Homework 6 Solutions

CAS CS 132

Fall 2024

$\boxed{\text{Problem 1.1}}$

$$\begin{bmatrix} 2 & -1 \\ 3 & 3 \end{bmatrix}^{-1} = \frac{1}{2(3)-(-1)3} \begin{bmatrix} 3 & 1 \\ -3 & 2 \end{bmatrix}$$

$$= \frac{1}{9} \begin{bmatrix} 3 & 1 \\ -3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & 1/9 \\ -1/3 & 2/9 \end{bmatrix}$$

Problem 1.2

$$\left[\begin{array}{cccccc} 1 & 0 & -2 & 1 & 0 & 0 \\ -3^{+3} & 1^{+0} & 4^{-6} & 0^{+3} & 1^{+0} & 0^{+0} \\ 2^{-2} & -3^{-0} & 4^{+4} & 0^{-2} & 0^{-0} & 1^{-0} \end{array}\right] \sim$$

$$\left[\begin{array}{cccccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -3^{+3} & 8^{-6} & -2^{+9} & 0^{+3} & 1^{+0} \end{array}\right] \sim$$

$$\left[\begin{array}{cccccc} 1 & 0 & -2^{+2} & 1^{+7} & 0^{+3} & 0^{+1} \\ 0 & 1 & -2^{+2} & 3^{+7} & 1^{+3} & 0^{+1} \\ 0 & 0 & 2 & 7 & 3 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 2_{/2} & 7_{/2} & 3_{/2} & 1_{/2} \end{array}\right] \sim$$

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & 7/2 & 3/2 & 1/2 \end{array}\right]$$

$$\begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

Problem 1.3

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \cos\theta + \sin\theta & \cos\theta - \sin\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \\ x+y \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \cos\theta + \sin\theta & \cos\theta - \sin\theta \end{bmatrix}^{-1} =$$

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta - \sin\theta & \sin\theta \\ -\cos\theta - \sin\theta & \cos\theta \end{bmatrix}$$

note: $\cos^2\theta - \cos\theta\sin\theta + \sin\theta\cos\theta + \sin^2\theta = 1$

Problem 2.1

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^n \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}^{-n}$$

$$= \begin{bmatrix} 1 & 0 & 2n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3n & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 6n^2 & 2n \\ 0 & 1 & 0 \\ 0 & 3n & 1 \end{bmatrix}$$

$\boxed{\text{Problem 2.2}}$

This is rotation about the z-axis followed by reflection across the yz-plane. Therefore its square is I.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\boxed{\text{Problem 2.3}}$

$$\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}^3 = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}^6 = \left(\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}^3 \right)^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$n = 6$$

Note: $A^4 \neq I$ and $A^5 \neq I$ because this would imply $A = -I$ or $A^2 = -I$.

Problem 3.1

$$A\left(C^{-1}(AB)^T\right)^T C =$$

$$A(AB)^{TT}\, C^{-1\,T}\, C =$$

$$A(AB)(C^T)^{-1} C =$$

$$\cancel{A^{-1}}(\cancel{A}B(\cancel{C^T})^{-1}\cancel{C^T} =$$

$$B$$

Problem 3.2

$$X^2 + \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} X + \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} =$$

$$\left(X + \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \right) \left(X + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

(1) $\begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}^2 + \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \checkmark$$

$$= \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -6 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

(2) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}^2 + \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

(3) $\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}^2 + \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Problem 4.1

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$$L = \begin{bmatrix} 2 & 0 & 0 & -1 & -1 & 0 \\ 0 & 2 & -1 & 0 & 0 & -1 \\ 0 & -1 & 2 & 0 & 0 & -1 \\ -1 & 0 & 0 & 2 & -1 & 0 \\ -1 & 0 & 0 & -1 & 2 & 0 \\ 0 & -1 & -1 & 0 & 0 & 2 \end{bmatrix}$$

$$L \sim \begin{bmatrix} \boxed{1} & 0 & 0 & 0 & -1 & 0 \\ 0 & \boxed{1} & 0 & 0 & 0 & -1 \\ 0 & 0 & \boxed{1} & 0 & 0 & -1 \\ 0 & 0 & 0 & \boxed{1} & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4 pivots

$\boxed{\text{Problem 4.2}}$

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$$L = \begin{bmatrix} 2 & 0 & 0 & -1 & -1 & 0 \\ 0 & 2 & -1 & 0 & 0 & -1 \\ 0 & -1 & 3 & -1 & 0 & -1 \\ -1 & 0 & -1 & 3 & -1 & 0 \\ -1 & 0 & 0 & -1 & 2 & 0 \\ 0 & -1 & -1 & 0 & 0 & 2 \end{bmatrix}$$

$$L \sim \begin{bmatrix} \boxed{1} & 0 & 0 & 0 & 0 & -1 \\ 0 & \boxed{1} & 0 & 0 & 0 & -1 \\ 0 & 0 & \boxed{1} & 0 & 0 & -1 \\ 0 & 0 & 0 & \boxed{1} & 0 & -1 \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

5 pivots