Midterm

CAS CS 132: Geometric Algorithms

October 12, 2023

Name:

BUID:

Location:

- You will have approximately 70 minutes to complete this exam.
- Make sure to read every question, some are easier than others.
- Please write your name and BUID **on every page**.
- Your work will only be seens on the **front facing pages**. You may write on the backs of pages if you are using a pencil, but you will not be evaluated on what is written there.

1 Span and Linear Independence

Consider the following vectors in $\mathbb{R}^4$.

$$\mathbf{v}_1 = \begin{bmatrix} -3 \\ 4 \\ 3 \\ 7 \end{bmatrix} \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 2 \end{bmatrix} \mathbf{v}_3 = \begin{bmatrix} 0 \\ -2 \\ -1 \\ -1 \end{bmatrix} \mathbf{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -2 \end{bmatrix}$$

A. (6 points) Determine if $\mathbf{v}_1$ is in $\mathsf{span}\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$. Justify your answer. In particular, if $\mathbf{v}_1$ is in $\mathsf{span}\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$, then write $\mathbf{v}_1$ as a linear combination of $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$.

B. (6 points) Determine if the vectors $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ are linearly independent. Justify your answer. In particular, if they are linearly dependent, then write a dependence relation for them (that is, write the zero vector $\mathbf{0}$ as a linear combination of the vectors $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$).

C. (6 points) Determine if the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent. Justify your answer. In particular, if they are linearly dependent, then write a dependence relation for them.

Solution.

Extra Page.

2 True/False Questions

For each of the following statements, determine if they are true or false. You do not have to show your work or justify your answer. You just have to write "true" or "false."

A. (2 points) For any $m \times n$ matrix A, if $m > n$ then the columns of A must be linearly independent.

B. (2 points) Every matrix has a unique echelon form.

C. (2 points) Any set of distinct standard basis vectors in $\mathbb{R}^n$ is linearly independent.

D. (2 points) For any 20×24 matrix A, the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

E. (2 points) For any nonzero real values a and b, the matrix $\begin{bmatrix} a & 2a \\ b & 2b \end{bmatrix}$ has a pivot in every column and every row.

F. (2 points) To show that a transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is linear, it is enough to show that $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ for any two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^m$.

G. (2 points) If A is the augmented matrix of an inconsistent system, then A has a pivot in its last (rightmost) column.

H. (2 points) For any vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ in $\mathbb{R}^n$, if $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent, then so is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Solution.

A.

B.

C.

D.

E.

F.

G.

H.

Extra Page.

3 Inner Products and Matrix Equations

Consider the following matrix and vector.[1]

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ -2 & 1 & -2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

A. (3 points) Compute the following matrix-vector multiplications.

$$\begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \left(A \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right)$$

B. (6 points) Write down a general form solution for the solution set of the matrix equation $A\mathbf{x} = \mathbf{b}$.

C. (6 points) Use your solution to the previous part to find a **nonzero** vector $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ such that

$$\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \left(A \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \right) = 0$$

Solution.

[1] Credit to Vishesh Jain for suggesting a version of this problem.

Extra Page.

4 Linear Transformations

(5 points) Consider the following linear transformation T.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} x_1 + 2x_2 - x_3 \\ x_3 \\ 2x_3 \end{bmatrix}$$

Find a set of linearly independent vectors which span the range of T. *Hint.* First find the matrix implementing T.

Solution.

5 Matrix Equations

Consider the following matrices.

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \qquad \mathbf{e}_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

A. (3 points) Explain why the columns of B span $\mathbb{R}^6$.

B. (7 points) Find a solution to the equation $B\mathbf{x} = \mathbf{e}_6$.

C. **(3 points, Extra Credit)** Find a solution to the equation $C\mathbf{x} = \mathbf{e}_6$.

Solution.

Extra Page.