Homework 10 Solutions

CAS CS 132

Fall 2024

Problem 1.1

$$A - \lambda I = \begin{bmatrix} 9-\lambda & 4 \\ -14 & -6-\lambda \end{bmatrix}$$

$\det(A-\lambda I) =$

$(9-\lambda)(-6-\lambda) + 56 =$

$-54 - 9\lambda + 6\lambda + \lambda^2 + 56 =$

$\lambda^2 - 3\lambda + 2$

$(\lambda - 2)(\lambda - 1)$

$$A - I = \begin{bmatrix} 8 & 4 \\ -14 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix}$$

$x_1 = -1/2\, x_2$
x_2 is free $\quad \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad \lambda = 1$

$$A - 2I = \begin{bmatrix} 7 & 4 \\ -14 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 4/7 \\ 0 & 0 \end{bmatrix}$$

$x_1 = -4/7\, x_2$
x_2 is free $\quad \begin{bmatrix} -4 \\ 7 \end{bmatrix}, \quad \lambda = 2$

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -4 & -1 \\ 7 & 2 \end{bmatrix}$$

$$P^{-1} = \frac{1}{-8+7} \begin{bmatrix} 2 & 1 \\ -7 & -4 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 7 & 4 \end{bmatrix}$$

$$\begin{bmatrix} -4 & -1 \\ 7 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & -1 \\ 7 & 4 \end{bmatrix}$$

Problem 1.2

$$A - \lambda I = \begin{bmatrix} -6-\lambda & -7 \\ 7 & 8-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (-6-\lambda)(8-\lambda) + 49 =$$

$$-48 + 6\lambda - 8\lambda + \lambda^2 + 49 =$$

$$\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$$

$$A - I = \begin{bmatrix} -7 & -7 \\ 7 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\dim(\text{Nul}(A - I)) = 1$$

not diagonalizable

Problem 1.3

$$(2-\lambda)^2(3-\lambda)(-1-\lambda) =$$

$$(\lambda-2)^2(\lambda-3)(\lambda+1)$$

$$(A-3I) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 3 & -3 & -1 & 0 \\ -3 & 18 & -6 & -4 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & 18 & -6 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & 0 & -12 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 1 & 1/3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & -1/3 \\ 0 & 0 & 1 & 1/3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1/9 \\ 0 & 0 & 1 & 1/3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1/9 \\ 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \\ -3 \\ 9 \end{bmatrix} \quad \lambda = 3$$

$$(A - 2I) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 3 & -3 & 0 & 0 \\ -3 & 18 & -6 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ -3 & 18 & -6 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 15 & -6 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -2/5 & -1/5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & -2/5 & -1/5 \\ 0 & 1 & -2/5 & -1/5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = \frac{2}{5} x_3 + \frac{1}{5} x_4$

$x_2 = \frac{2}{5} x_3 + \frac{1}{5} x_4$

x_3 is free

x_4 is free

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 2 \end{bmatrix} \quad \lambda = 2$$

$$A + I = \begin{bmatrix} 3 & 0 & 0 & 0 \\ -1 & 4 & 0 & 0 \\ 3 & -3 & 3 & 0 \\ -3 & 18 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1/3 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 0$

$x_2 = 0$

$x_3 = 0$

x_4 is free

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \lambda = -\underline{1}$$

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -3 & 0 & -1 & 0 \\ 9 & 5 & 2 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 3 & -3 & -1 & 0 \\ -2 & -3 & 2 & 1 \end{bmatrix}$$

Note: there are principled ways to ensure that P^{-1} has integer entries, guess-and-check works too.

$$\left[\begin{array}{cccc|cccc} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -3 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 9 & 5 & 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{cccc|cccc} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ -3 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 9 & 5 & 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -3 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 9 & 5 & 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccccccc} 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 3 & -3 & -1 & 0 \\ 9 & 5 & 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 3 & -3 & -1 & 0 \\ 0 & 0 & 0 & 1 & -2 & -3 & 2 & 1 \end{array}\right]$$

Problem 1.4

$$\det(A-\lambda I) = \det \begin{bmatrix} -1-\lambda & -3 & 0 \\ 2 & 4-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}$$

$$= (-1-\lambda)(4-\lambda)(1-\lambda) + -3(0)(0) + 0(2)(0)$$

$$- 0(4-\lambda)0 - (-1-\lambda)(0)(0) -$$

$$(1-\lambda)(-3)(2)$$

$$= (-1-\lambda)(4-\lambda)(1-\lambda) + 6(1-\lambda)$$

$$= (1-\lambda)\big((-1-\lambda)(4-\lambda) + 6\big)$$

$$= (1-\lambda)(-4+\lambda-4\lambda+\lambda^2+6)$$

$$= (1-\lambda)(\lambda^2 - \lambda 3 + 2)$$

$$= (1-\lambda)(\lambda-2)(\lambda-1)$$

$$= -(\lambda-1)^2(\lambda-2) \qquad \lambda = 2, 1$$

$$A - 2I = \begin{bmatrix} -3 & -3 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -x_2$

x_2 is free

$x_3 = 0$

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \lambda = 2$$

$$A - I = \begin{bmatrix} -2 & -3 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3/2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \lambda = 1$$

$$P = \begin{bmatrix} -1 & -3 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} 2 & 3 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -3 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 & 3 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Problem 2.1

$$\det \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

$$\det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$$

$$\det \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = 1$$

Probem 2.2

$$\begin{bmatrix} -6 & -7 \\ 7 & 8 \end{bmatrix}$$ by problem 1.2

Problem 2.2

$$\begin{bmatrix} -6 & -7 \\ 7 & 8 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is diagonalizable, $\begin{bmatrix} -6 & -7 \\ 7 & 8 \end{bmatrix}$ is not

Problem 3

$$\lambda = 9, 1, 4$$

$$A - 9I \sim \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \lambda = 9$$

$$A - I \sim \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \lambda = 1$$

$$A - 4I \sim \begin{bmatrix} 1 & 0 & -3/7 \\ 0 & 1 & 4/7 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 3 \\ -4 \\ 7 \end{bmatrix}, \quad \lambda = 4$$

$$P = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & -4 \\ 2 & 2 & 7 \end{bmatrix} \quad D = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$B = P \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} P^{-1} = \begin{bmatrix} 25 & -2 & -11 \\ 8 & 1 & -4 \\ 46 & -4 & -20 \end{bmatrix}$$