Homework 4 Solutions

CAS CS 132

Fall 2024

Problem 1.1

$$\begin{bmatrix} \vec{r}_1 & \vec{v}_2 & \vec{v}_3 & \vec{r}_4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 & -2 \\ 0 & 4 & -4 & 12 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -9 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$-9\vec{r}_1 + 4\vec{r}_2 + \vec{r}_3 = r_4$$

Problem 1.2

$$\begin{bmatrix} \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -2 \\ 4 & -4 & 12 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 0 & -9 \\ 1 & 0 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & -9 \\ 0 & 0 & 0 \end{bmatrix}$$

not in span

$\boxed{\text{Problem 2.1}}$

$$\begin{bmatrix} 1 & 1 & -2 \\ -3 & 3 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix}$$

$$9(0) + 4(-2) = h$$

$$h = -8$$

Explanation: We need to choose values so that the first and third equations in the associated linear system are satisfied. We can then use those to determine h.

Problem 2.2

$$\begin{aligned} x_1 + 4x_2 &= 13 \\ -x_1 &= h \\ x_1 h + 2x_2 &= -1 \end{aligned} \Rightarrow \begin{aligned} x_1 &= -h \\ -h^2 + 2x_2 &= -1 \end{aligned}$$

$$\Rightarrow x_2 = \frac{h^2-1}{2} \quad \Rightarrow \quad -h + 2\left(\frac{h^2-1}{2}\right) = 13$$

$$2h^2 - h - 15 = 0$$

$$h = \frac{1 \pm \sqrt{1 - 4(2)(-15)}}{4}$$

$$h = 3, \frac{-5}{2}$$

Note: None of the vectors can be colinear, so they must be coplanar. In particular, it must be that $\vec{v}_3 \in \text{span}\{\vec{v}_1, \vec{v}_2\}$

Problem 3.1

$$\boxed{\vec{r}_3 = 2\vec{r}_1}$$

Note: $\vec{r}_2$ is not a scalar multiple of $\vec{r}_1$ so $\vec{r}_2 \notin \operatorname{span}\{\vec{r}_1\}$

Problem 3.2

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 & \vec{v}_5 & \vec{v}_6 & \vec{v}_7 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -2x_3$

$x_2 = -x_7$

x_3 is free

$x_4 = -2x_7$

$x_5 = 2x_7$

$x_6 = 0$

x_7 is free

$$-2\vec{v}_1 - \vec{v}_2 + v_3 - 2\vec{v}_4 + 2\vec{v}_5 + \vec{x}_7 = \vec{0}$$

Note: There are two answers, you can take the negation as well.

Problem 4

$$\vec{v}_1 = \begin{bmatrix} -3 \\ 9 \\ 5 \\ -4 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} -9 \\ -5 \\ 0 \\ -7 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 9 \\ -1 \\ -2 \\ 4 \end{bmatrix} \quad v_4 = \begin{bmatrix} 30 \\ 0 \\ -7 \\ 22 \end{bmatrix}$$

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$T(\vec{r}_4) = T(-\vec{r}_1 - 2\vec{r}_2 + \vec{r}_3)$$

$$= -T(\vec{r}_1) - 2T(\vec{r}_2) + T(\vec{r}_3)$$

$$= -\begin{bmatrix} -4 \\ 3 \\ 4 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ -3 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ -3 \\ -4 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 \\ -3 \\ 7 \end{bmatrix} = \begin{bmatrix} 7 \\ -8 \\ -1 \end{bmatrix}$$

$\boxed{\text{Problem 5.1}}$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} \sqrt[3]{2} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \neq \begin{bmatrix} \sqrt[3]{2} \\ 1 \end{bmatrix}$$

Does not satisfy additivity

<u>Note</u>: There are many possible solutions

Problem 5.2

$$\begin{bmatrix} cx \\ cy \\ cz \end{bmatrix} \longmapsto \begin{bmatrix} \left((cx)^3 + (cy)^3 + (cz)^3\right)^{1/3} \\ cy + cz \end{bmatrix}$$

$$= \begin{bmatrix} \left(c^3x^3 + c^3y^3 + c^3z^3\right)^{1/3} \\ cy + cz \end{bmatrix}$$

$$= \begin{bmatrix} \left(c^3\left(x^3 + y^3 + z^3\right)\right)^{1/3} \\ c(y+z) \end{bmatrix}$$

$$= \begin{bmatrix} c\left(x^3 + y^3 + z^3\right)^{1/3} \\ c(y+z) \end{bmatrix}$$

$$= c \begin{bmatrix} \left(x^3 + y^3 + z^3\right)^{1/3} \\ y+z \end{bmatrix}$$