Homework 11 Solutions

CAS CS 132

Fall 2024

Problem 1.1

$$\|v\| = \left(3^2 + 0^2 + 3^2 + (-5)^2 + (-7)^2\right)^{1/2}$$
$$= (9 + 9 + 25 + 49)^{1/2}$$
$$= \sqrt{92}$$

Problem 1.2

$$\|u\| = \left(1^2 + (-1)^2 + 5^2 + 7^2 + 4^2\right)^{1/2}$$
$$= (1 + 1 + 25 + 49 + 16)^{1/2}$$
$$= \sqrt{92}$$

$$\frac{1}{\sqrt{92}} \vec{u} = \frac{1}{\sqrt{92}} \begin{bmatrix} 1 \\ -1 \\ 5 \\ 7 \\ 4 \end{bmatrix}$$

Problem 1.3

$$\|u - r\| = \left\| \begin{bmatrix} 1-3 \\ -1-0 \\ 5-3 \\ 7+5 \\ 4+7 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -2 \\ -1 \\ 2 \\ 12 \\ 11 \end{bmatrix} \right\|$$

$$= \left((-2)^2 + (-1)^2 + 2^2 + 12^2 + 11^2 \right)^{1/2}$$

$$= \left(4 + 1 + 4 + 144 + 121 \right)^{1/2}$$

$$= \sqrt{274}$$

Problem 1.4

$$\cos\theta = \left\langle \frac{u}{\|u\|}, \frac{r}{\|r\|} \right\rangle = \frac{1}{92} \langle u, r \rangle$$

$$= \frac{1}{92} \left(1(3) + (-1)0 + 5(3) + 7(-5) + 4(-7) \right)$$

$$= \frac{1}{92} \left(3 + 0 + 15 - 35 - 28 \right)$$

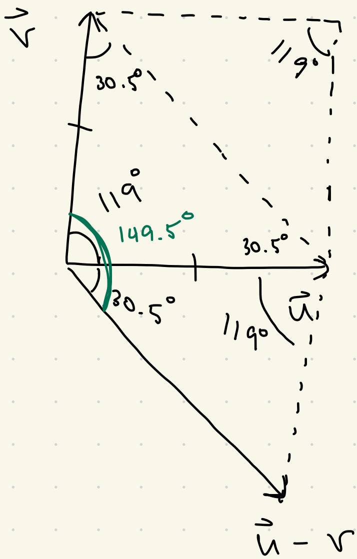
$$= \frac{-45}{92}$$

$$\theta = \cos^{-1}\left(\frac{-45}{92} \right) \approx 2.0819$$

$$\approx 119°$$

Problem 1.5

$149.5°$



Problem 2.1

$$S = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \middle| \; 2v_1 + 3v_2 - 4v_3 + 5v_3 = 0 \right\}$$

$$\vec{n} = \begin{bmatrix} 2 \\ 3 \\ -4 \\ 5 \end{bmatrix} \qquad \langle \vec{u}, \vec{r} \rangle = 0 \text{ for } \vec{r} \in S$$

Problem 2.2

$$A^T = \begin{bmatrix} 1 & 2 & 1 & -2 \\ -3 & -6 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & -4 \end{bmatrix}$$

$x_1 = -2x_2 - 2x_4$

x_2 is free

$x_3 = 4x_4$

x_4 is free

$$x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 4 \\ 1 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 4 \\ 1 \end{bmatrix} \right\}$$

Problem 2.3

By rank-nullity:

$$\text{rank}(A^T) + \dim(\text{Nul}(A^T)) = m$$

(since $A^T \in \mathbb{R}^{n \times m}$)

$$\text{rank}(A) = m - \dim(\text{Nul}(A^T))$$

$$= m - (m - \text{rank}(A^T))$$

$$= \text{rank}(A^T)$$

Problem 3.1

$$v_2' = v_2 - \hat{v}_2 = v_2 - \frac{\langle v_1, v_2 \rangle}{\langle v_1, v_1 \rangle} v_1$$

$$= \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \frac{6}{6} \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Problem 3.2

$$v_3' = v_3 - \hat{v}_3 = v_3 - \frac{\langle v_1, v_3 \rangle}{\langle v_1, v_1 \rangle} v_1$$

$$= \begin{bmatrix} -4 \\ 9 \\ 0 \\ -11 \end{bmatrix} - \frac{(-8+9+11)}{6} \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} -4 \\ 9 \\ 0 \\ -11 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix} =$$

$$\begin{bmatrix} -4 \\ 9 \\ 0 \\ -11 \end{bmatrix} + \begin{bmatrix} -4 \\ -2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -8 \\ 7 \\ 0 \\ -9 \end{bmatrix}$$

Problem 3.3

$$v_3'' = v_3' - \hat{v}_3' = v_3' - \frac{\langle v_3', v_2' \rangle}{\langle v_2', v_2' \rangle} v_2'$$

$$= v_3' - \frac{\ }{3}$$

$$= \begin{bmatrix} -8 \\ 7 \\ 0 \\ -9 \end{bmatrix} - \left(\frac{-24}{3} \right) \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -8 \\ 7 \\ 0 \\ -9 \end{bmatrix} + \begin{bmatrix} 8 \\ -8 \\ 0 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \end{bmatrix}$$

Problem 3.4

$$\langle v_1, v_2' \rangle = 2(1) + 1(-1) + 0 + (-1)(1) = 0 \checkmark$$

$$\langle v_2', v_3'' \rangle = 1(0) + (-1)(-1) + 0 + 1(-1) = 0 \checkmark$$

$$\langle v_1, v_3'' \rangle = 2(0) + 1(-1) + 0 + (-1)(-1) = 0 \checkmark$$

Problem 3.5

$$u = \frac{\langle u, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 + \frac{\langle u, v_2' \rangle}{\langle v_2', v_2' \rangle} v_2' + \frac{\langle u, v_3'' \rangle}{\langle v_3'', v_3'' \rangle} v_3''$$

$$[u]_\beta = \begin{bmatrix} 7/6 \\ -1/3 \\ 7/2 \end{bmatrix}$$