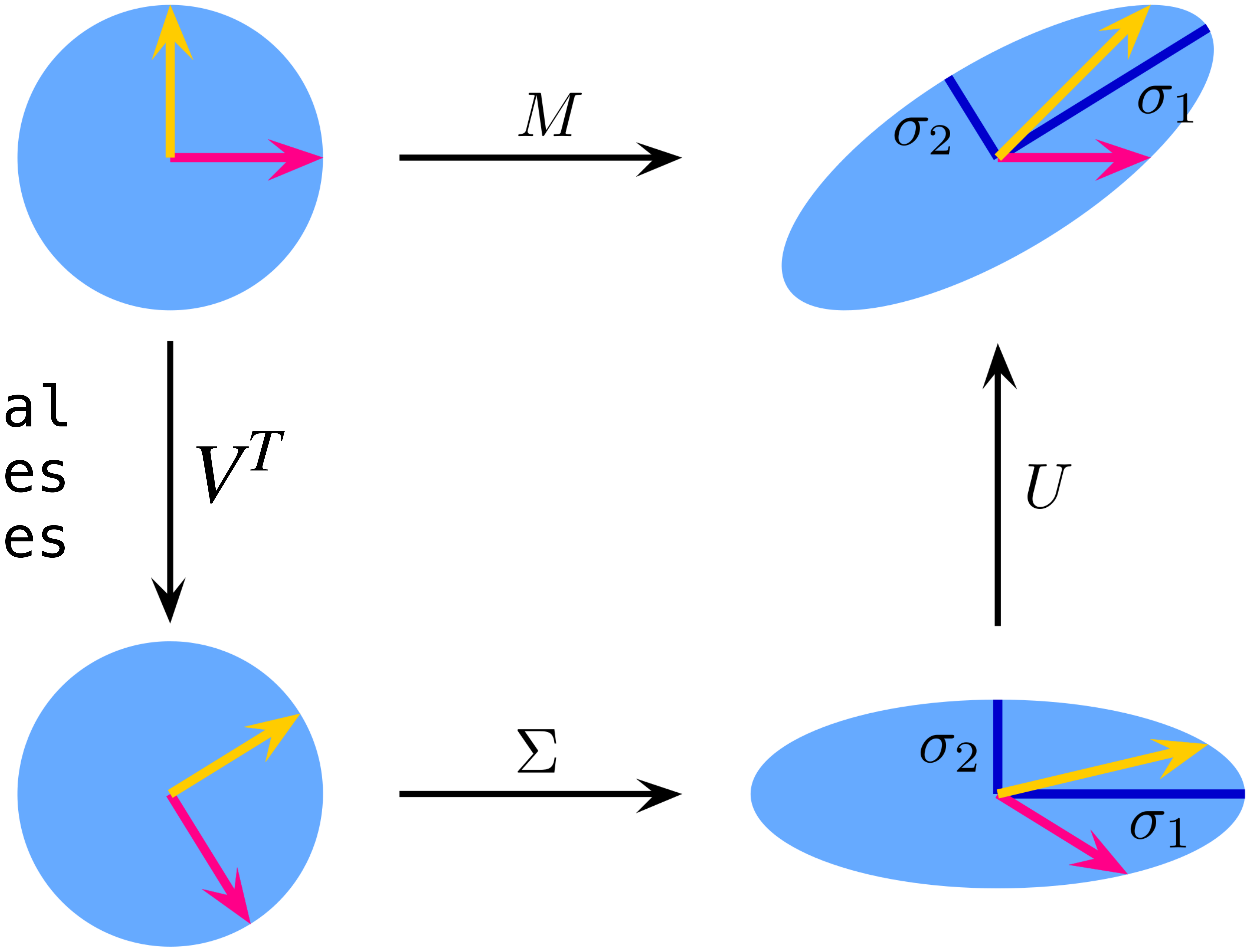
» We can find orthonormal vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ in $\mathbb{R}^n$ such that $A\mathbf{v}_1, \ldots, A\mathbf{v}_r$ in $\mathbb{R}^m$ form an orthogonal basis for $\mathrm{Col}(A)$.

» So if we take $\mathbf{u}_i = \dfrac{A\mathbf{v}_i}{\|A\mathbf{v}_i\|}$, we get an **orthonormal** basis of $\mathrm{Col}(A)$

» And we can extend this to $\mathbf{u}_1, \ldots, \mathbf{u}_m$ an orthonormal basis of $\mathbb{R}^m$ (via Gram-Schmidt).

High Level View of the Decomposition



<u>recall</u>: Orthogonal matrices preserves lengths and angles

$$M = U \cdot \Sigma \cdot V^T$$

https://commons.wikimedia.org/wiki/File:Singular-Value-Decomposition.svg

The Important Equality

$$\mathbf{u}_i = \frac{A\mathbf{v}_i}{\|A\mathbf{v}_i\|}$$

$$A\mathbf{v}_i = \|A\mathbf{v}_i\|\mathbf{u}_i = \sigma_i\mathbf{u}_i$$

The Important Equality

$$\mathbf{u}_i = \frac{A\mathbf{v}_i}{\|A\mathbf{v}_i\|}$$

$$A\mathbf{v}_i = \|A\mathbf{v}_i\|\mathbf{u}_i = \sigma_i\mathbf{u}_i$$

Remember that $\sigma_i = \sqrt{\lambda_i}$ is the singular value, which is the length $\|A\mathbf{v}_i\|$.

The Important Equality

$$\mathbf{u}_i = \frac{A\mathbf{v}_i}{\|A\mathbf{v}_i\|}$$

$$A\mathbf{v}_i = \|A\mathbf{v}_i\|\mathbf{u}_i = \sigma_i\mathbf{u}_i$$

Remember that $\sigma_i = \sqrt{\lambda_i}$ is the singular value, which is the length $\|A\mathbf{v}_i\|$.

What happens when we write this in matrix form?

The Important Equality

$$A[\mathbf{v}_1 \ \ \dots \ \ \mathbf{v}_n] = [\sigma_1\mathbf{u}_1 \ \ \dots\sigma_n\mathbf{u}_n]$$

Remember that $\sigma_i = \sqrt{\lambda_i}$ is the singular value, which is the length $\|A\mathbf{v}_i\|$.

The Important Equality

$$A[\mathbf{v}_1 \ \ \ldots \ \ \mathbf{v}_n] = [\sigma_1 \mathbf{u}_1 \ \ \ldots \sigma_n \mathbf{u}_n]$$

Remember that $\sigma_i = \sqrt{\lambda_i}$ is the singular value, which is the length $\|A\mathbf{v}_i\|$.

Let's take $V = [\mathbf{v}_1 \ \ \ldots \ \ \mathbf{v}_n]$ and $U = [\mathbf{u}_1 \ \ \ldots \ \ \mathbf{u}_m]$ and

The Important Equality

$$A[\mathbf{v}_1 \ \ \dots \ \ \mathbf{v}_n] = [\sigma_1\mathbf{u}_1 \ \ \dots\sigma_n\mathbf{u}_n]$$

Remember that $\sigma_i = \sqrt{\lambda_i}$ is the singular value, which is the length $\|A\mathbf{v}_i\|$.

Let's take $V = [\mathbf{v}_1 \ \dots \ \mathbf{v}_n]$ and $U = [\mathbf{u}_1 \ \dots \ \mathbf{u}_m]$ and

$$\overset{m>n}{\Sigma = \begin{bmatrix} \sigma_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_n \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix}} \text{ or } \overset{m<n}{\Sigma = \begin{bmatrix} \sigma_1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_m & 0 & \dots & 0 \end{bmatrix}} \text{ or } \overset{m=n}{\Sigma = \begin{bmatrix} \sigma_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_n \end{bmatrix}}$$

The Important Equality

$$A[\mathbf{v}_1 \ \dots \ \mathbf{v}_n] = [\sigma_1\mathbf{u}_1 \ \dots \sigma_n\mathbf{u}_n]$$

Remember that $\sigma_i = \sqrt{\lambda_i}$ is the singular value, which is the length $\|A\mathbf{v}_i\|$.

Let's take $V = [\mathbf{v}_1 \ \dots \ \mathbf{v}_n]$ and $U = [\mathbf{u}_1 \ \dots \ \mathbf{u}_m]$ and

remember: U is orthonormal

$m > n$

$$\Sigma = \begin{bmatrix} \sigma_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_n \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix}$$

or ($m < n$)

$$\Sigma = \begin{bmatrix} \sigma_1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_m & 0 & \dots & 0 \end{bmatrix}$$

or ($m = n$)

$$\Sigma = \begin{bmatrix} \sigma_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_n \end{bmatrix}$$

The Important Equality

$$AV = U\Sigma$$

Remember that $\sigma_i = \sqrt{\lambda_i}$ is the singular value, which is the length $\|A\mathbf{v}_i\|$.

Let's take $V = [\mathbf{v}_1 \; \dots \; \mathbf{v}_n]$ and $U = [\mathbf{u}_1 \; \dots \; \mathbf{u}_m]$ and

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$m < n$

or $$\Sigma = \begin{bmatrix} \sigma_1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_m & 0 & \dots & 0 \end{bmatrix}$$

$m = n$

or $$\Sigma = \begin{bmatrix} \sigma_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_n \end{bmatrix}$$

The Important Equality

$$\overset{m\times n}{A}\underset{n\times n}{V} = \overset{m\times m}{U}\underset{m\times n}{\Sigma}$$

Remember that $\sigma_i = \sqrt{\lambda_i}$ is the singular value, which is the length $\|A\mathbf{v}_i\|$.

Let's take $V = [\mathbf{v}_1 \ \dots \ \mathbf{v}_n]$ and $U = [\mathbf{u}_1 \ \dots \ \mathbf{u}_m]$ and

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The Important Equality

$$AVV^T = U\Sigma V^T$$

Remember that $\sigma_i = \sqrt{\lambda_i}$ is the singular value, which is the length $\|A\mathbf{v}_i\|$.

Let's take $V = [\mathbf{v}_1 \ \dots \ \mathbf{v}_n]$ and $U = [\mathbf{u}_1 \ \dots \ \mathbf{u}_m]$ and

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The Important Equality

$$A = U\Sigma V^T$$

Remember that $\sigma_i = \sqrt{\lambda_i}$ is the singular value, which is the length $\|A\mathbf{v}_i\|$.

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$m > n$ $\quad$ $m < n$ $\quad$ $m = n$

$$\Sigma = \begin{bmatrix} \sigma_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_n \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix} \text{ or } \Sigma = \begin{bmatrix} \sigma_1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_m & 0 & \dots & 0 \end{bmatrix} \text{ or } \Sigma = \begin{bmatrix} \sigma_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_n \end{bmatrix}$$

The Important Equality

singular value decomposition

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Singular Value Decomposition

Theorem. For a $m \times n$ matrix A, there are *orthogonal* matrices $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ such that

$$A = \overset{m \times m}{\boxed{U}} \underset{m \times n}{\Sigma} \overset{n \times n}{\boxed{V^T}}$$

where diagonal entries* of Σ are $\sigma_1, \ldots, \sigma_n$ the singular values of A.

* these are diagonal entries in a <u>non-square</u> matrix.

Singular Value Decomposition

Theorem. For a $m \times n$ matrix A, there are *orthogonal* matrices $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ such that

left singular vectors

right singular vectors

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where diagonal entries* of Σ are $\sigma_1, \ldots, \sigma_n$ the singular values of A.

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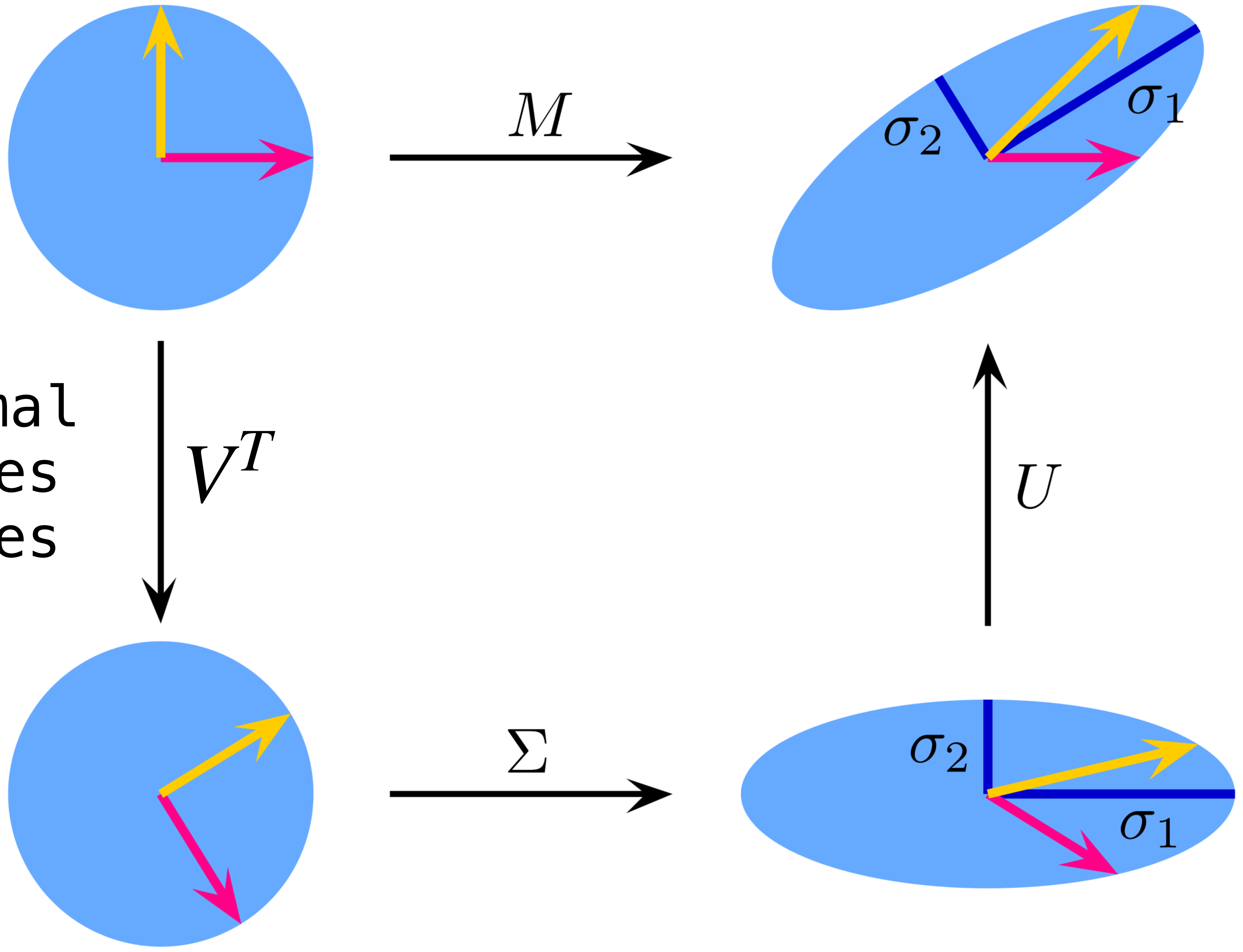
The Picture (Again)



<u>recall</u>: Orthonormal matrices preserves lengths and angles

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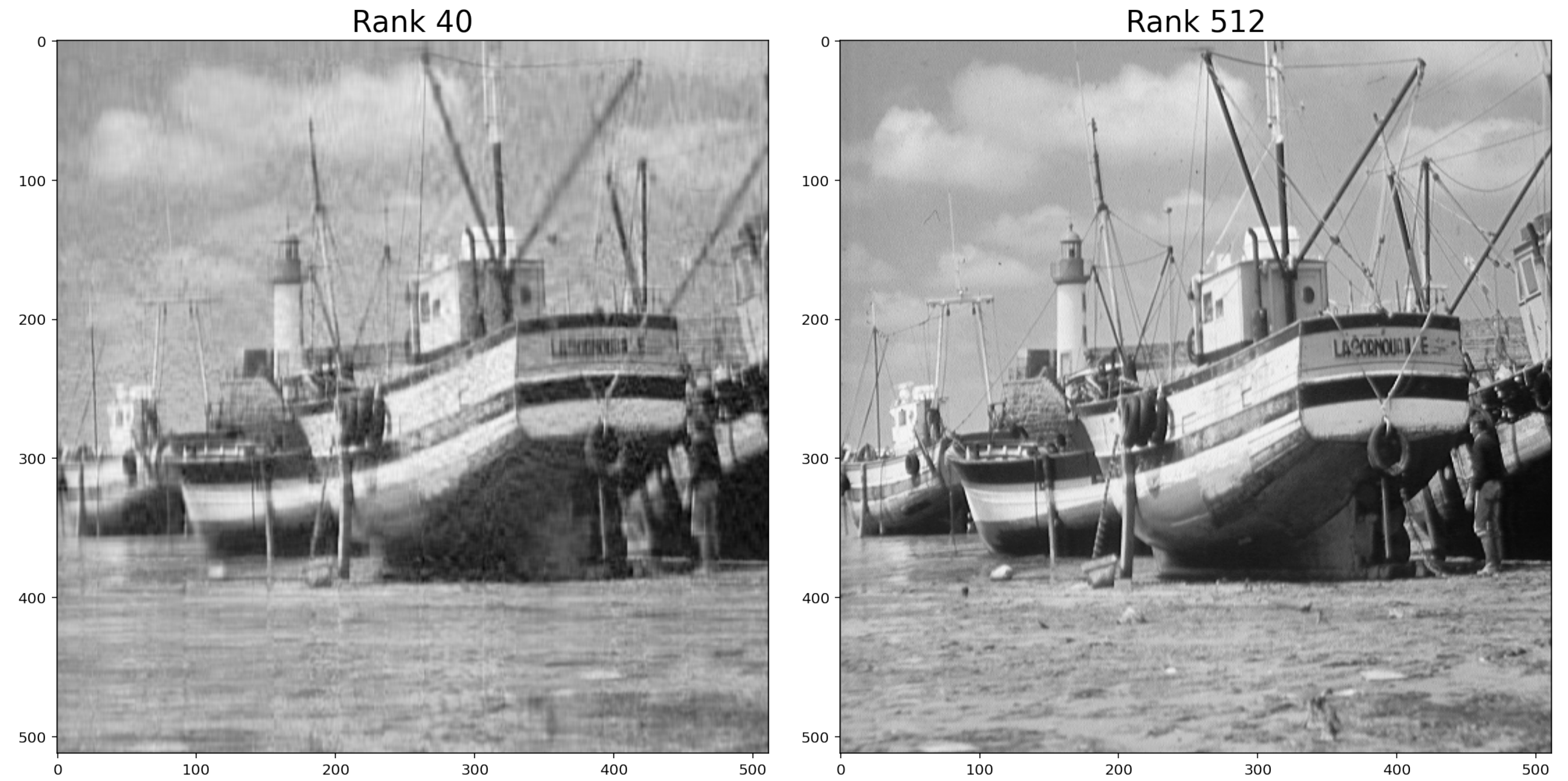
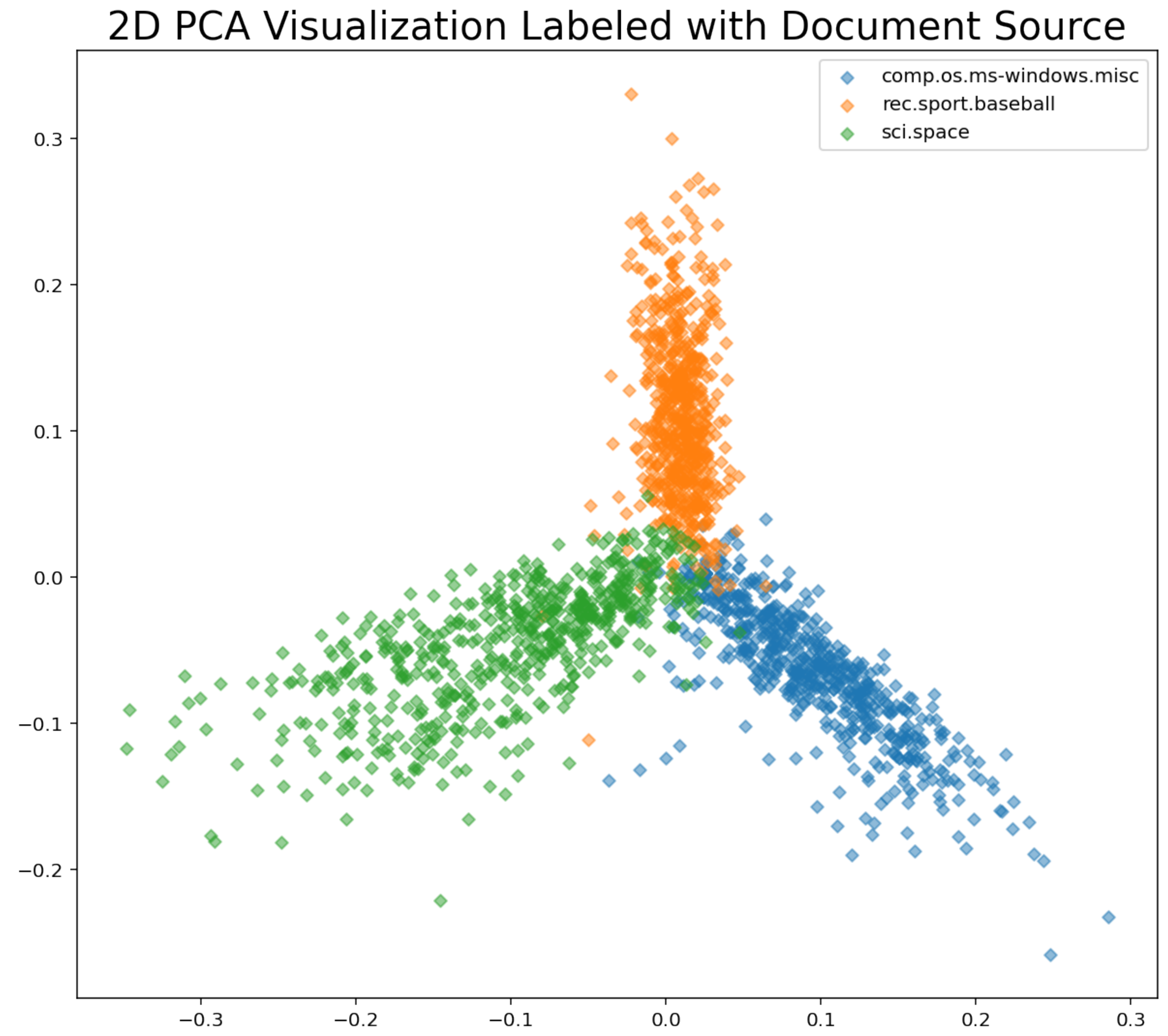
https://commons.wikimedia.org/wiki/File:Singular-Value-Decomposition.svg

What's next?
A couple final thoughts

Applications of SVD

image compression



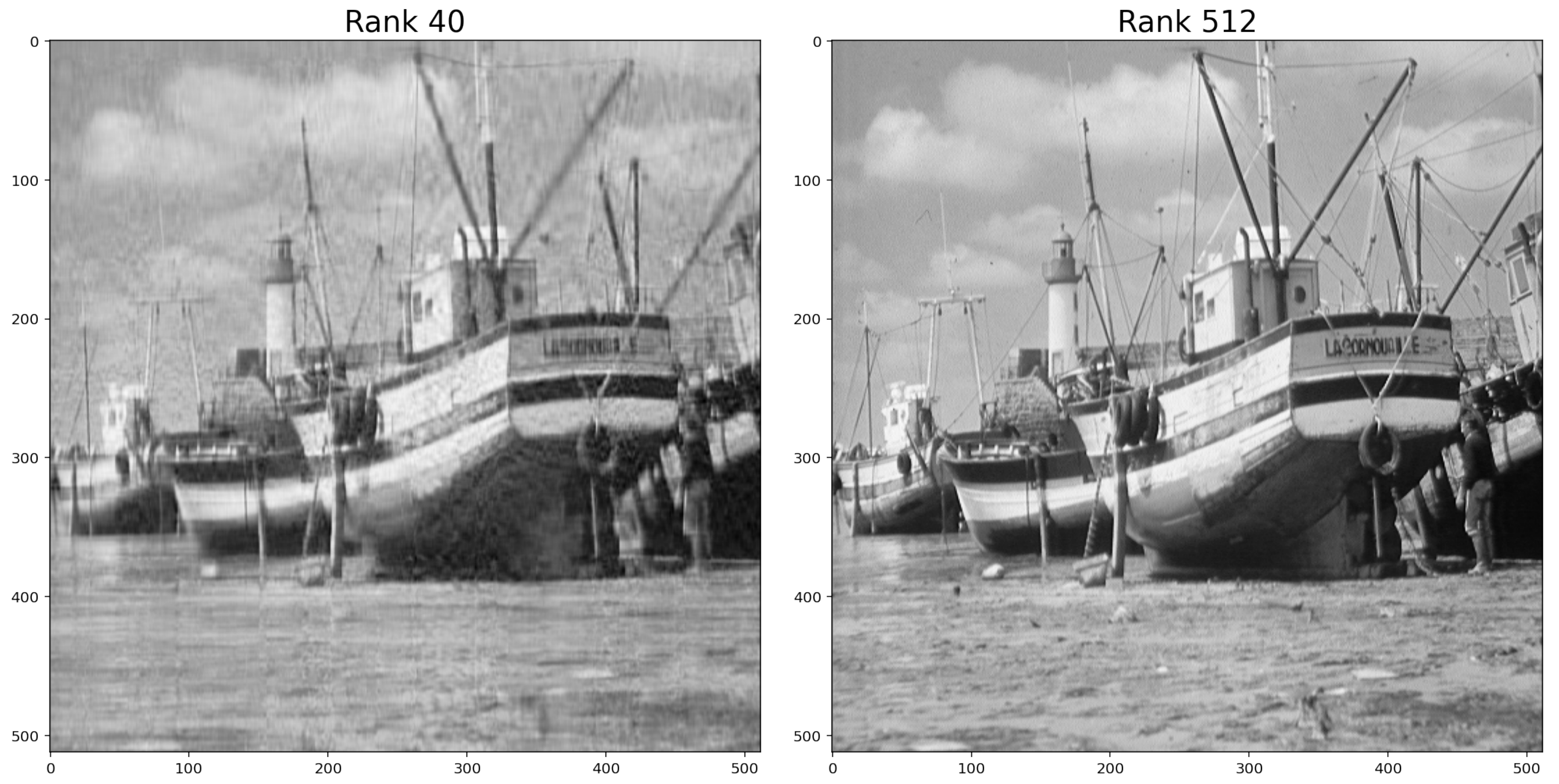


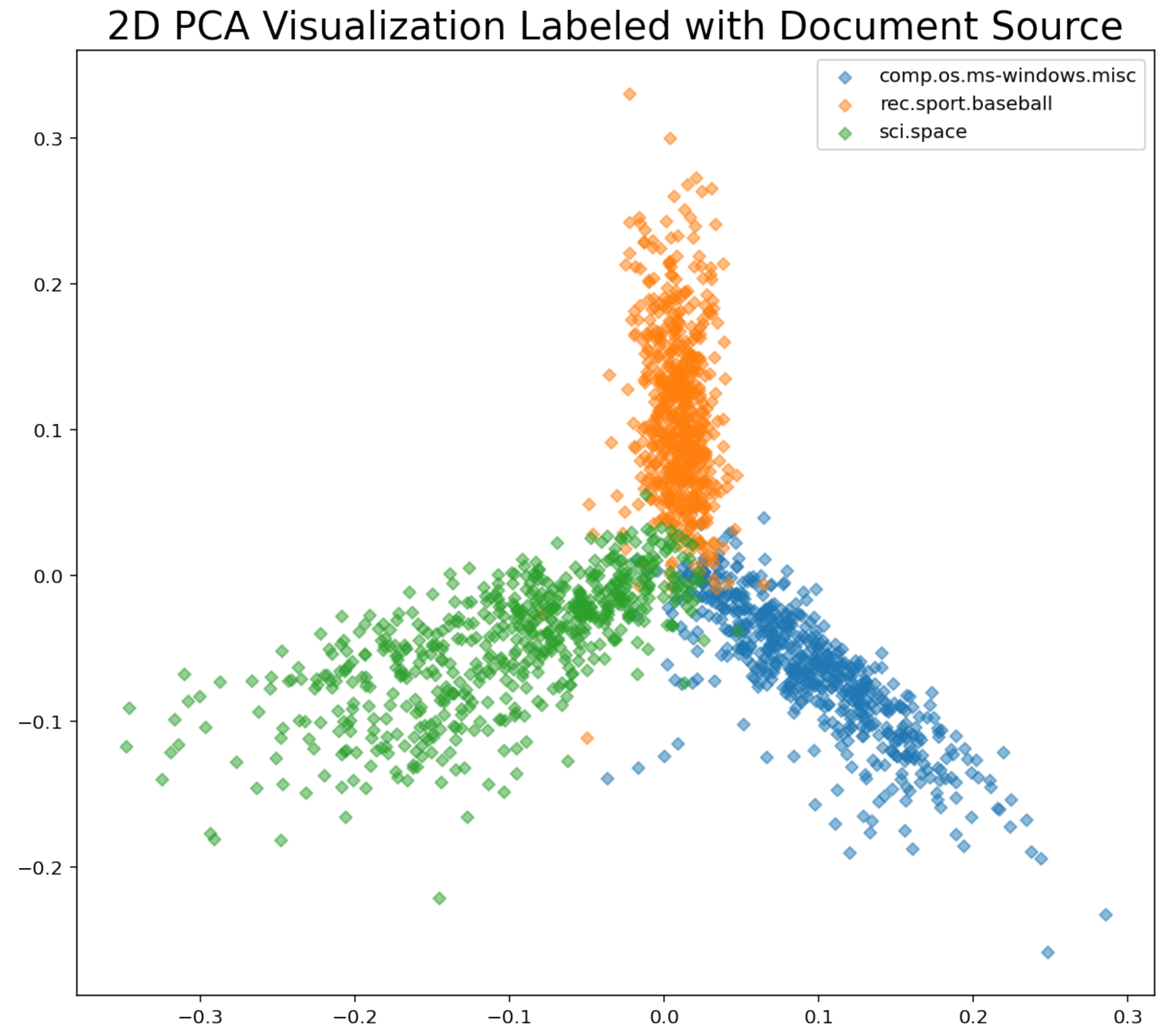
document classification

Applications of SVD

- Reduced SVD, pseudoinverses and least squares

image compression





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Applications of SVD

- Reduced SVD, pseudoinverses and least squares
  - If $A^+ = V\Sigma^{-1}U^T$, then $A^+\mathbf{b}$ is a least squares solution of minimum length

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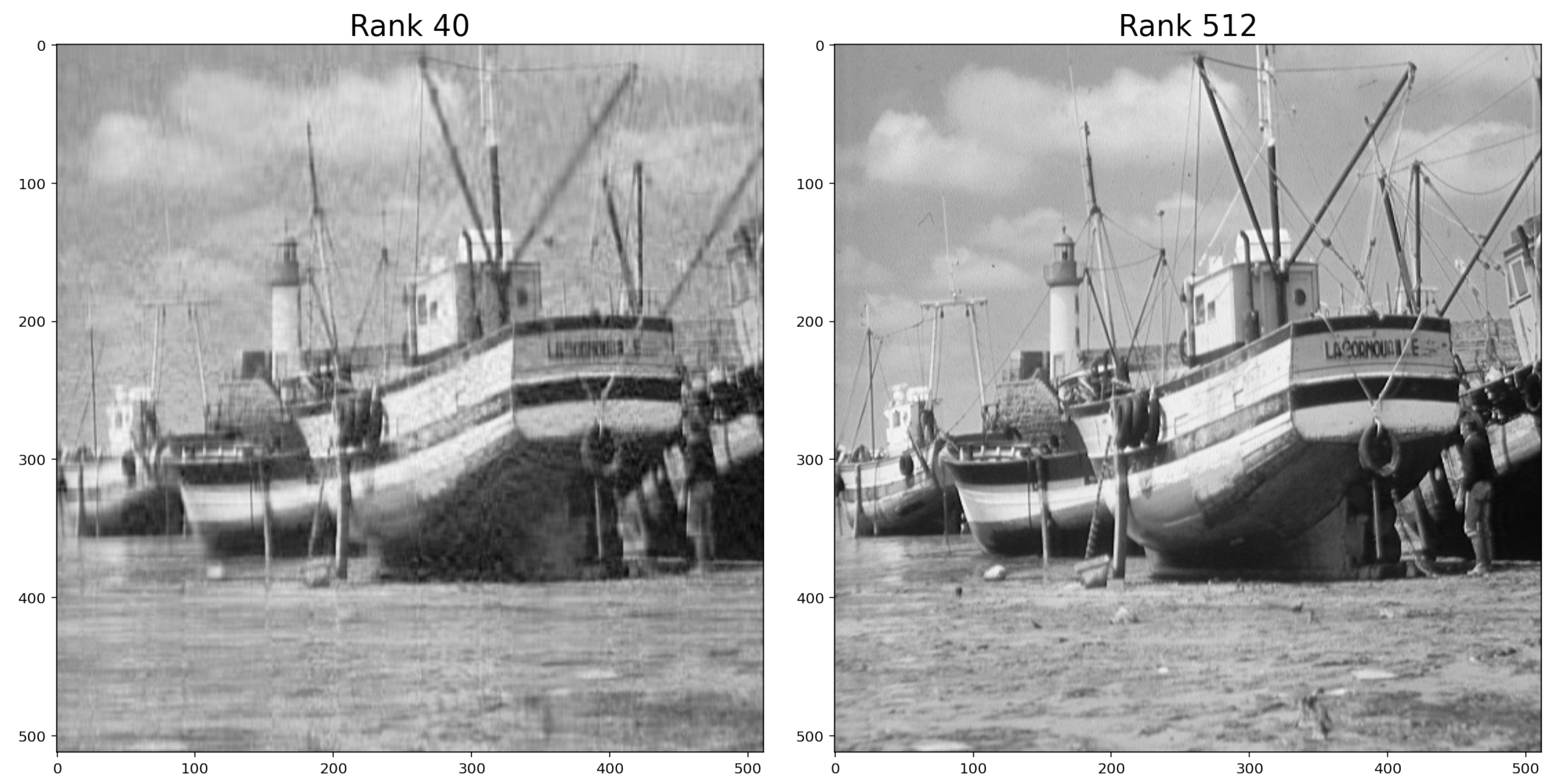

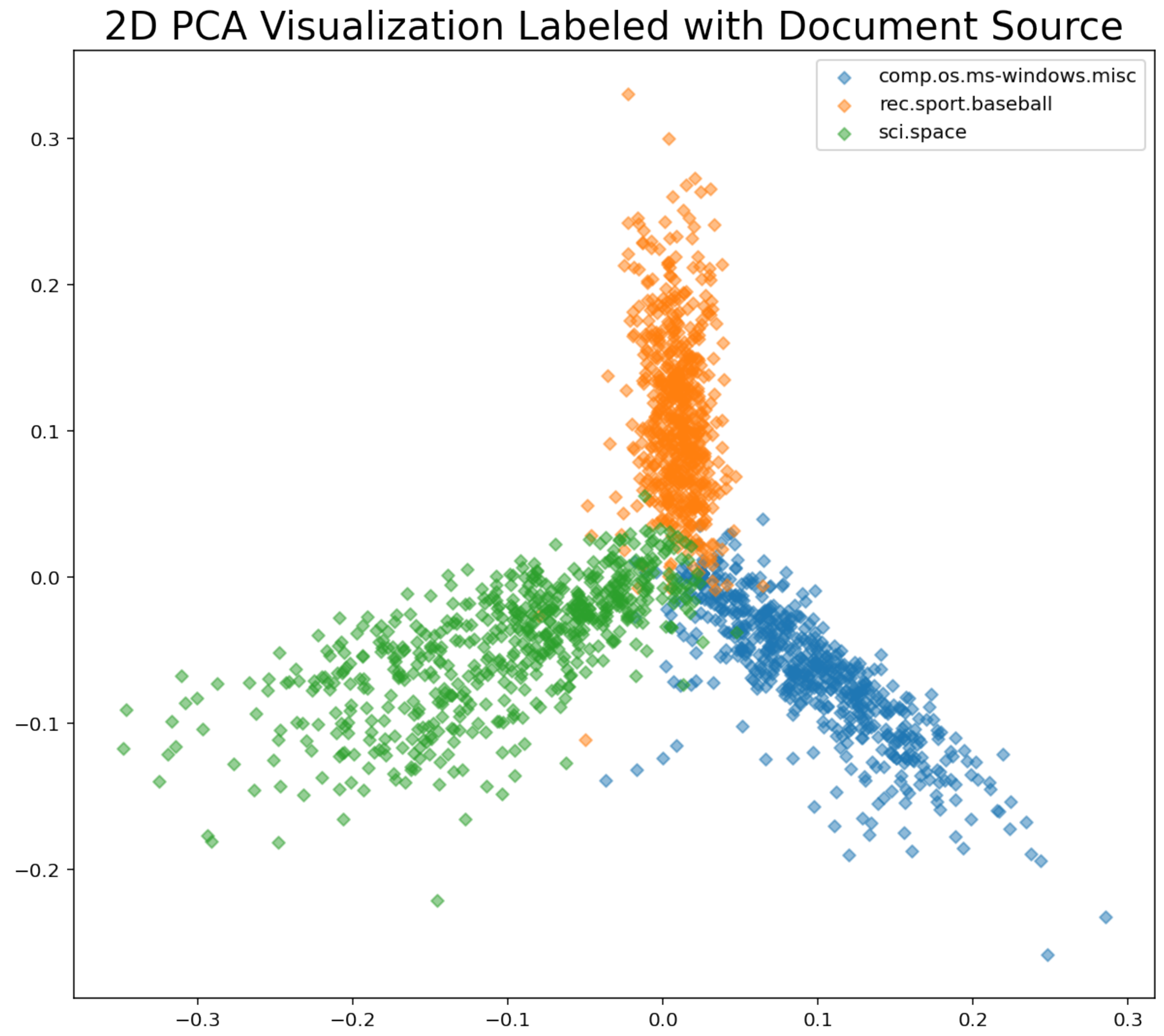

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Applications of SVD

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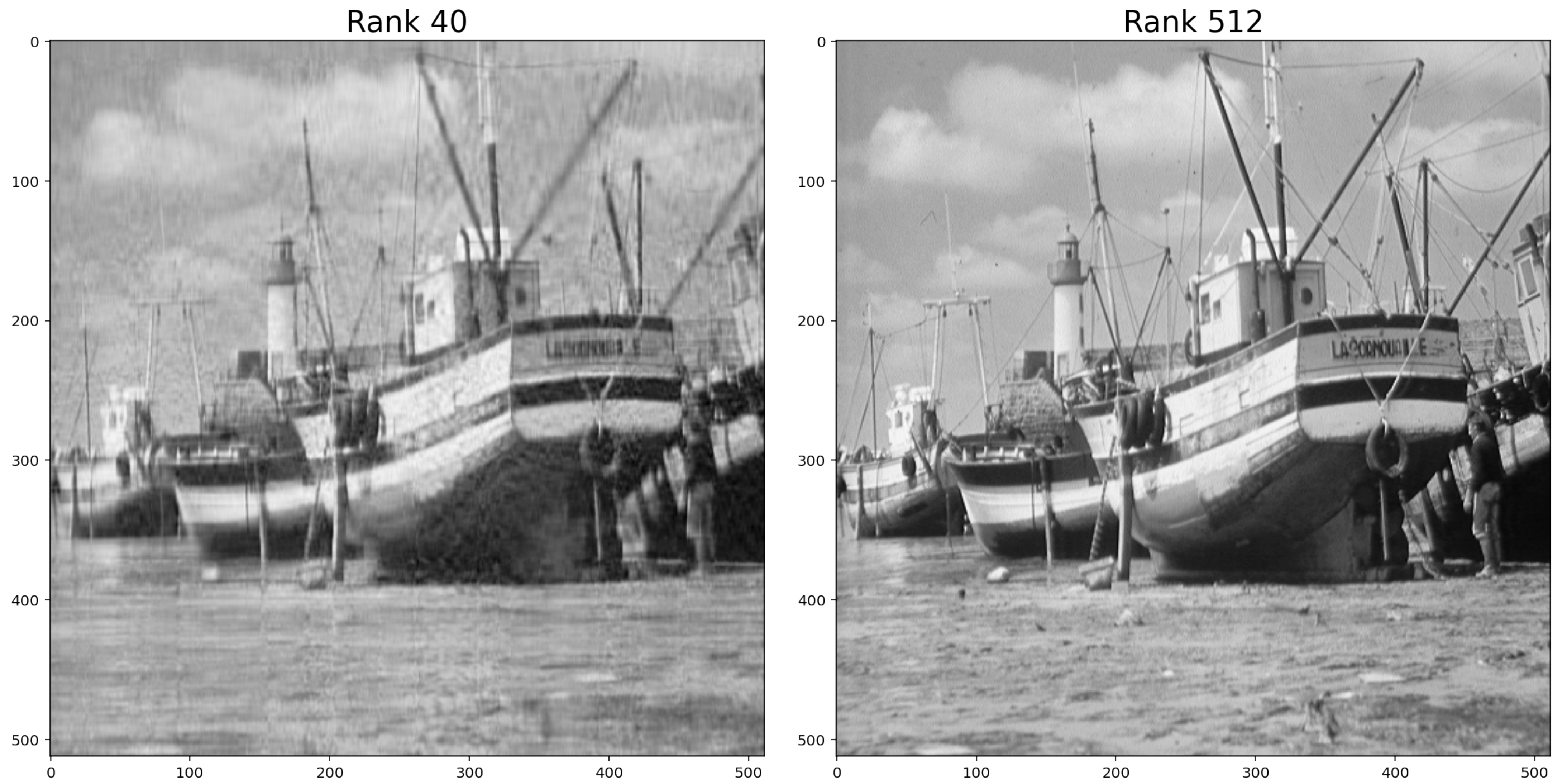

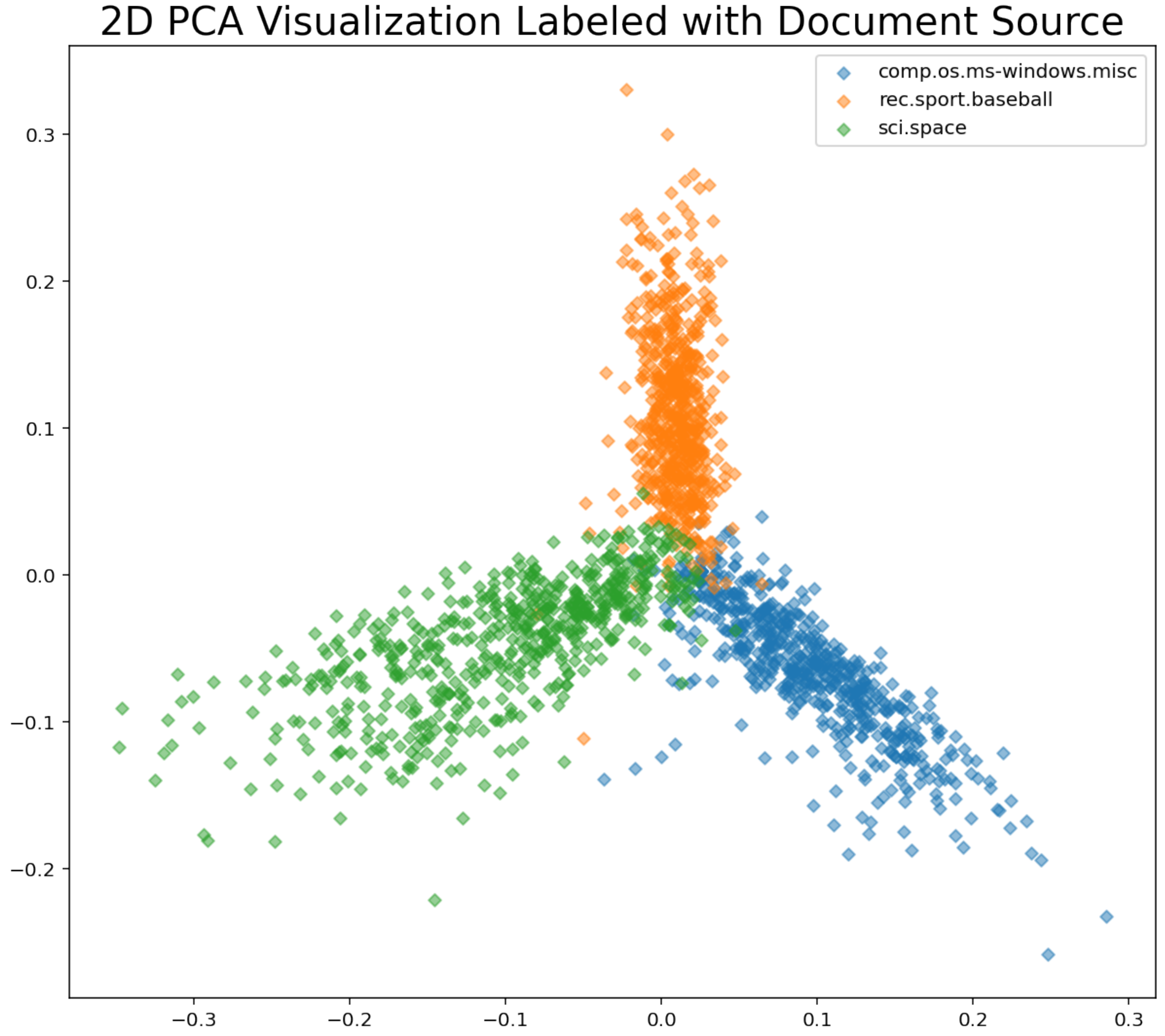

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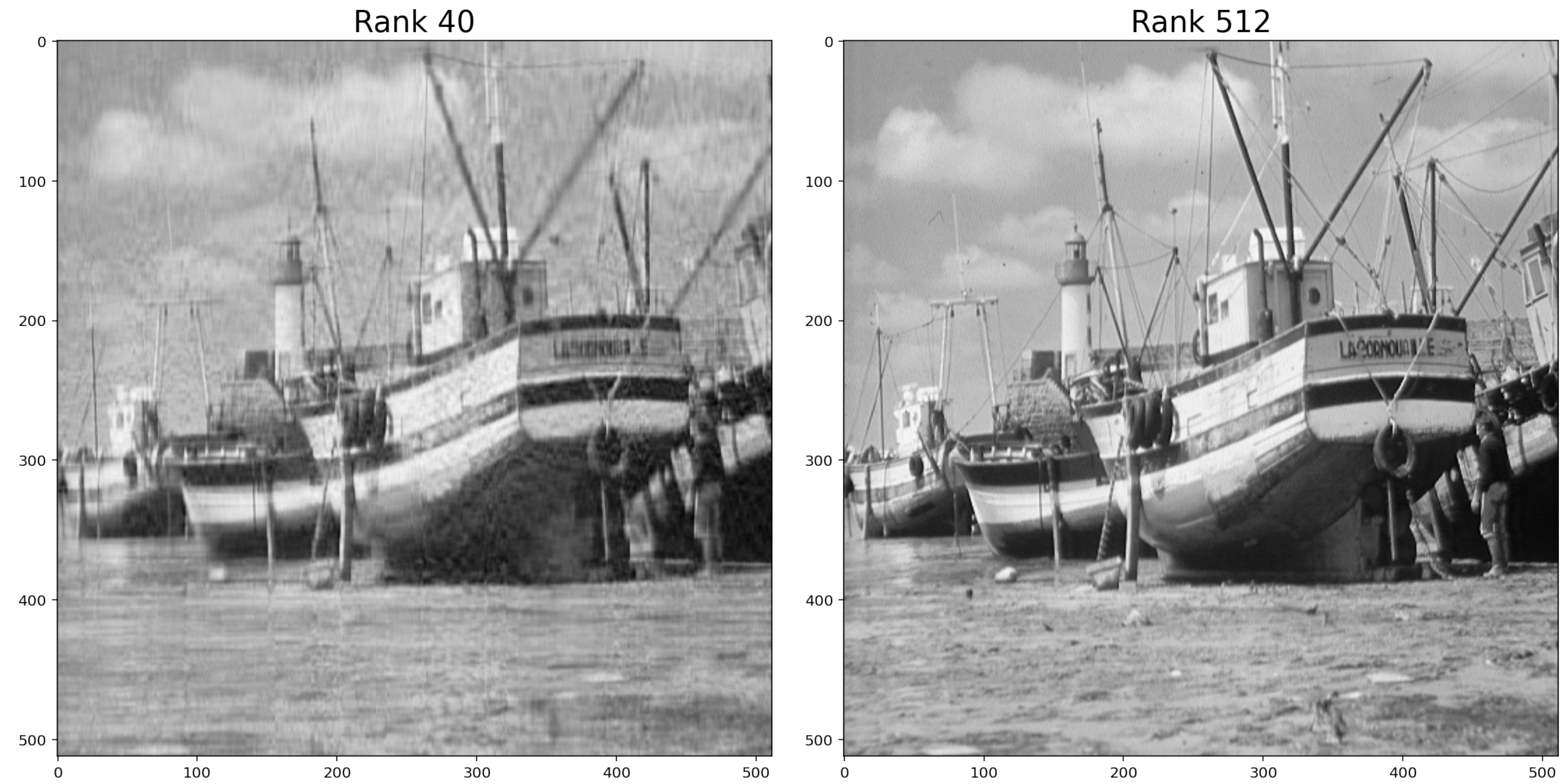

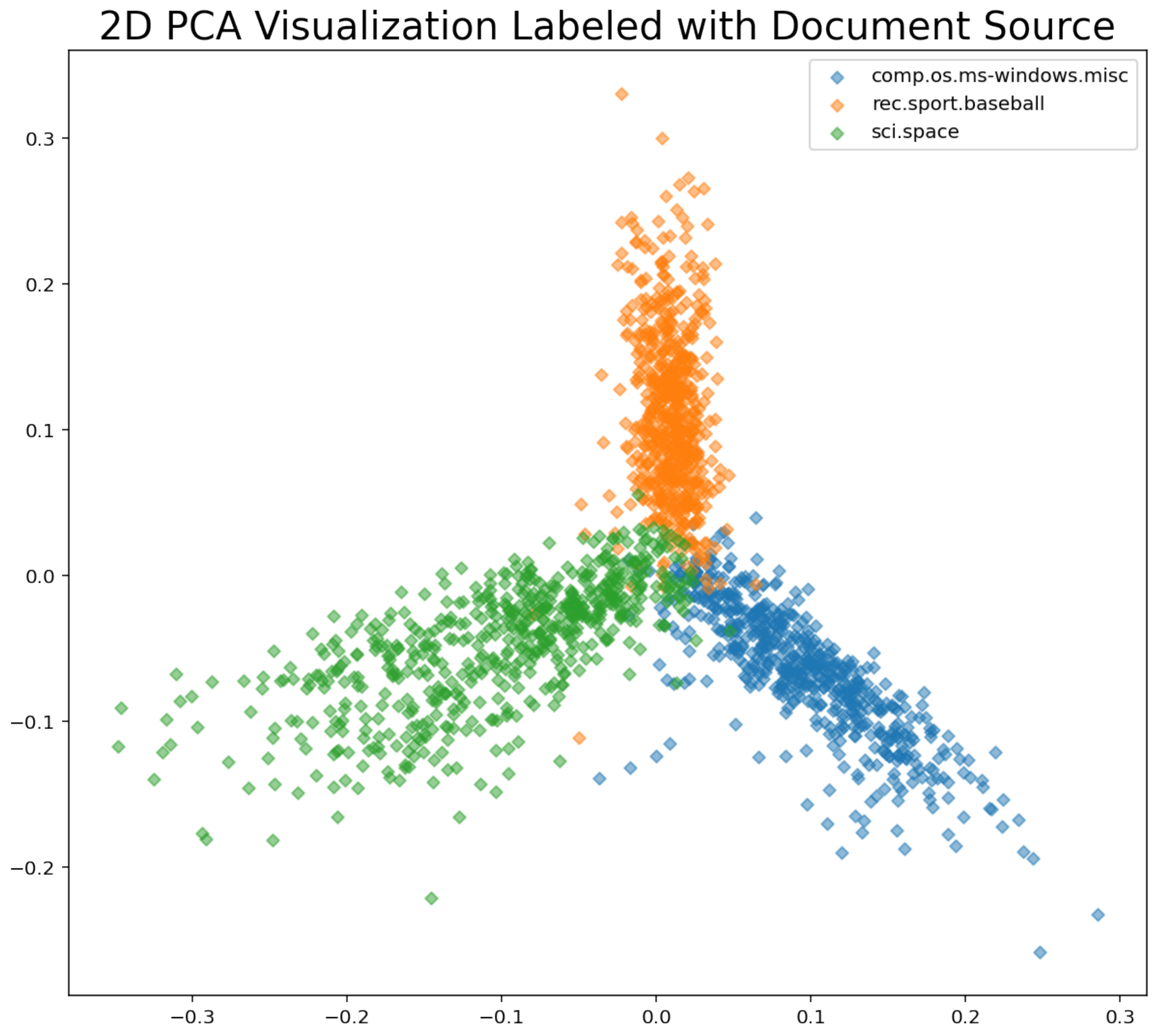

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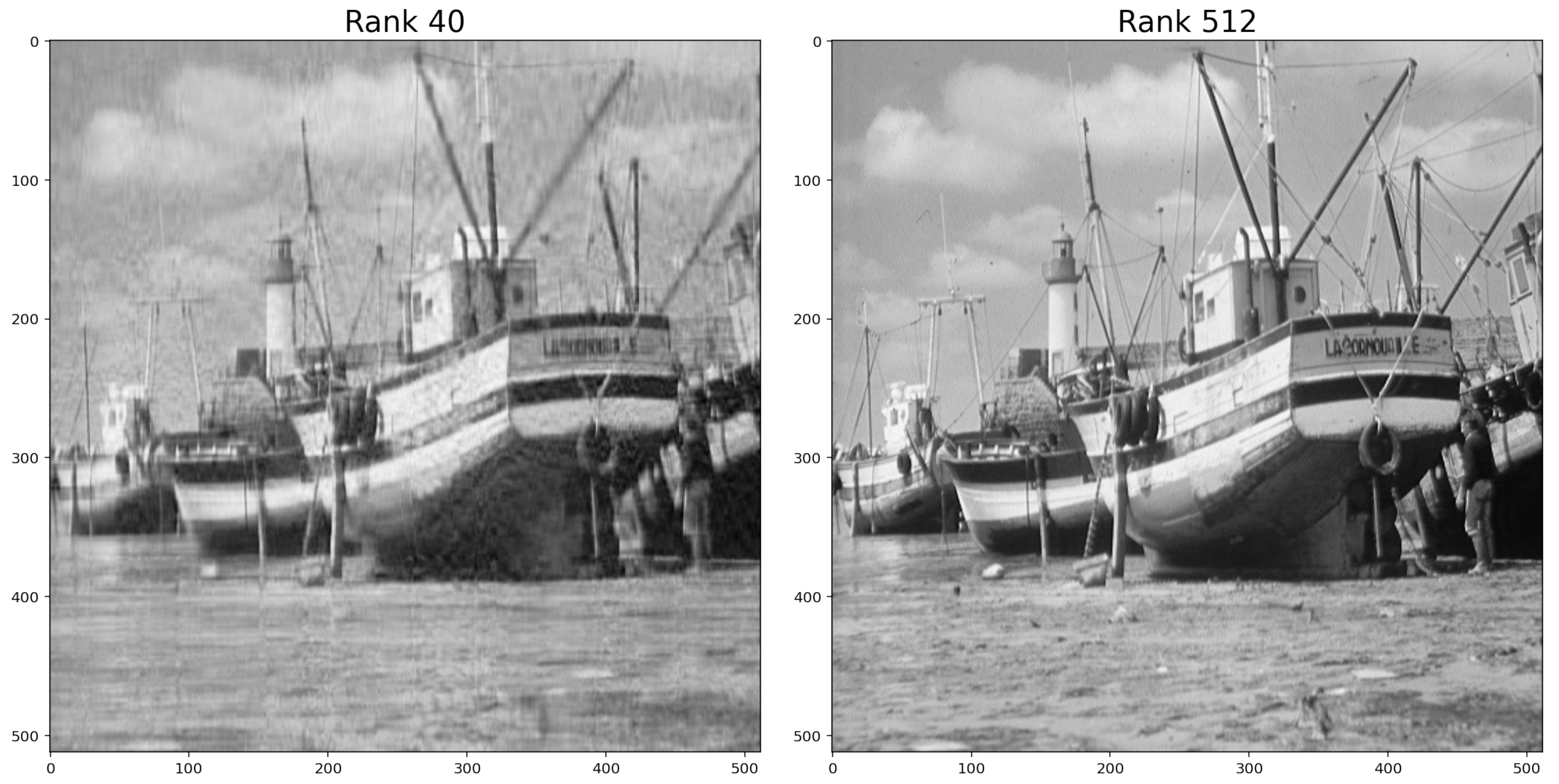
image compression



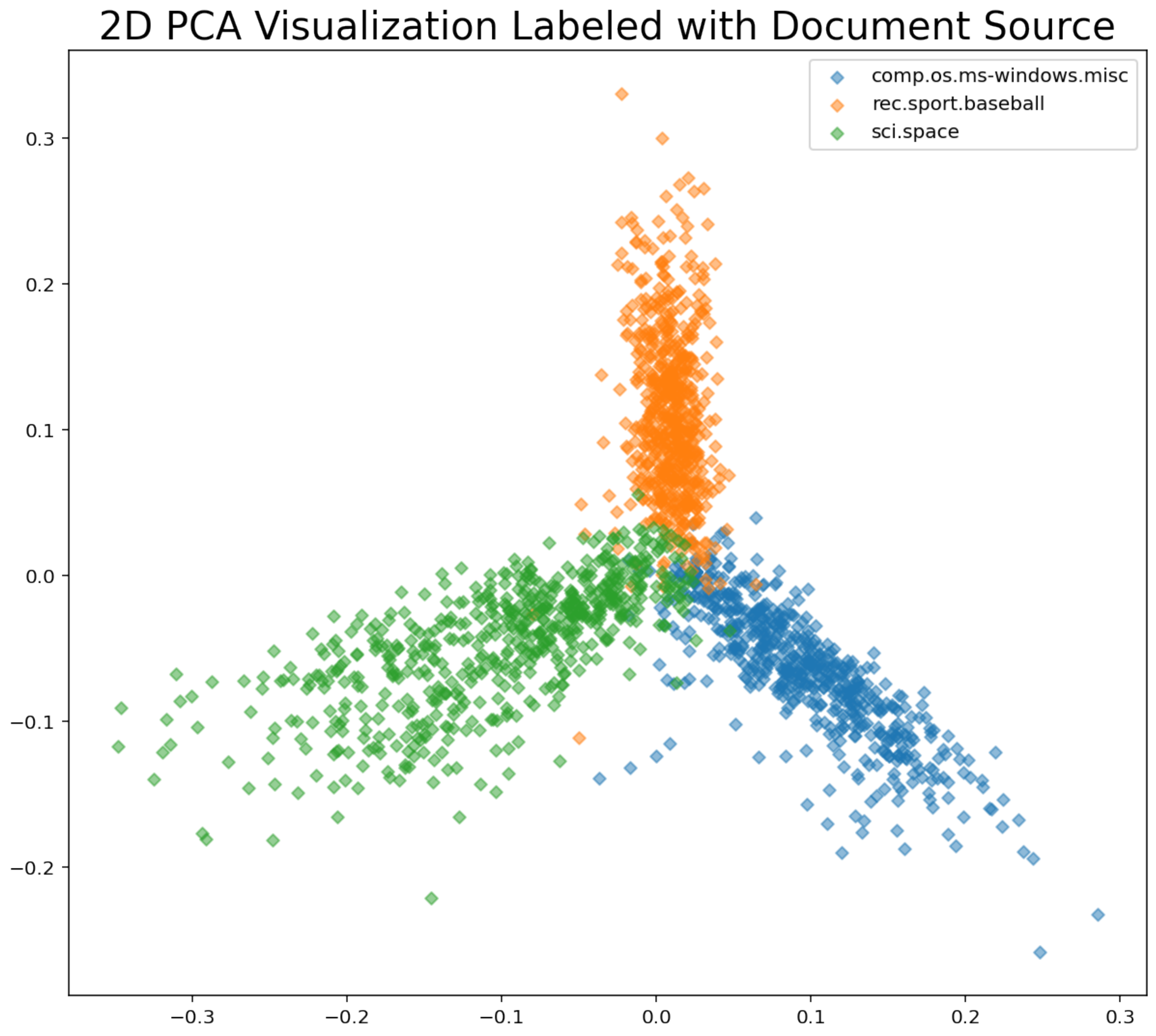


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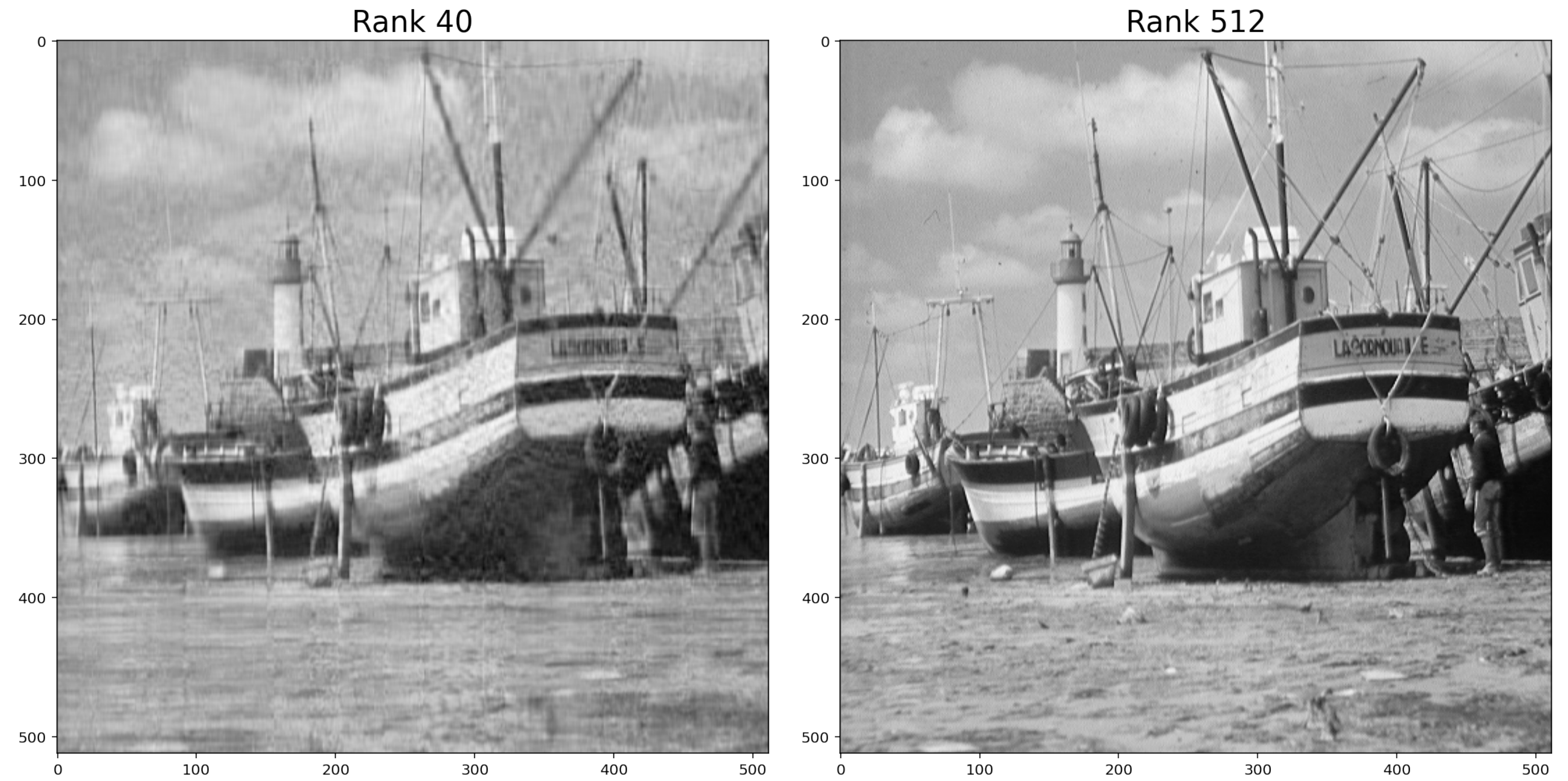
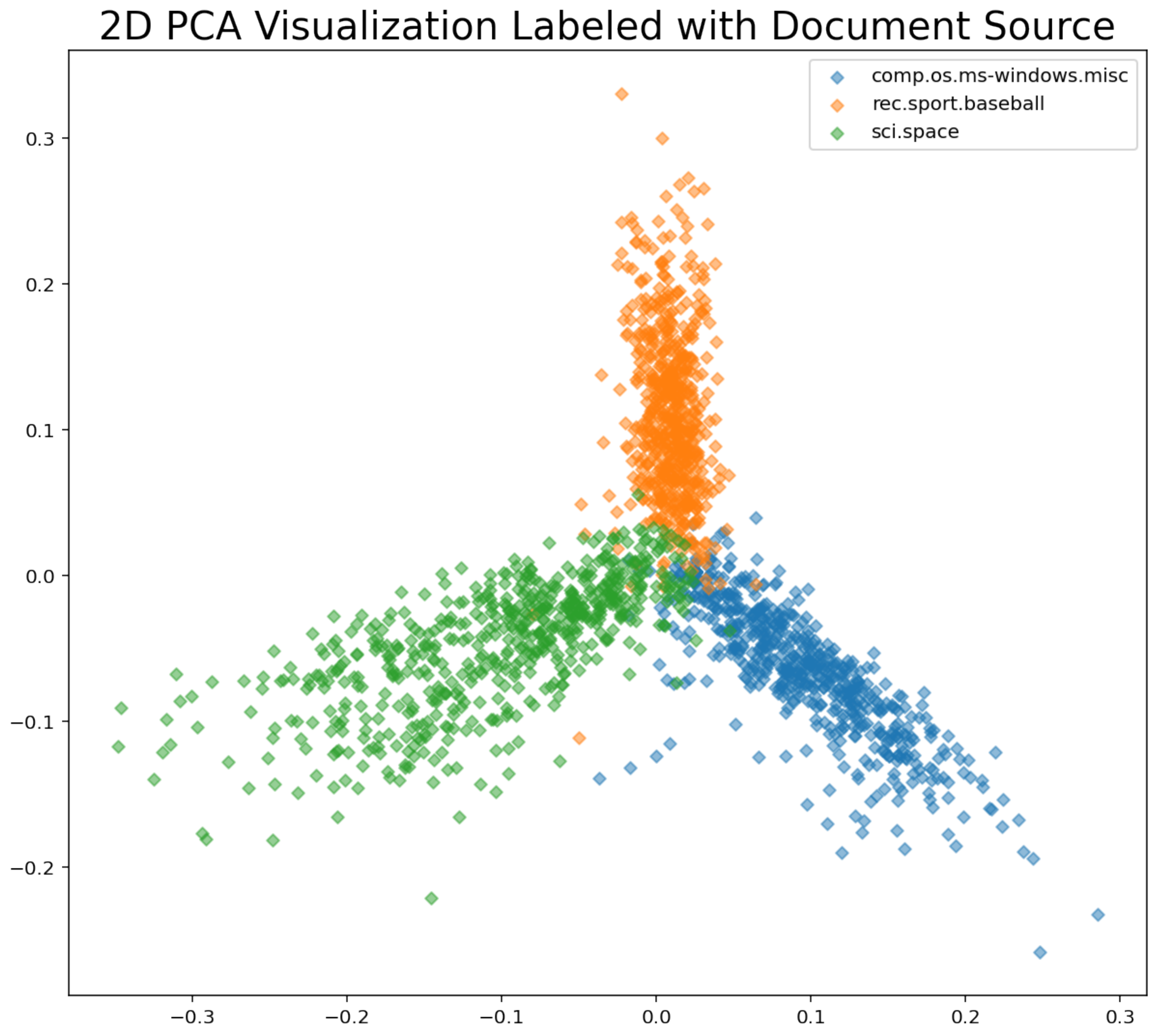
image compression





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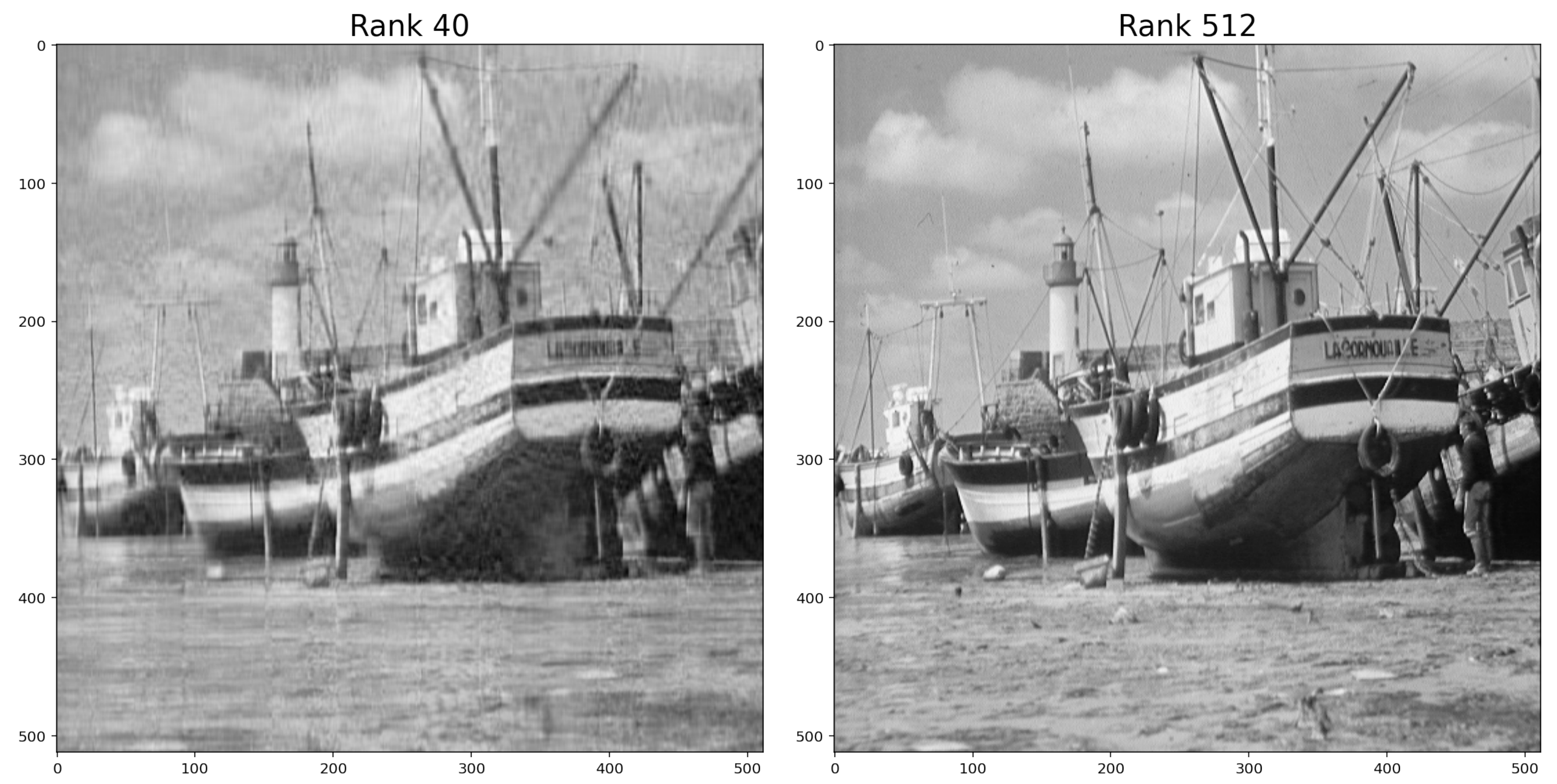
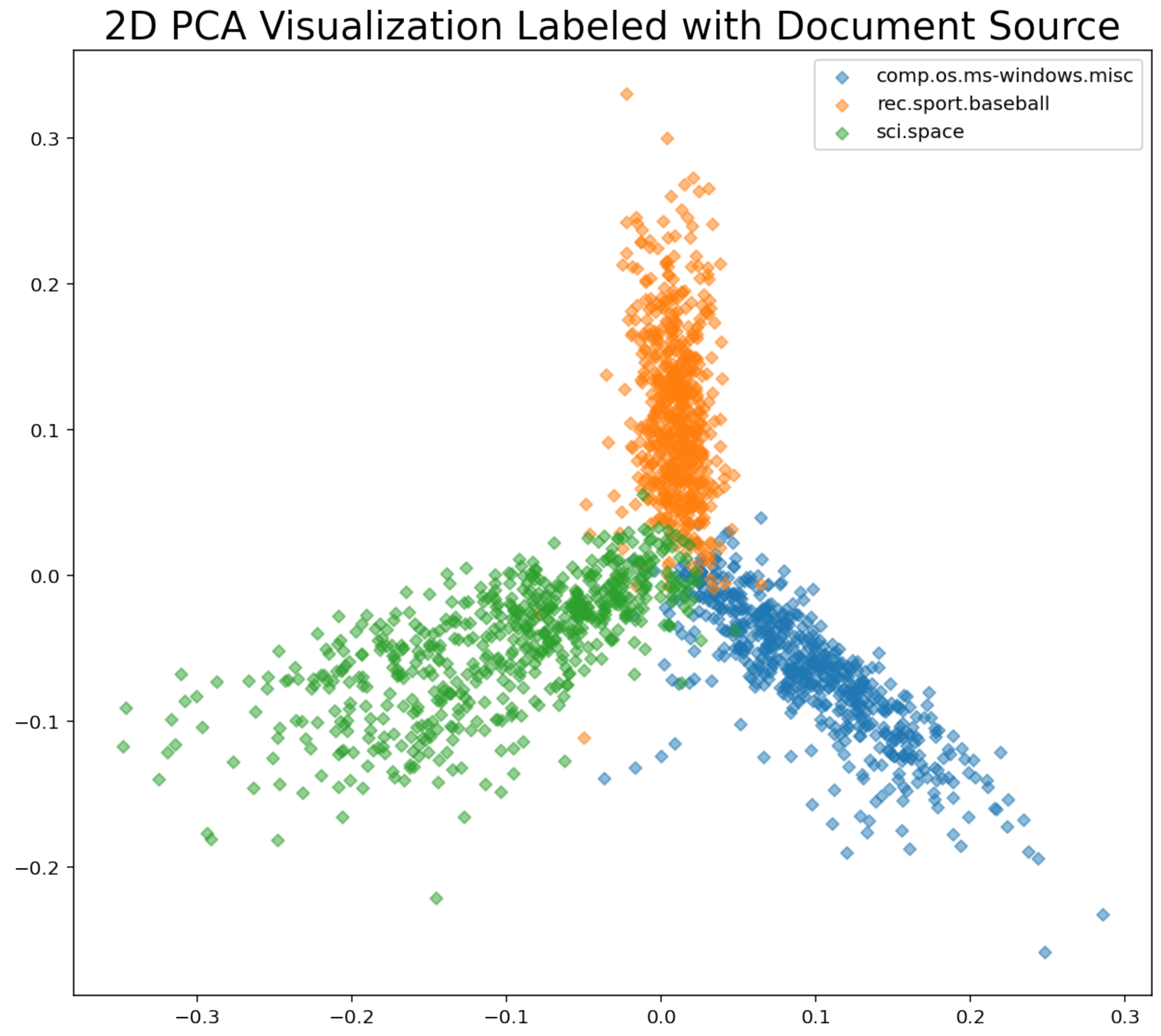
- Principle Component Analysis
  - Large singular vectors are "most affected."

image compression





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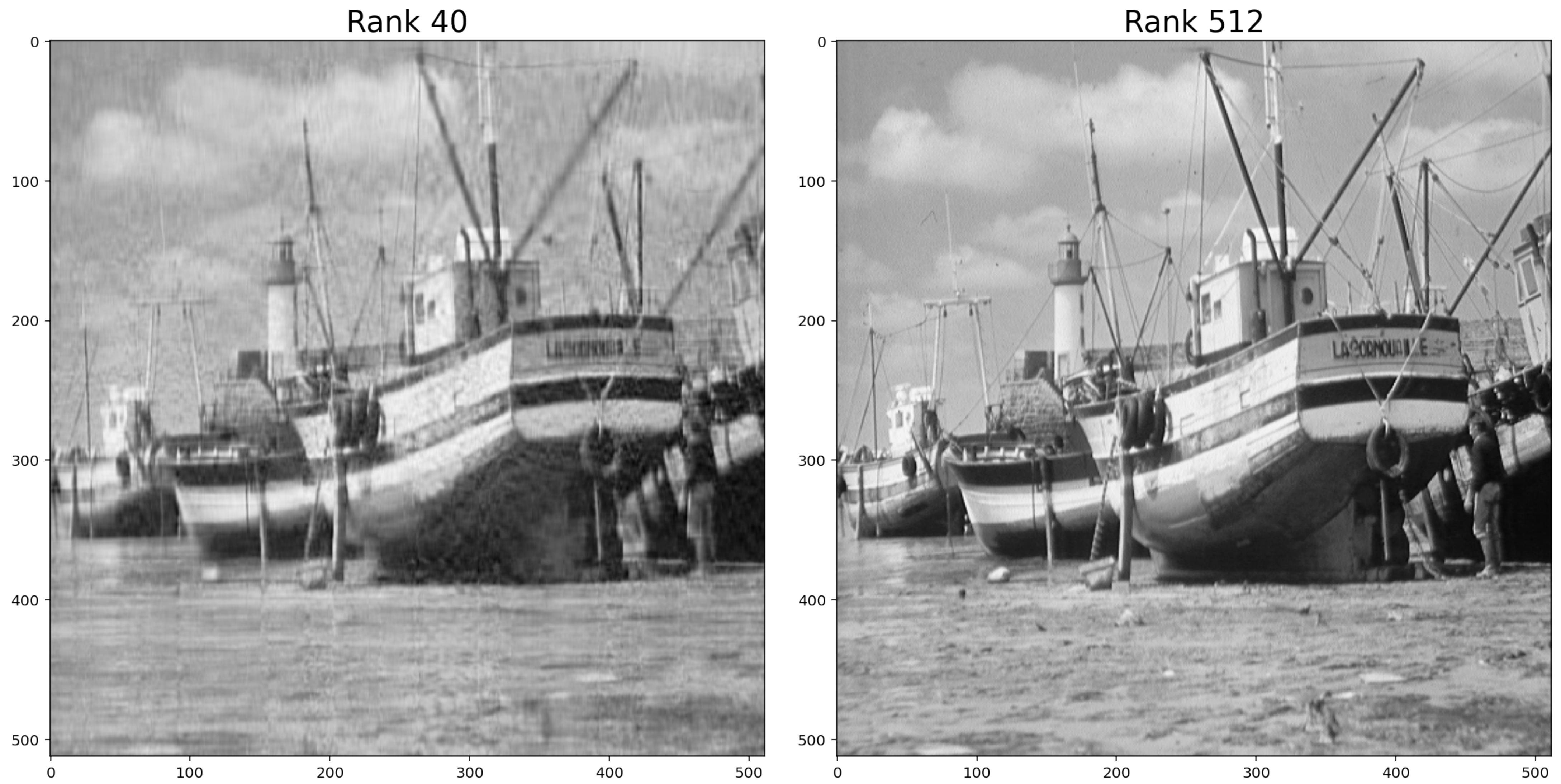
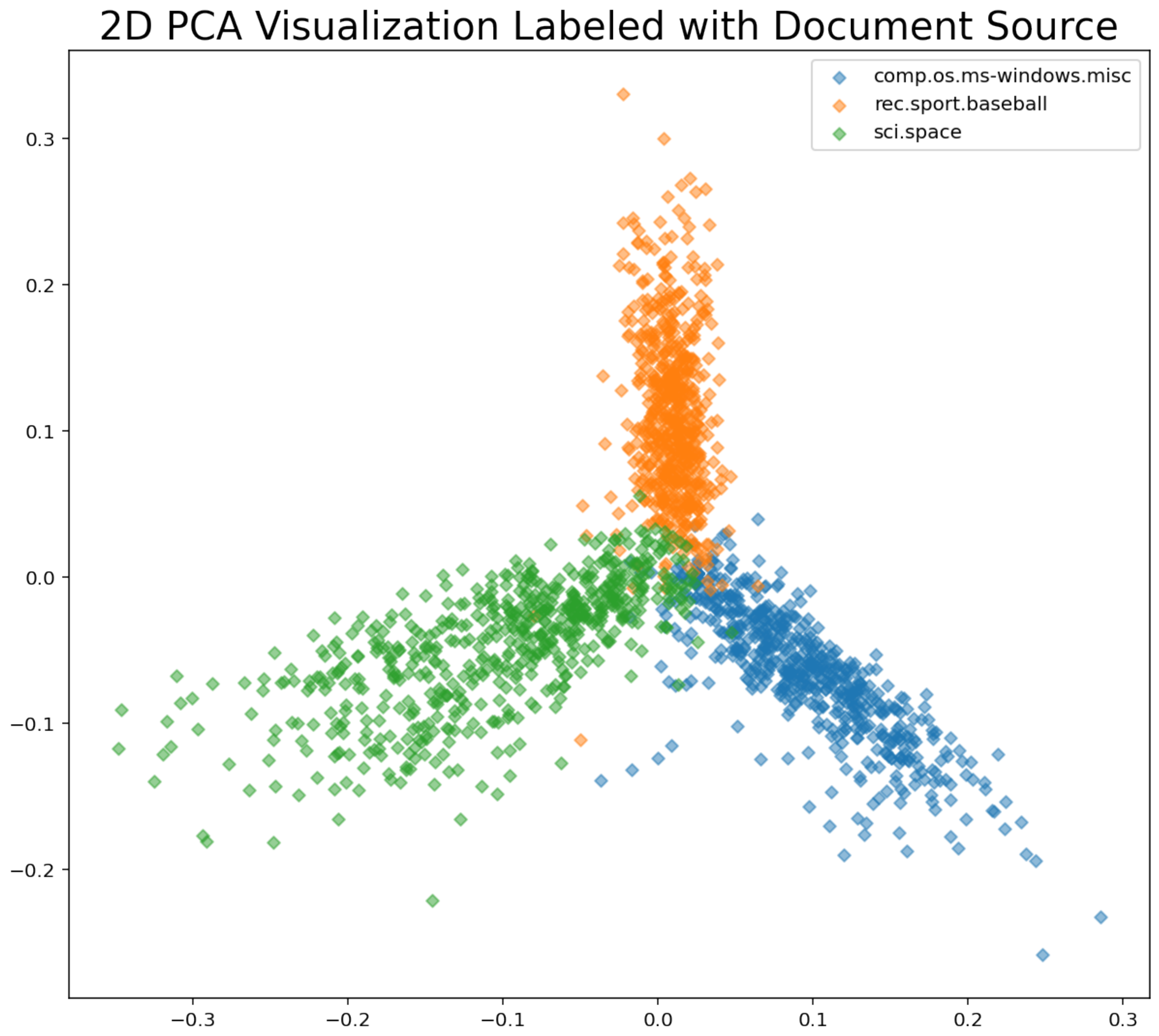
- Principle Component Analysis
  - Large singular vectors are "most affected."
  - These are good vectors to look at for classifying data

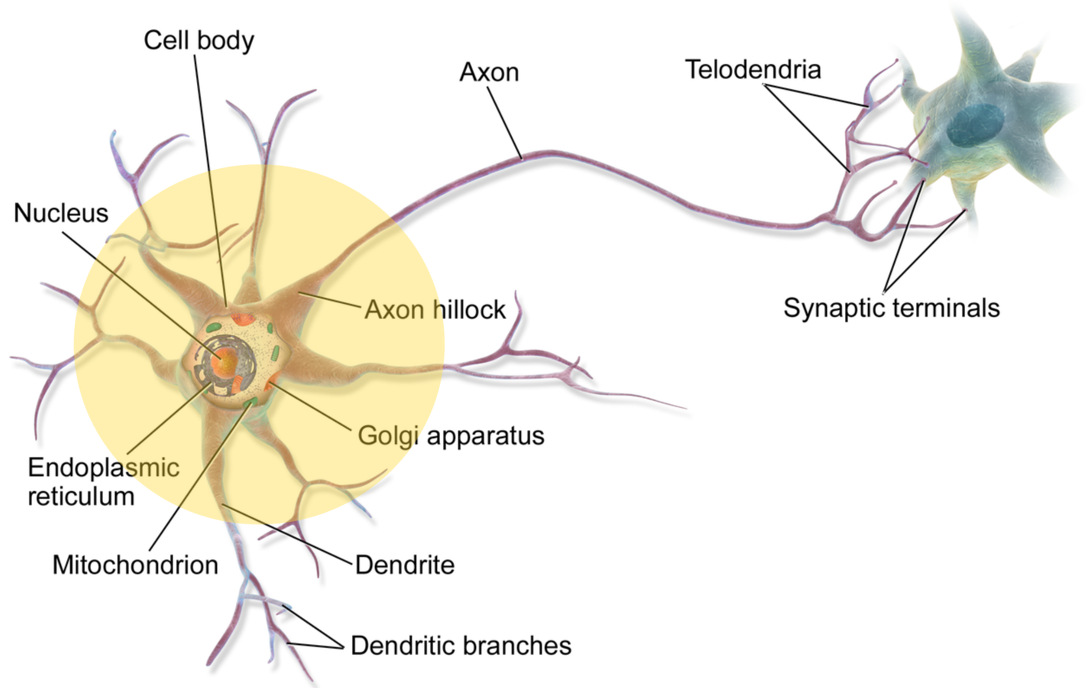
image compression

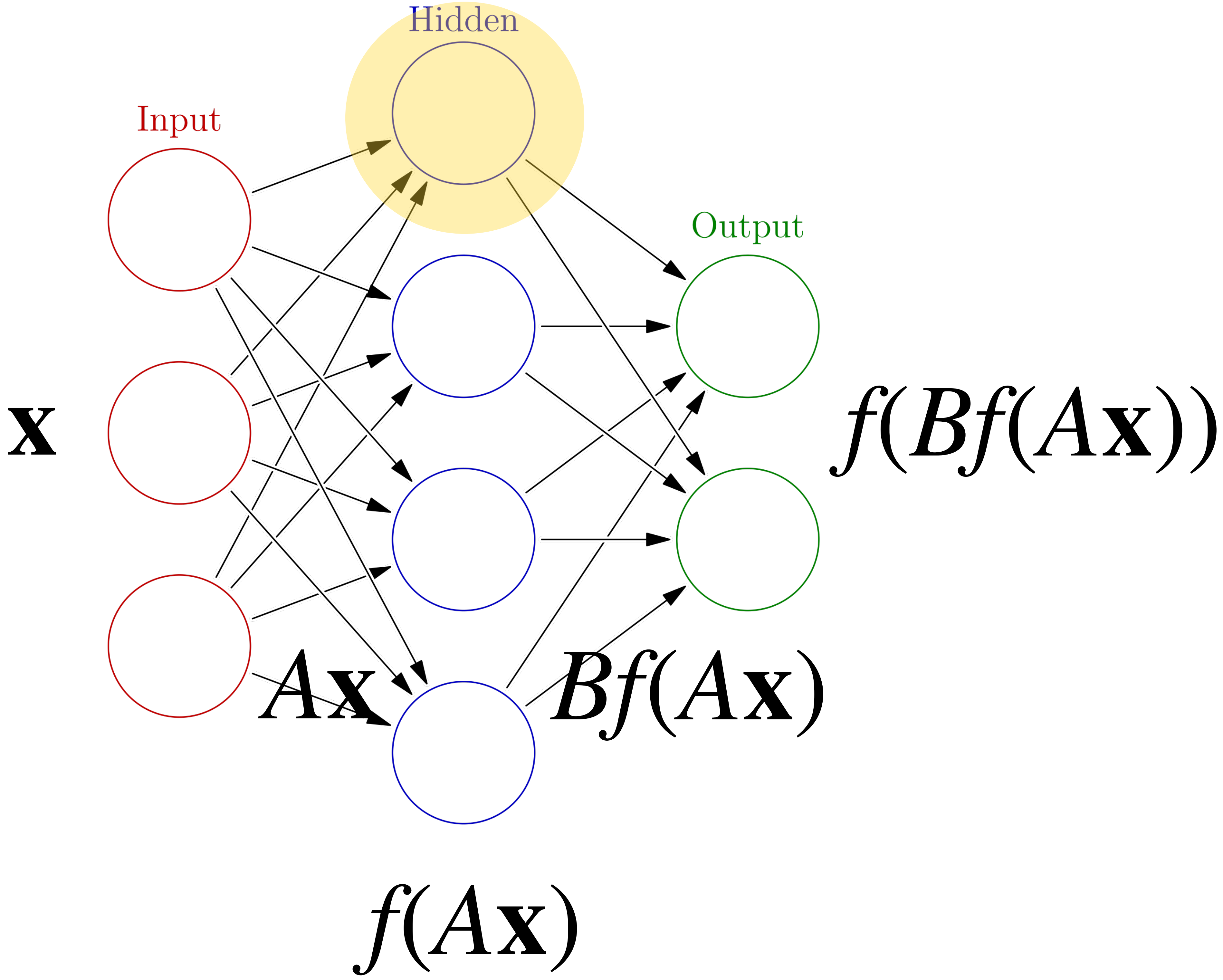




document classification

Neural Networks (Non-Linearity)

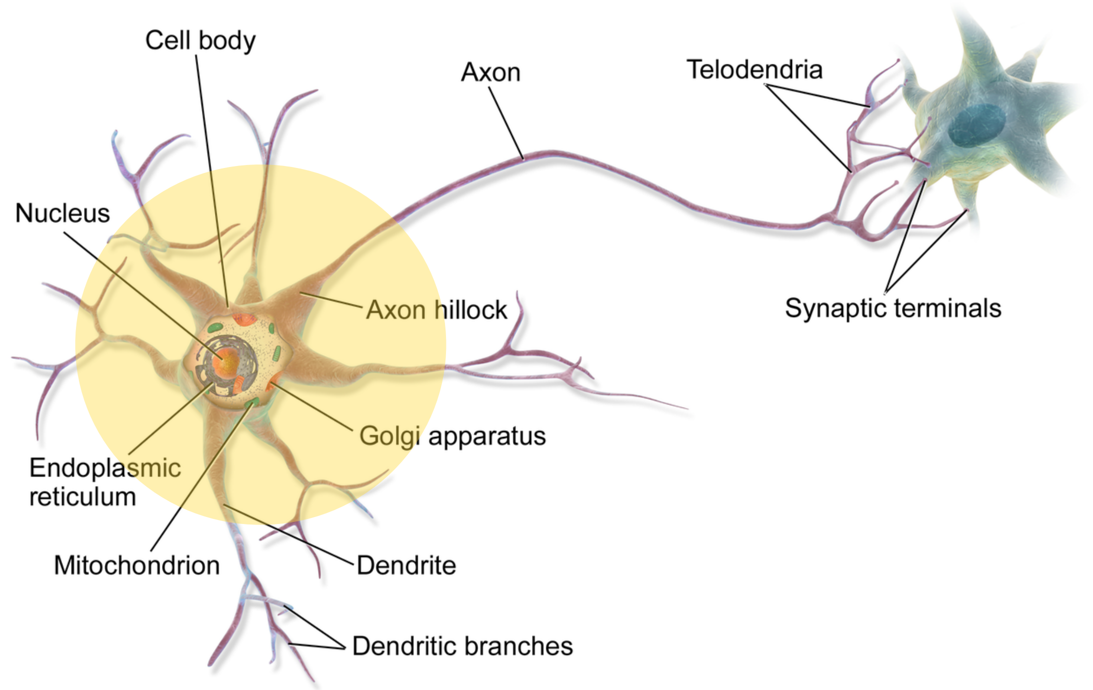


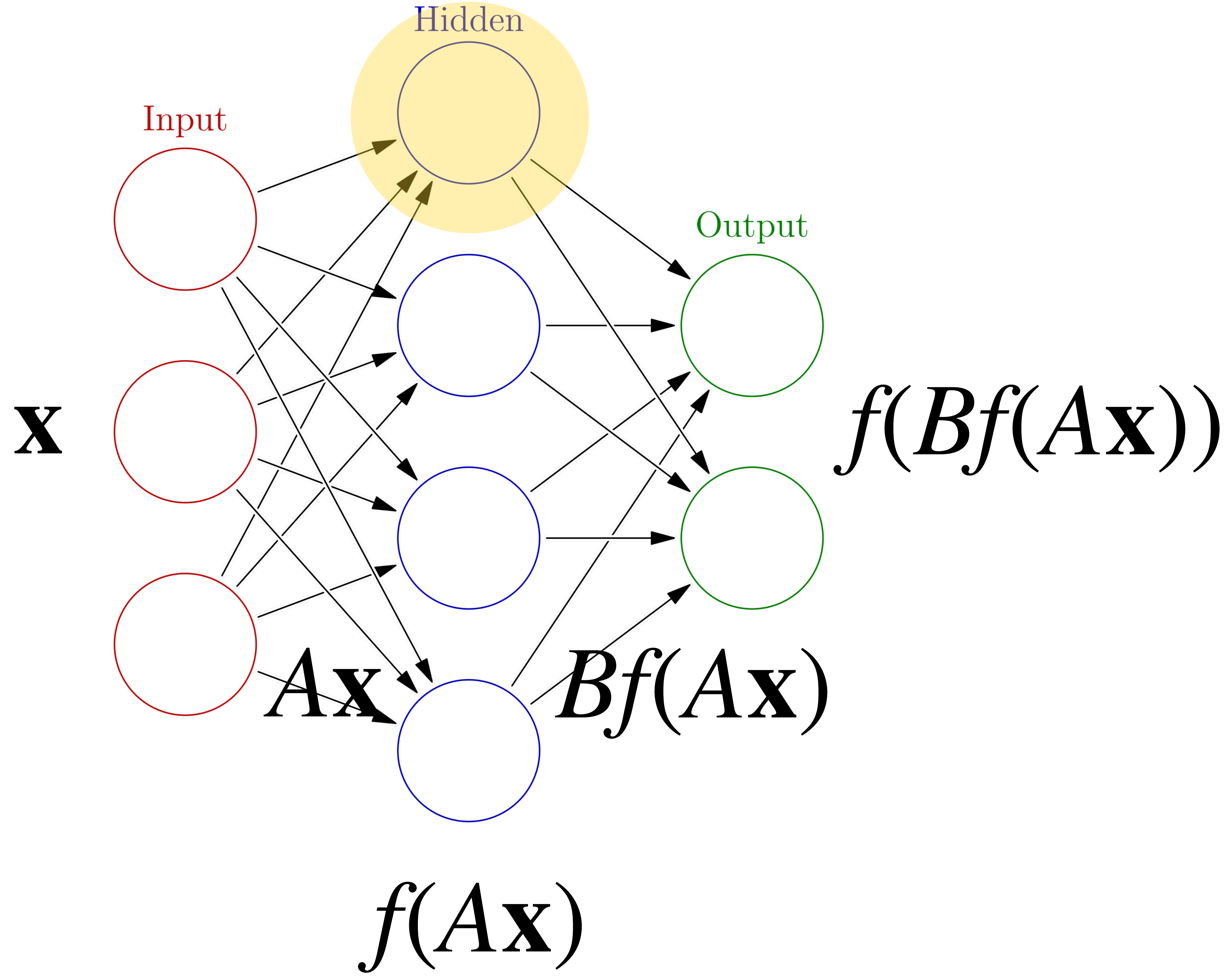


https://commons.wikimedia.org/wiki/File:Blausen_0657_MultipolarNeuron.png
https://commons.wikimedia.org/wiki/File:Colored_neural_network.svg

Neural Networks (Non-Linearity)

Neural networks are models of artificial neurons bundles.





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Neural Networks (Non-Linearity)

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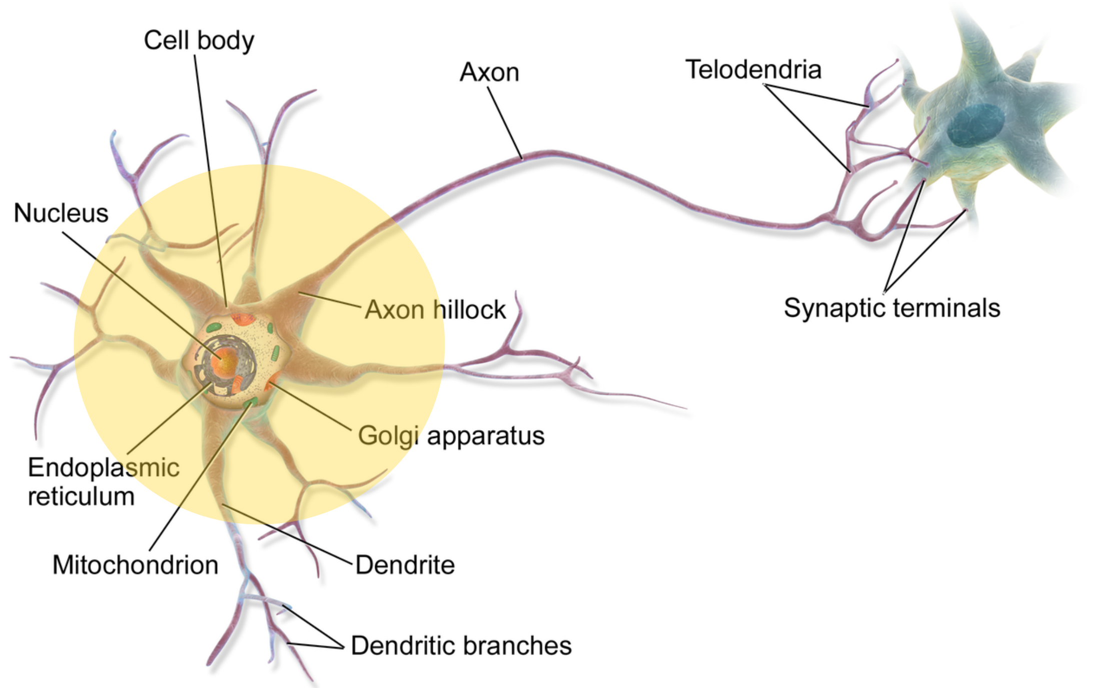
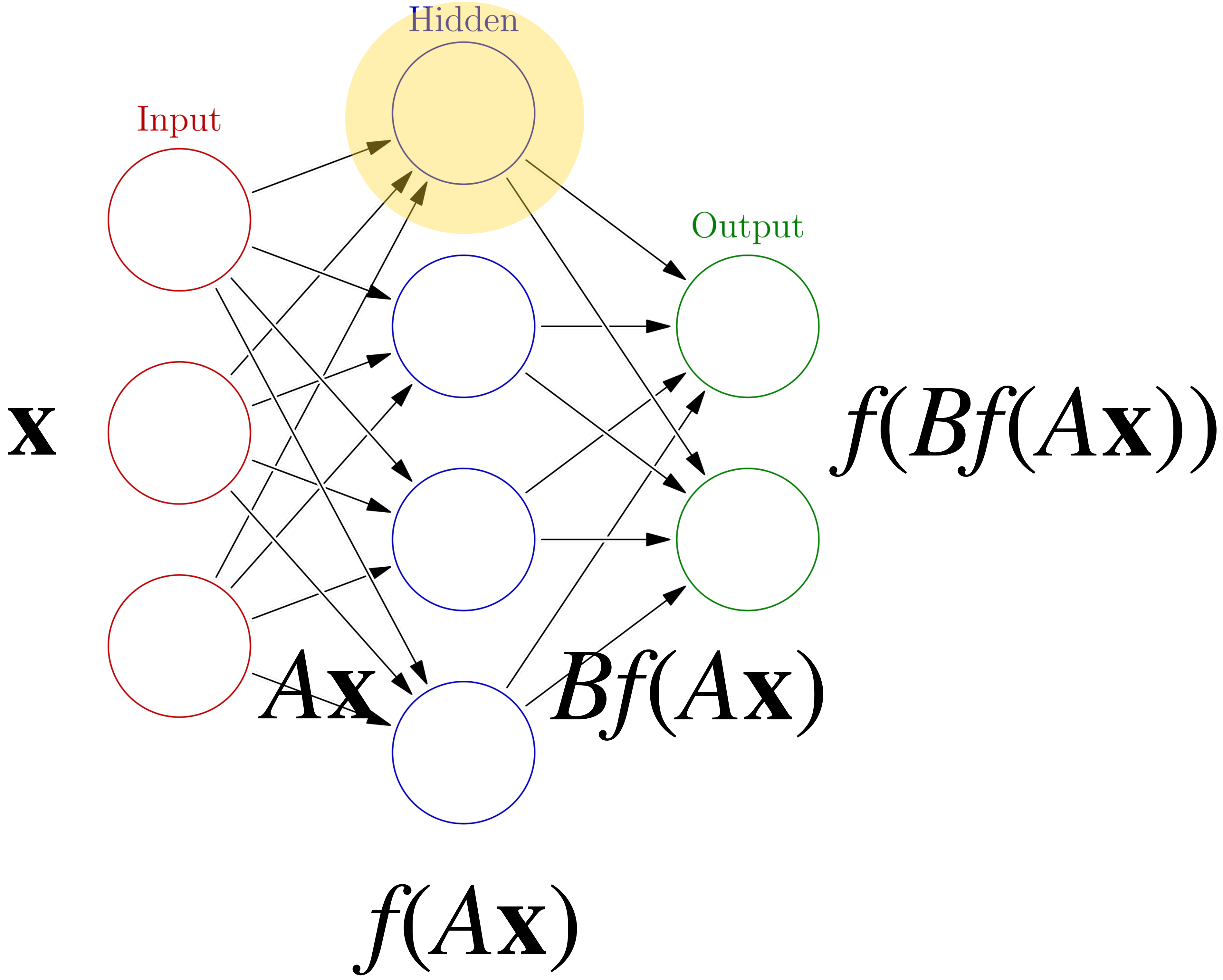
Given an input vector $\mathbf{x}$, it is transformed into a *hidden* vector $A\mathbf{x}$ by a linear transformation, and then an *activation function* f is applied to the result.





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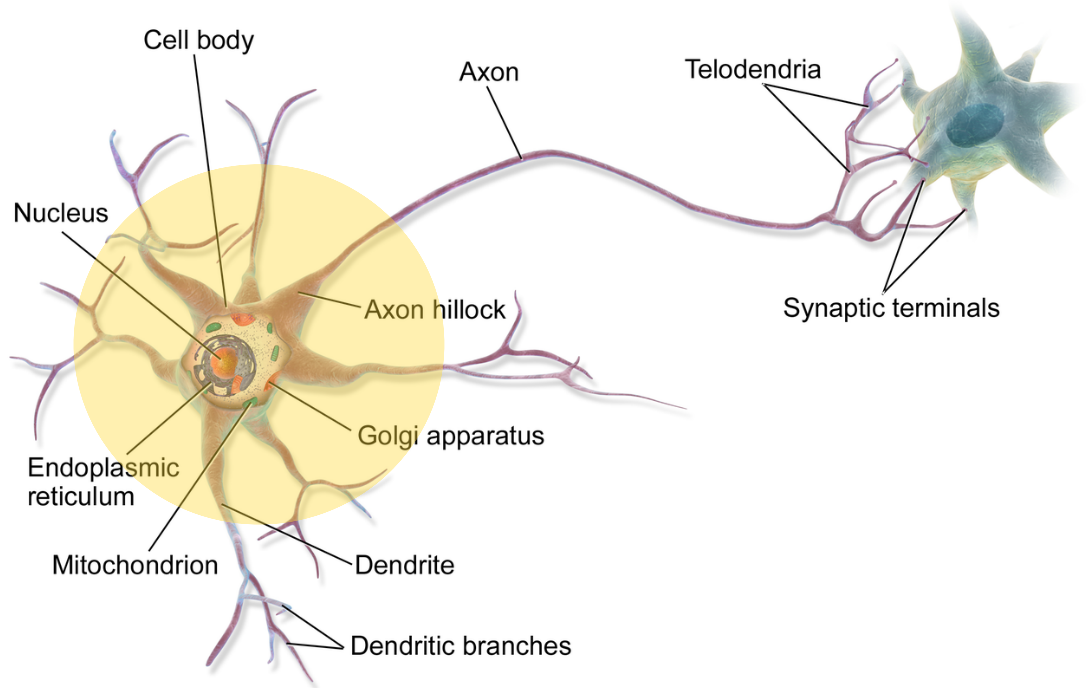

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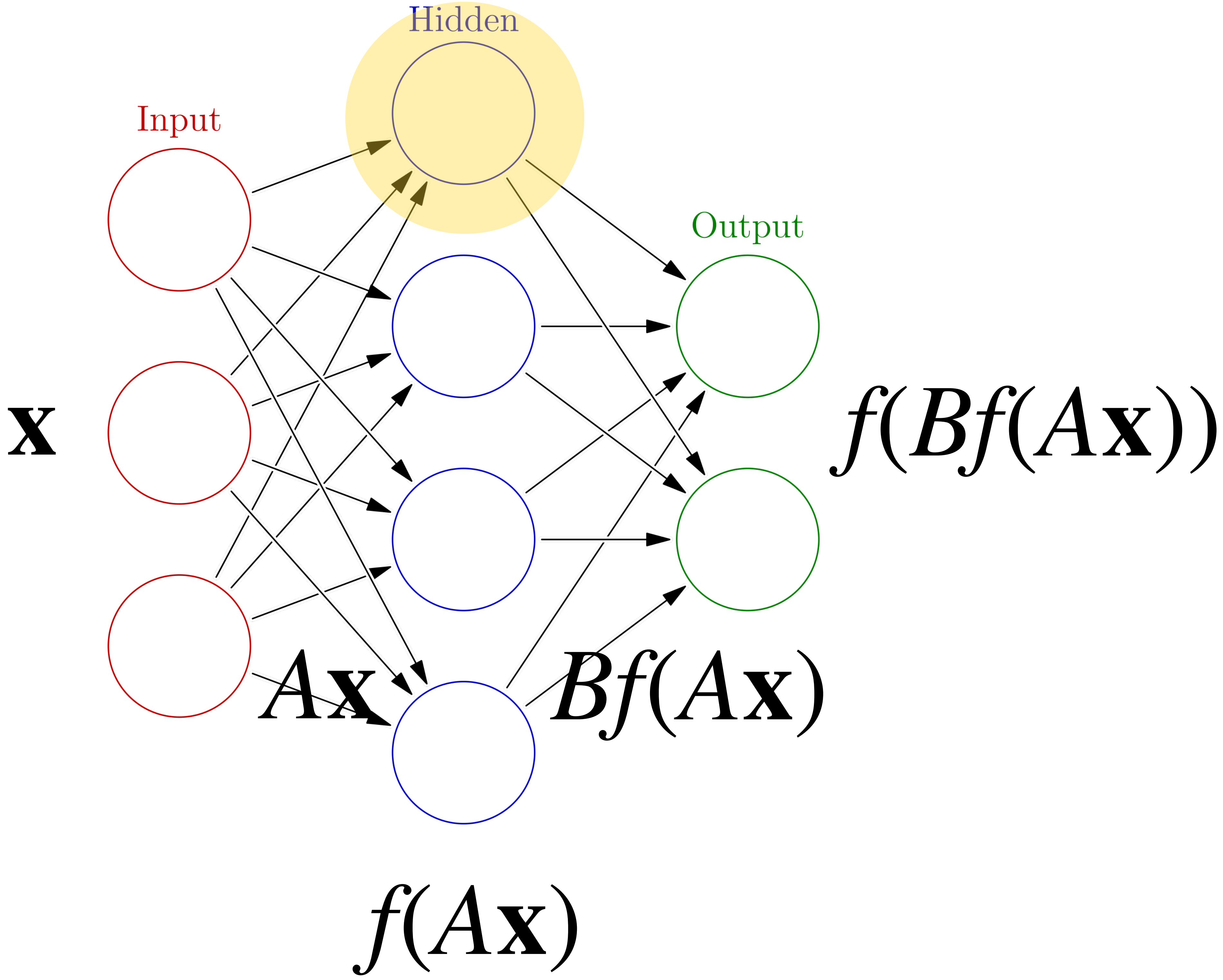
Neural networks are just matrix multiplications with intermediate calls to a nonlinear function f.



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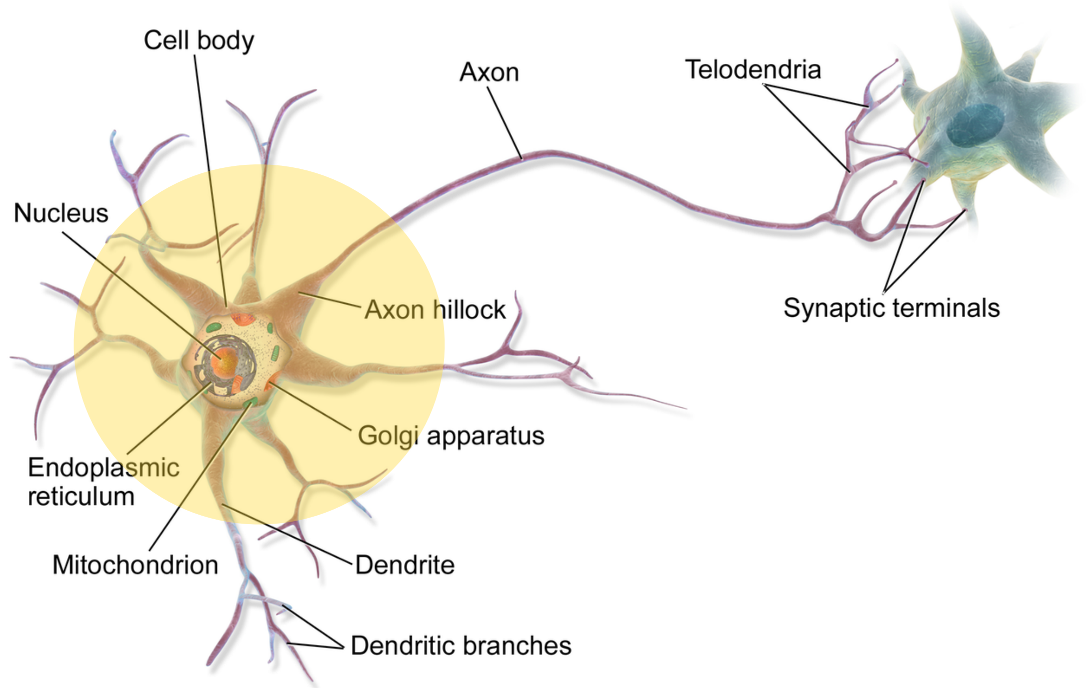

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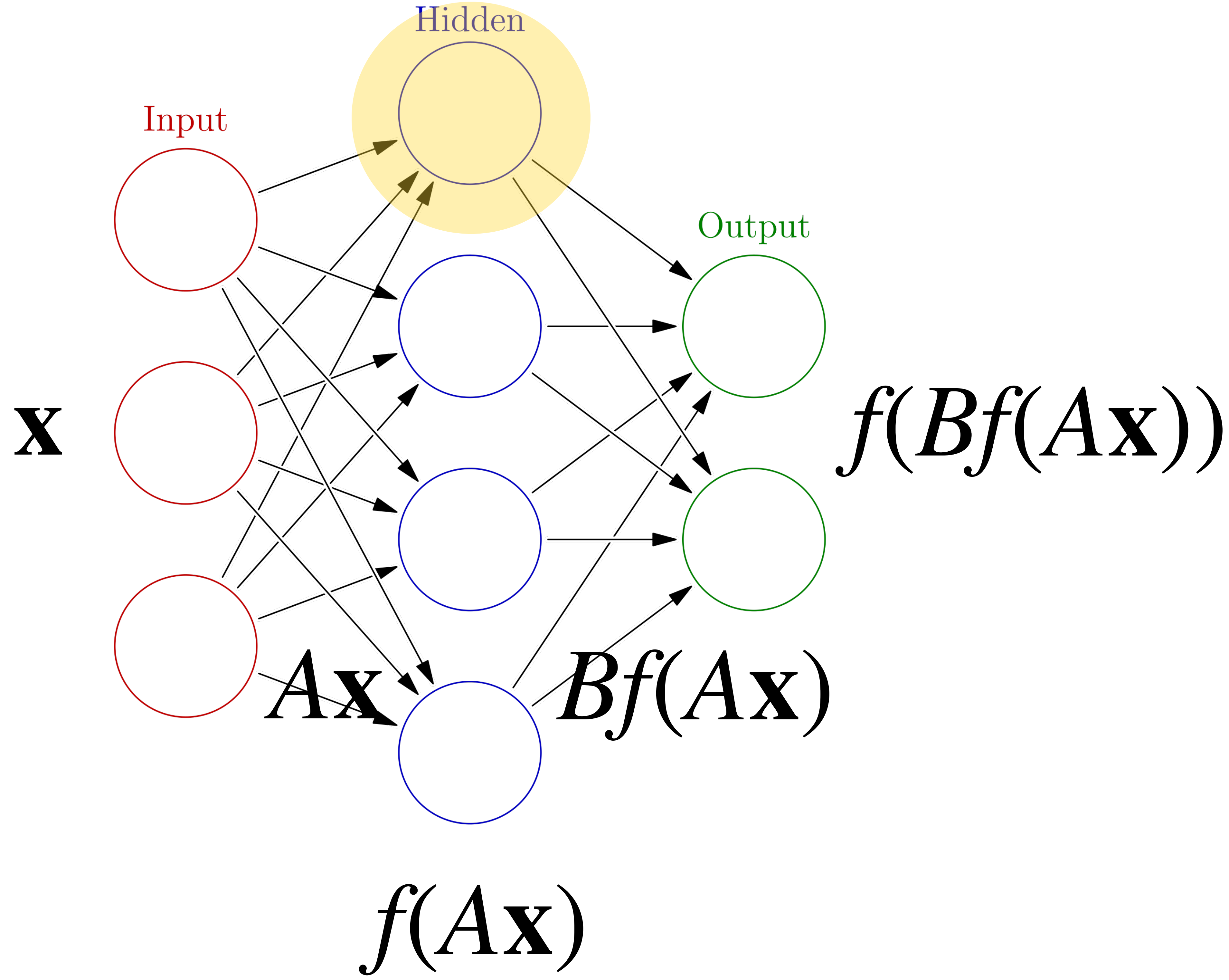

$$\mathrm{NN}(\mathbf{x}) = f(A_k(f(A_{k-1} \ldots f(A_1\mathbf{x}))$$

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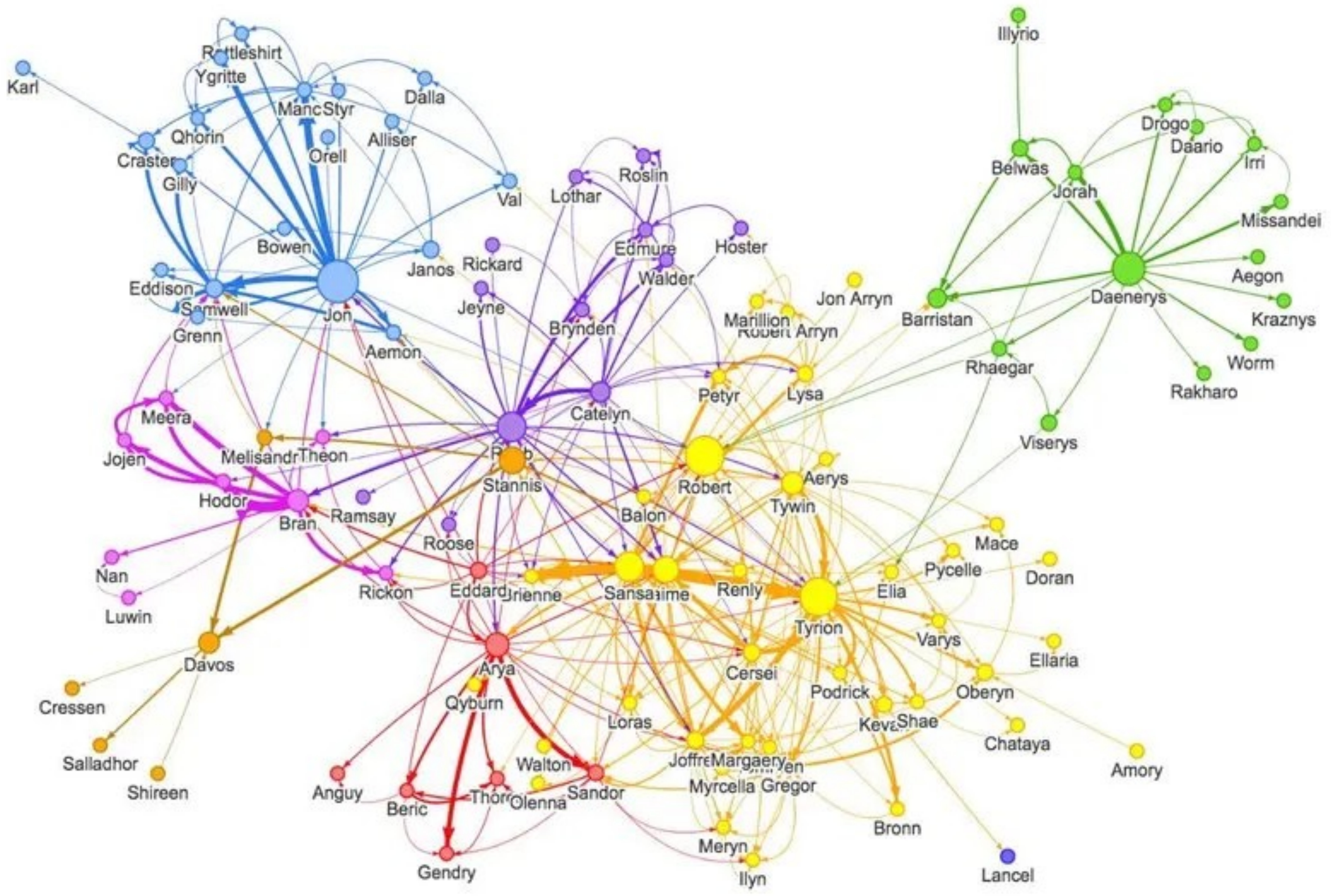
Spectral/Algebraic Graph Theory

Graphs can be viewed as matrices.

The finding eigenvalues in graphs can gives use better clustering and cutting algorithms.



Abstract Algebra

$$\frac{U}{\mathsf{Nul}(f)} \cong \mathsf{Range}(f)$$

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d] & V \\ U/\mathsf{Nul}(f) \arrow[ur] & \end{tikzcd}$$

There's a lot of beautiful structure in the algebra we've done in this course.

And there are lots of directions to go from here (infinite dimensional spaces, less restrictive settings like groups and modules,...)

Course List

- CS 365 Foundations of Data Science
- CS 440 Intro to Artificial Intelligence
- CS 480 Intro to Computer Graphics
- CS 505 Intro to Natural Language Processing
- CS 506 Tools for Data Science
- CS 507 Intro to Optimization in ML
- CS 523 Deep Learning
- CS 530 Advanced Algorithms
- CS 531 Advanced Optimization Algorithms
- CS 542 Machine Learning
- CS 565 Algorithmic Data Mining
- CS 581 Computational Fabrication
- CS 583 Audio Computation

Some of these may not exist anymore...

Appreciations

The Course Staff

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If you see them around you should thank them as well.

The CS Department Staff

If you're ever in the CS Department office, be kind to the people who work there. They work very hard to keep all our courses running.

The Students of CS132

`Thanks for sticking with it.`

`For giving feedback.`

`For adjusting and re-adjusting.`

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