Midterm Exam

CAS CS 132: Geometric Algorithms

October 22, 2024

Name:

BUID:

- ▷ You will have approximately 75 minutes to complete this exam. Make sure to read every question, some are easier than others.
- ▷ Please do not remove any pages from the exam.
- ▷ If there is a solution box for a problem, please put your *final* solution in the box and nothing else. You should do your work outside of the box.
- ▷ We will not look at any work on the pages marked "*This page is intentionally left blank.*" You should use these pages for scratch work.

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1 Systems of Linear Questions

$$x - 2y - 7z = -3$$
$$2x + y + 6z = 14$$
$$x + y + 5z = 9$$

A. (2 points) Write down the augmented matrix of the above system of linear equations.

Solution.

B. (7 points) Using row reductions, determine the row-reduced echelon form of the augmented matrix from the previous part. You must write the row reductions you use and the intermediate matrices of those reductions below. You are allowed to apply multiple row reduction in a single step if it is sufficiently clear.

The final RREF.

Problem 1B (continued).

C. (3 points) Write down the general form solution of the system of linear equations at the beginning of this problem. *For partial credit, write down any matrix in row-reduced echelon form with two pivot positions and write down its general form solution.*

Solution.

2 True/False Questions

Determine if each of the following statements is **True** or **False**. Bubble in your answers below. You do not need to show your work.[1]

A. (2 points) For any square matrices A and B in $\mathbb{R}^{n\times n}$, if $A = A^T$ and $B = B^T$ then $AB = (AB)^T$.

○ True
○ False

B. (2 points) For any matrix A in $\mathbb{R}^{m\times n}$ and distinct vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in $\mathbb{R}^n$, if $A\mathbf{v}_1 = \mathbf{0}$ and $A\mathbf{v}_2 = \mathbf{0}$, then the columns of A are linearly dependent.

○ True
○ False

C. (2 points) For any matrix A, if the matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto, then it is also one-to-one.

○ True
○ False

D. (2 points) For any vector $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ in $\mathbb{R}^n$, $\mathsf{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \mathsf{span}\{\mathbf{v}_1 + \mathbf{v}_3, \mathbf{v}_2\}$.

○ True
○ False

E. (2 points) For any real numbers a and b, the matrix $\begin{bmatrix} a & a^2 \\ b & ab \end{bmatrix}$ is singular (i.e., not invertible).

○ True
○ False

F. (2 points) For any matrix A in $\mathbb{R}^{m\times n}$, if the columns of A^T are linearly dependent, then the columns of A do not span $\mathbb{R}^m$.

○ True
○ False

G. (2 points) If the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ are linearly independent then $\mathbf{v}_1 \notin \mathbb{R}^3$.

○ True
○ False

H. (2 points) If A is the augmented matrix of a linear system and it has a pivot position in every column, then the system is inconsistent.

○ True
○ False

I. (2 points) If the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are linearly dependent, then $\mathbf{v}_3 \in \mathsf{span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

○ True
○ False

[1] Credit to Ieva Sagaitis for suggesting some parts in this question.

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3 Echelon Forms

(5 points) Suppose that A is the augmented matrix of a system of linear equations with **three variables** and **three equations**. Furthermore, suppose that A has **two pivot positions** and that $(-3, 0, 1)$ are $(-3, 1, 1)$ are solutions to the system. Write down the row-reduced echelon form of A.

Solution.

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4 Linear Transformation

Consider the transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given as follows.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} 5x_1 + 7x_2 + 10x_3 \\ 2x_1 - x_2 + 4x_3 \\ -3x_2 \end{bmatrix}$$

A. (4 points) Determine the matrix which implements T.

Solution.

B. (4 points) Determine if T is one-to-one and/or onto. Bubble in your answer below. You must also justify your answer **without doing row operations.**

- ◯ onto, but **not** one-to-one
- ◯ one-to-one, but **not** onto
- ◯ **both** one-to-one and onto
- ◯ **neither** one-to-one nor onto

Justification.

C. (4 points) Determine a set of **linearly independent** vectors whose span is the range of T. Recall that the range of a transformation is set of all possible images, i.e., $\mathsf{ran}(T) = \{T(\mathbf{v}) : \mathbf{v} \in \mathbb{R}^3\}$.

Solution.

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5 Matrix Inverses

A. (6 points) Suppose that A is a 3×3 matrix such that applying the following row operations to A yields the identity matrix.

$$
\begin{aligned}
R_2 &\leftarrow R_2 - R_1 \\
R_3 &\leftarrow R_3 - R_1 \\
R_3 &\leftarrow R_3 - 4R_2 \\
R_1 &\leftarrow R_1 - R_3 \\
R_1 &\leftarrow R_1 + 2R_2
\end{aligned}
$$

Determine the inverse of A. Show your work below.

Solution.

Problem 5A (continued).

B. (6 points) Suppose that A is a matrix such that

$$A^T \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$$\det(A) = 1$$

Determine the inverse of A. Show your work below. *Hint.* $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$

Solution.

Problem 5B (continued).