Homework 9 Solutions

CAS CS 132

Fall 2024

Problem 1.1

$$\begin{bmatrix} 1 & -4 & -10 \\ -4 & 17 & 42 \\ 1 & -2 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathrm{Col}(A) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ 17 \\ -2 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 1 & -4 & b_1 \\ \overset{+4}{-4} & \overset{-16}{17} & \overset{+4b_1}{b_2} \\ \overset{-1}{1} & \overset{+4}{-2} & \overset{-b_1}{b_3} \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & b_1 \\ 0 & 1 & b_2 + 4b_1 \\ 0 & \underset{-2}{2} & \underset{-2(b_2+4b_1)}{b_3 - b_1} \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & -4 & b_1 \\ 0 & 1 & b_2 + b_1 \\ 0 & 0 & b_3 - b_1 - 2b_2 - 8b_1 \end{bmatrix}$$

$$9x_1 - 2x_2 + x_3 = 0$$

check:

$$-9(1) - 2(-4) + 1 = -9 + 8 + 1 = 0$$

$$-9(-4) - 2(17) + (-2) = 36 - 34 - 2 = 0$$

Problem 1.2

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 + 2R_1} \begin{bmatrix} b_1 \\ b_2 + 2b_1 \\ b_3 \\ b_4 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - 7R_1}$$

$$\begin{bmatrix} b_1 \\ b_2 + 2b_1 \\ b_3 - 7b_1 \\ b_4 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - 5R_2} \begin{bmatrix} b_1 \\ b_2 + 2b_1 \\ b_3 - 7b_1 - 5b_2 - 10b_1 \\ b_4 \end{bmatrix}$$

$$\begin{bmatrix} b_1 \\ b_2 + 2b_1 \\ b_3 - 5b_2 - 17b_1 \\ b_4 \end{bmatrix} \xrightarrow{R_4 \leftarrow R_4 - R_3}$$

$$\begin{bmatrix} b_1 \\ b_2 + 2b_1 \\ b_3 - 5b_2 - 3b_1 \\ b_4 - b_3 + 5b_2 + 17b_1 \end{bmatrix}$$

$$17x_1 + 5x_2 - x_3 + x_4 = 0$$

Idea: L describes the sequence of row operations to get from A to U. Since rank(A) = 3, there is a row of all zeros at the end of U.

Problem 2.1

$\vec{h}_1 \in H \quad \vec{h}_2 \in H$

$\vec{h}_1 = \vec{u}_1 + \vec{v}_1$ where $u_1 \in Col(A)$, $v_1 \in Col(B)$

$\vec{h}_2 = \vec{u}_2 + \vec{v}_2$ where $u_2 \in Col(A)$, $v_2 \in Col(B)$

Additivity:

$$\vec{h}_1 + \vec{h}_2 = \vec{u}_1 + \vec{v}_1 + \vec{u}_2 + \vec{v}_2$$
$$= (\vec{u}_1 + \vec{u}_2) + (\vec{v}_1 + \vec{v}_2) \in H$$

since $\vec{u}_1 + \vec{u}_2 \in Col(A)$ and $\vec{v}_1 + \vec{v}_2 \in Col(B)$

Scaling:

$$\alpha \vec{h}_1 = \alpha(\vec{u}_1 + \vec{v}_1) = \alpha\vec{u}_1 + \alpha\vec{v}_1 \in H$$

since $\alpha\vec{u}_1 \in H$ and $\alpha\vec{v}_1 \in H$

Problem 2.2

$$[A|B] \sim \begin{bmatrix} 1 & 0 & 0 & 4 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 2 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\dim(H) = 4$

Idea: The columns of A and B combined span H, so we need to find a basis for this combined column spaces

Problem 3.1

$$\begin{bmatrix} 1 & 3 & -17 \\ -5 & -1 & 15 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -5 \end{bmatrix}$$

$$\begin{bmatrix} -17 \\ 15 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} -2 \\ -5 \end{bmatrix}$$

Problem 3.2

$$\begin{bmatrix} b_1 & b_2 & b_3 & v \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} -4 \\ -3 \\ 4 \end{bmatrix}$$

Problem 4.1

$$A+3I \sim \begin{bmatrix} 1 & -6 \\ 0 & 0 \end{bmatrix}$$

$x_1 = 6x_2$

x_2 is free

$$\text{Nul}(A+3I) = \text{span}\left\{ \begin{bmatrix} 6 \\ 1 \end{bmatrix} \right\}$$

$\left\{ \begin{bmatrix} 6 \\ 1 \end{bmatrix} \right\}$ is a basis

check:

$$\begin{bmatrix} 4 & -42 \\ 1 & 9 \end{bmatrix} \begin{bmatrix} 6 \\ 1 \end{bmatrix} = \begin{bmatrix} 24-42 \\ 6-9 \end{bmatrix} = \begin{bmatrix} -18 \\ -3 \end{bmatrix}$$

Problem 4.2

$$A + 5I \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = x_2 - x_3$

x_2 is free

x_3 is free

$$x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis

check:

$$\begin{bmatrix} 3 & -8 & 8 \\ 8 & -13 & 8 \\ 2 & -2 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 - 8 \\ 8 - 13 \\ 2 + 2 \end{bmatrix} = \begin{bmatrix} -5 \\ -5 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -8 & 8 \\ 8 & -13 & 8 \\ 2 & -2 & -3 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 + 8 \\ -8 + 8 \\ -2 + -3 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ -5 \end{bmatrix}$$

Problem 4.3

$$A - 4I \sim I$$

4 is not an eigenvalue

Problem 5.1

$$A^5 \begin{bmatrix} 3 \\ 0 \\ -11 \end{bmatrix} = A^5 (\vec{b}_1 + \vec{b}_2 + \vec{b}_3)$$

$$= A^5 \vec{b}_1 + A^5 \vec{b}_2 + A^5 \vec{b}_3$$

$$= \lambda_1^5 \vec{b}_1 + \lambda_2^5 \vec{b}_2 + \lambda_3^5 \vec{b}_3$$

$$= (-1)^5 \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 32 \begin{bmatrix} 2 \\ -2 \\ -9 \end{bmatrix}$$

$$= \begin{bmatrix} 64 \\ -64 \\ -288 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 63 \\ -65 \\ -285 \end{bmatrix}$$

Problem 5.2

$$A\vec{v} = A\,[\vec{b}_1\,\vec{b}_2\,\vec{b}_3]\,[v]_\beta$$

$$= A(\alpha_1\vec{b}_1 + \alpha_2\vec{b}_2 + \alpha_3\vec{b}_3)$$

$$= \alpha_1\lambda_1\vec{b}_1 + \alpha_2\lambda_2\vec{b}_2 + \alpha_3\lambda_3\vec{b}_3$$

$$= [\lambda_1\vec{b}_1\;\; \lambda_2\vec{b}_2\;\; \lambda_3\vec{b}_3]\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{bmatrix}$$

$$= [\lambda_1\vec{b}_1\;\; \lambda_2\vec{b}_2\;\; \lambda_3\vec{b}_3]\,[v]_\beta$$

$$\begin{bmatrix}-1 & 0 & 4\\ -1 & 0 & -4\\ 3 & 1 & -18\end{bmatrix}$$

Problem 5.3

Since A is triangular the eigenvalues are 1, 0, 2

A has at least 3 L.I. eigenvectors, one for each distinct eigenvalue.

$$A - I = \begin{bmatrix} 0 & \boxed{2} & -4 & 1 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & \boxed{-1} & 3 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

so $\dim(\operatorname{Nul}(A-I)) = 1$

A has exactly 3 L.I. eigenvectors

so **NO**