

Homework 12 Solutions

CAS CS 132

Fall 2024

Problem 1.1

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$$

Problem 1.2

$$x_1 = 2$$

$$x_2 = 1$$

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Problem 1.3

Yes. The columns of A are L.I.

Problem 2.1

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0.54 & 0.84 & 1 & 1 \\ -0.42 & 0.91 & 2 & 1 \\ -0.99 & 0.14 & 3 & 1 \\ 0.54 & -0.84 & -1 & 1 \\ -0.46 & -0.91 & -2 & 1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix} = \begin{bmatrix} 13 \\ 9 \\ 5 \\ 0 \\ 5 \\ 0 \end{bmatrix}$$

Problem 2.2

$$\beta_1 = 6.32$$

$$\beta_2 = 1.89$$

$$\beta_3 = 0.36$$

$$\beta_4 = 4.89$$

(determined using `numpy.linalg.lstsq`)

Problem 3.1

$$A = \begin{bmatrix} 3 & 3 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$$

Problem 3.2

7, this is the largest eigenvalue of A , determined using numpy.

Problem 3.3

$$A - 7I = \begin{bmatrix} -4 & 3 & 1 \\ 3 & -4 & 1 \\ 1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 \\ 3 & -4 & 1 \\ -4 & 3 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -2 \\ 0 & -7 & 7 \\ 0 & 7 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 &= x_3 \\ x_2 &= x_3 \\ x_3 &\text{ is free} \end{aligned}$$

$$\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

Problem 4.1

$$A^T A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \\ 2 & -2 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$$

$$\det(A^T A - \lambda I) = (\lambda - 17)^2 - 64 = \lambda^2 - 34\lambda + 225 \\ = (\lambda - 9)(\lambda - 25)$$

$$\lambda = 9, 25$$

$$\Sigma = \begin{bmatrix} 5 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}$$

$$A^T A - 25 I = \begin{bmatrix} -8 & 8 \\ 8 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$A^T A - 9 I = \begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$u_1 = \frac{1}{5} A \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 5/\sqrt{2} \\ 5/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$u_2 = \frac{1}{3} A \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 4/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/3\sqrt{2} \\ -1/3\sqrt{2} \\ 4/3\sqrt{2} \end{bmatrix}$$

$$u_3 = \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix} \quad \left(\text{by solving } \begin{array}{l} x + y = 0 \\ x - y + 4z = 0 \end{array} \right)$$

$$U = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{6} & -2/3 \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{6} & 2/3 \\ 0 & \frac{2\sqrt{2}}{3} & 1/3 \end{bmatrix}$$

Problem 2.2

$$A^T = (U \Sigma V^T)^T = V \Sigma^T U^T$$

$$U' = V$$

$$\Sigma' = \Sigma^T$$

$$V = U$$