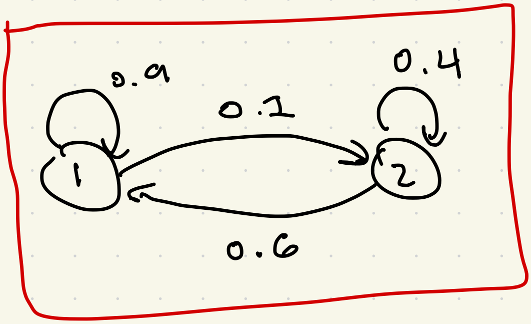
Homework 7 Solutions

CAS CS 132

Fall 2024

Problem 1.1



A is regular because $A^{\boxed{1}}$ has strictly positive entries

$$(A - I) \sim \begin{bmatrix} 1 & -6 \\ 0 & 0 \end{bmatrix}$$

$$\boxed{\begin{array}{l} x_1 = 6x_2 \\ x_2 \text{ is free} \end{array}}$$

$x_1 + x_2 = 1$

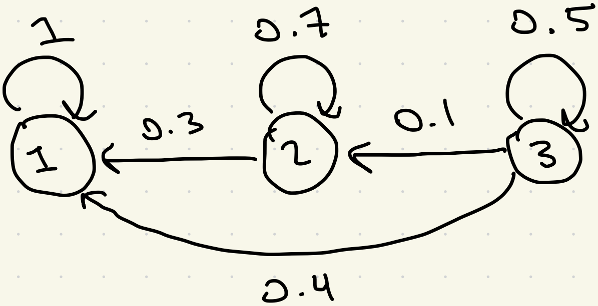
$6x_2 + x_2 = 1$

$7x_2 = 1$

$x_2 = 1/7$

$\boxed{\begin{bmatrix} 6/7 \\ 1/7 \end{bmatrix}}$ is the $\boxed{\text{unique}}$ steady state vector

Problem 1.2



A is not regular because the first column of A^k is $\begin{bmatrix}1\\0\\0\end{bmatrix}$ for any k.

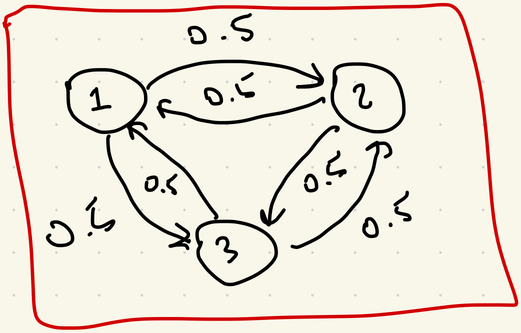
(Also, the product of upper triangular matrices is upper triangular)

$$A - I \sim \begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}$$

$$\boxed{\begin{array}{l}x_1 \text{ is free}\\ x_2 = 0\\ x_3 = 0\end{array}}$$

$\boxed{\begin{bmatrix}1\\0\\0\end{bmatrix}}$ is the unique steady state vector.

Problem 1.3



A is regular because $A^{\boxed{2}}$ has strictly positive entries.

$$(A - I) \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{\begin{aligned} x_1 &= x_3 \\ x_2 &= x_3 \\ x_3 &\text{ is free} \end{aligned}}$$

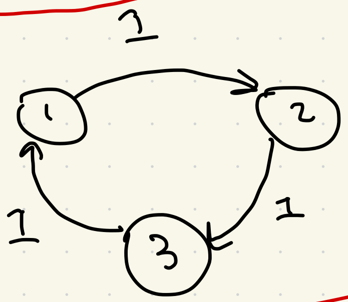
$$x_1 + x_2 + x_3 = 1$$

$$3x_3 = 1$$

$$x_3 = 1/3$$

$\boxed{\begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}}$ is the $\boxed{\text{unique}}$ steady state vector

Problem 1.4



A is $\boxed{\text{not regular}}$ because

$$A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

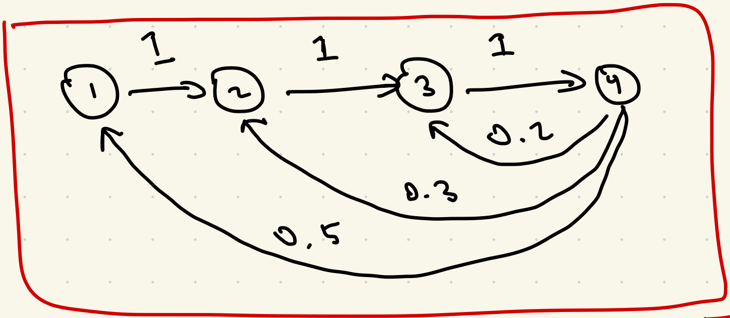
So every power of A is the same as A^0, A^1 or A^2

$$A - I \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{\begin{array}{l} x_1 = x_3 \\ x_2 = x_3 \\ x_3 \text{ is free} \end{array}}$$

$\boxed{\begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}}$ is the $\boxed{\text{unique}}$ steady state vector

Problem 1.5



A is regular because $A^{\boxed{6}}$ has strictly positive entries

$$A - I \sim \begin{bmatrix} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & -4/5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{\begin{aligned} x_1 &= \tfrac{1}{2} x_4 \\ x_2 &= 4/5\, x_4 \\ x_3 &= x_4 \\ x_4 &\text{ is free} \end{aligned}}$$

$$x_1 + x_2 + x_3 + x_4 = 1$$

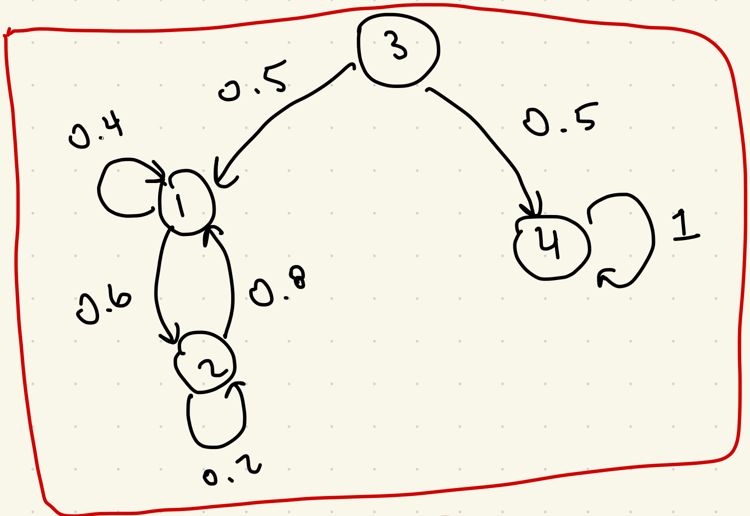
$$\frac{5}{10} x_4 + \frac{8}{10} x_4 + \frac{10}{10} x_4 + \frac{10}{10} x_4 = 1$$

$$\frac{33}{10} x_4 = 1$$

$$x_4 = \frac{10}{33}$$

$$\boxed{\begin{bmatrix} 5/33 \\ 8/33 \\ 10/33 \\ 10/33 \end{bmatrix}}$$ is the unique steady state vector

Problem 1.6



A is **not regular** because the last column of A^k is $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ for any k.

$$A - I \sim \begin{bmatrix} 1 & -4/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 4/3\, x_2$

x_2 is free

$x_3 = 0$

x_4 is free

$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ is a steady state vector, it is **not unique**

Problems 2

$$\begin{array}{c|ccc} & S & H & L \\ \hline S & 0.7 & 0.4 & 0.3 \\ H & 0.2 & 0.55 & 0.1 \\ L & 0.1 & 0.05 & 0.6 \end{array}$$

$$A - I \sim \begin{bmatrix} 1 & 0 & -35/11 \\ 0 & 1 & -18/11 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 35/11\, x_3$

$x_2 = 18/11\, x_3$

x_3 is free

$x_1 + x_2 + x_3 = 1$

$35/11\, x_3 + 18/11\, x_3 + 11/11\, x_3 = 1$

$64/11\, x_3 = 1$

$x_3 = 11/64$

$\boxed{x_1 = 35/64}$

$x_2 = 18/64$

$x_3 = 11/64$

it is more likely to **stay steady**

Problem 3.1

$$\begin{bmatrix} 2 & 3 \\ \overset{-1}{1} & \overset{-3/2}{-4} \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - \frac{1}{2}R_1} \begin{bmatrix} 2 & 3 \\ 0 & -11/2 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 2 & 3 \\ 0 & -11/2 \end{bmatrix}$$

Problem 3.2

$$\begin{bmatrix} -3 & 0 & -6 & -9 \\ 0 & 2 & 4 & -4 \\ 3^{-3} & 2 & 10^{-6} & 9^{-9} \\ -3^{+3} & 2 & -2^{+6} & -9^{+9} \end{bmatrix} \xrightarrow[R_3 \leftarrow R_3 - R_1]{R_3 \leftarrow R_3 + R_1}$$

$$\begin{bmatrix} -3 & 0 & -6 & -9 \\ 0 & 2 & 4 & -4 \\ 0 & 2^{-2} & 4^{-4} & 0^{+4} \\ 0 & 2^{-2} & 4^{-4} & 0^{+4} \end{bmatrix} \xrightarrow[R_4 \leftarrow R_4 - R_2]{R_3 \leftarrow R_3 - R_2}$$

$$\begin{bmatrix} -3 & 0 & -6 & -9 \\ 0 & 2 & 4 & -4 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 4^{-4} \end{bmatrix} \xrightarrow{R_4 \leftarrow R_4 - R_3}$$

$$\begin{bmatrix} -3 & 0 & -6 & -9 \\ 0 & 2 & 4 & -4 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Problem 3.3

$$U = \begin{bmatrix} 2 & 8 & -4 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & -7 \end{bmatrix}$$ is in echelon form

and $A \longrightarrow U$

inverse of row ops.

$$\left[\begin{array}{l} R_2 \leftarrow R_2 - R_1 \\ R_3 \leftarrow R_3 - R_1 \\ R_3 \leftarrow R_3 + 2R_2 \end{array}\right.$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

Problem 4.1

$$\begin{bmatrix} 4 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 4 & -1 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 4 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 4 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 4 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 4 \end{bmatrix}$$

Problem 4.2

nonzero entries in L: 5997

nonzero entries in U: 5997

nonzero entries in A^{-1}: 3792976

Problem 4.3

Yes it is consistent. It slower to invert, roughly by 3 times