Practice Final Exam

CAS CS 132: Geometric Algorithms

December 9, 2024

Name:

BUID:

- ▷ You will have approximately 120 minutes to complete this exam. Make sure to read every question, some are easier than others.
- ▷ Please do not remove any pages from the exam.
- ▷ If there is a solution box for a problem, please put your *final* solution in the box and nothing else. You should do your work outside of the box.
- ▷ We will not look at any work on the pages marked "*This page is intentionally left blank.*" You should use these pages for scratch work.

1 Interpreting Matrices

A.

$$A \sim \begin{bmatrix} 0 & 1 & 2 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Find a general-form solution for the equation $A\mathbf{x} = \mathbf{0}$. Fill in your solution by writing "is free" or "= *expression*" next to each variable below.

Solution.

x_1

x_2

x_3

x_4

B. Find a basis of $\mathsf{Nul}\, A$, where A is the matrix from the previous part.

Solution.

$$B = \begin{bmatrix} 2 & -3 & 1 & -4 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

C. Determine the characteristic polynomial of B. Your answer should be given in fully factored form.

Solution.

D. Is the matrix B from the previous part invertible? Circle your answer below and provide justification.

Solution.

◯ Yes

◯ No

Justification.

2 Roots of Matrices

Suppose A is a 2×2 matrices such that

$$A\begin{bmatrix}-3\\1\end{bmatrix} = \begin{bmatrix}3\\-1\end{bmatrix} \qquad \text{and} \qquad A\begin{bmatrix}-2\\1\end{bmatrix} = \begin{bmatrix}-16\\8\end{bmatrix}$$

A. Determine A by first determining a diagonalization of A (that is, find matrices P and D such that $A = PDP^{-1}$). Show your work and fill in your answer below.

Solution.

$P =$

$D =$

$P^{-1} =$

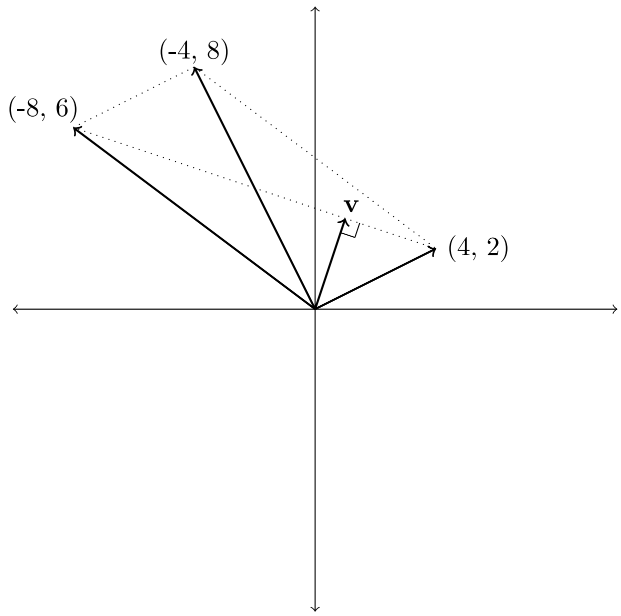
$A =$

B. Find a matrix B such that $BBB = A$. Show your work and fill in your answer below. *Hint:* B is diagonalizable.

Solution.

$B =$

3 Analytic Geometry



This picture is not be to scale.

A. Write down the two 2×2 matrices A and B which transforms the unit square into the parallelogram formed by $(0, 0)$, $(4, 2)$, $(-4, 8)$ and $(-8, 6)$. Recall that the unit square is the set of points in

$$\{(x, y) : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\}$$

Solution.

$A =$ $B =$

B. Compute the determinants of A and B. For partial credit, write down the general equation for the determinant of a 2×2 matrix.

Solution.

$\det A =$ $\det B =$

C. Find the distance between $(-8, 6)$ and $(4, 2)$.

Find the length of the vector $\mathbf{v}$, whose endpoint is on the line segment between $(-8, 6)$ and $(4, 2)$ and which forms a $90°$ angle with that line segment.

Use these values to determine the area of the triangle formed by $(0, 0)$, $(4, 2)$ and $(-8, 6)$.

Show your work and fill in your answer below.

Solution.

$$\mathsf{dist}((-8, 6), (4, 2)) =$$

$$\|\mathbf{v}\| =$$

$$\mathsf{area} =$$

4 True/False Questions

A. For any two symmetric matrices A and B, if AB is defined then $(AB)^T = A^T B^T$.

- ◯ True
- ◯ False

B. For any $n \times n$ orthogonal matrix A (that is, the columns of A are orthonormal) and any vector $\mathbf{v} \in \mathbb{R}^n$, it must be that $\|A\mathbf{v}\| = \|A^2\mathbf{v}\|$.

- ◯ True
- ◯ False

C. For any $m \times n$ matrix A, there is an eigenbasis of $\mathbb{R}^n$ for $A^T A$.

- ◯ True
- ◯ False

D. For any matrix A and quadratic form $Q(\mathbf{x})$, if $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$, then A is symmetric.

- ◯ True
- ◯ False

E. For any set X of vectors in $\mathbb{R}^n$ and any vector $\mathbf{v} \in \mathbb{R}^n$, if X is linearly dependent and $\mathbf{v} \in X$, then $\mathbf{v}$ can be written as a linear combination of the vectors in X not including $\mathbf{v}$.

- ◯ True
- ◯ False

F. For any two matrices A and B, if $A + B$ is defined then $A + B = B + A$.

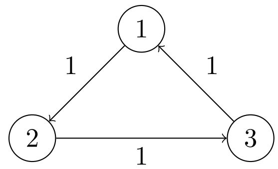
- ◯ True
- ◯ False

G. For any square matrix A, if $A\mathbf{x} = \mathbf{b}$ has a unique least squares solution for each choice of $\mathbf{b}$, then A is invertible.

- ◯ True
- ◯ False

5 Stochastic Matrices

A. Consider the following state diagram.



If an edge between two states is not present, the probability of transitioning in a single step is 0. For example, the probability of transitioning from state 3 to state 2 in a single step is 0.

Write down the transition matrix T for the above diagram. You should write T such that the ith column of T corresponds to the transitions from state i. (In the following parts, T refers to the matrix you determined here.)

Solution.

$$T =$$

B. Compute T^2, T^3 and T^{2023}. Write your answer below.

Solution.

$$T^2 =$$

$$T^3 =$$

$$T^{2023} =$$

C. Is T regular? Circle your answer below and provide justification. For partial credit, define regularity.

Solution.

◯ Yes

◯ No

Justification.

D. Does T have a unique steady-state distribution? Circle your answer below and provide justification.

Solution.

◯ Yes

◯ No

Justification.

6 Singular Values

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 3 & 1 & 1 & -2 \end{bmatrix}$$

Determine the singular values of the above matrix.

Solution.