Linear Equations

Geometric Algorithms
Lecture 1

CAS CS 132

Objectives

1. Motivation
2. Definitions
3. Solve systems of linear equations

Keywords

Systems of linear equations

Solutions

Coefficient matrix

Augmented matrix

Elimination and Back-substitution

Replacement, interchange, scaling

Row Equivalence

(In)consistency

Objectives

1. **Motivation**
2. Definitions
3. Solve systems of linear equations

Motivation

1. Lines and line intersections
2. An example from chemistry

Motivation

1. **Lines and line intersections**
2. An example from chemistry

Lines (Slope-Intercept Form)

$$y = mx + b$$

Lines (Slope-Intercept Form)

$$y = mx + b$$

slope y-intercept

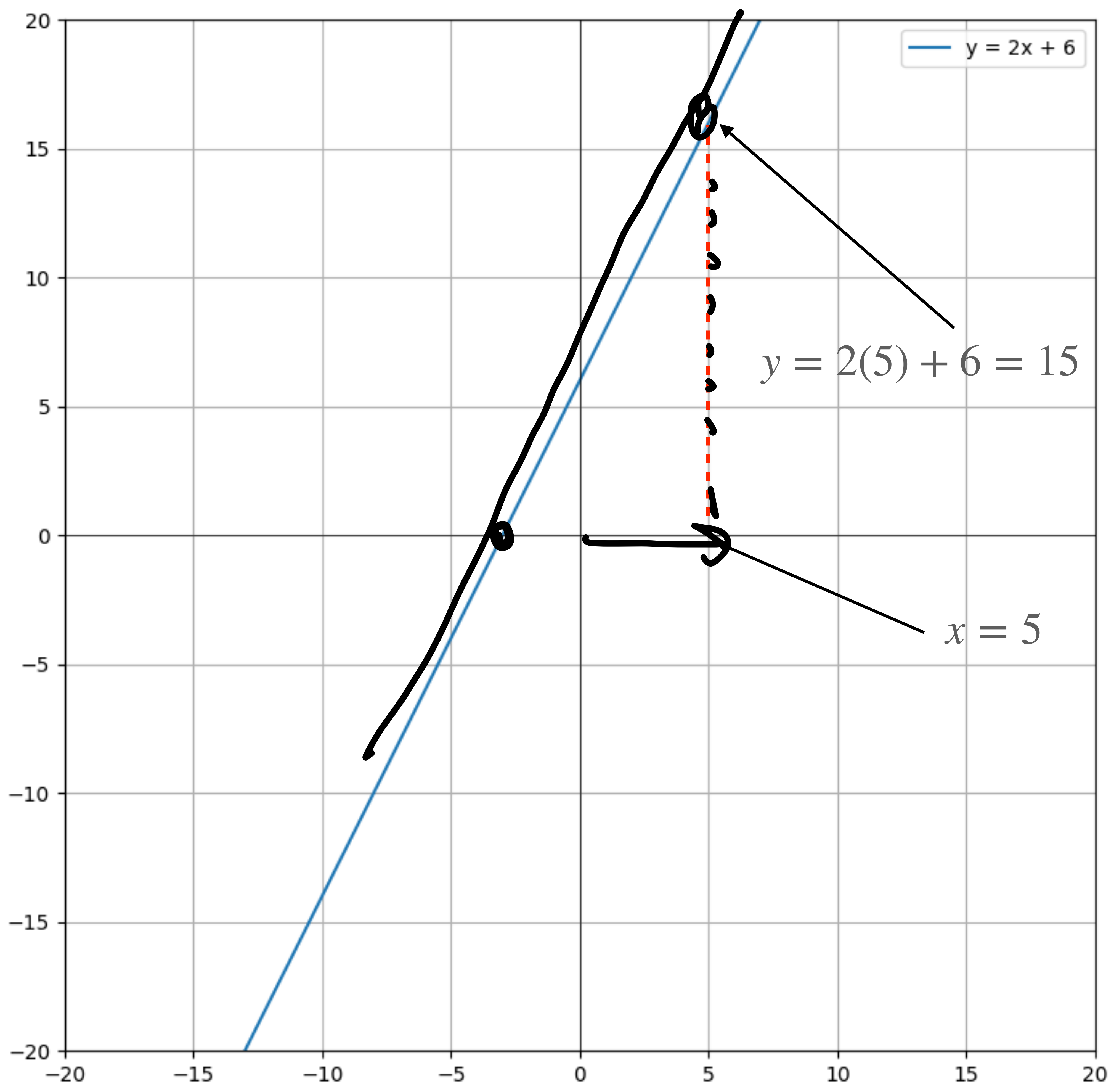
Lines (Slope-Intercept Form)

$$y = mx + b$$

slope   y-intercept

Given a value of x, I can compute a value of y

Lines (Graph)



$y = 2x + 6$

$x = -3$

$y = 2(-3) + 6$

$y = 0$

Lines (General Form)

$$ax + by = c$$

Lines (General Form)

$$ax + by = c$$

x-intercept: $\frac{c}{a}$

Lines (General Form)

$$ax + by = c$$

x-intercept: $\frac{c}{a}$

y-intercept: $\frac{c}{b}$

Lines (General Form)

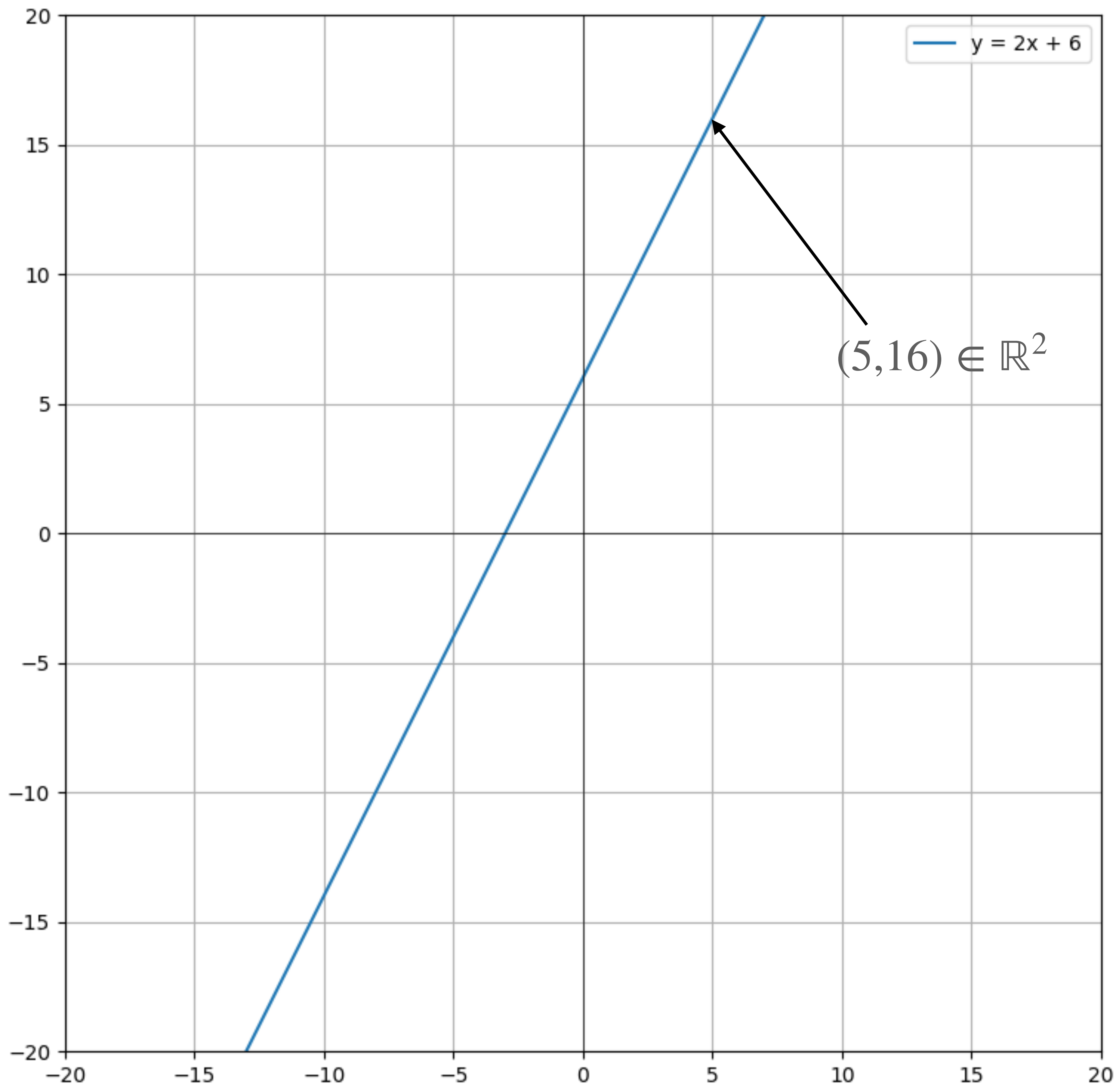
$$ax + by = c$$

x-intercept: $\frac{c}{a}$

y-intercept: $\frac{c}{b}$

What values of x and y make the equality hold?

Lines (Graph)



$$\{(x, y) : (-2)x + y = 6\}$$

(0, 6)

(-3, 0)

Lines

`slope-int → general`

$$(-m)x + y = b$$

`general → slope-int`

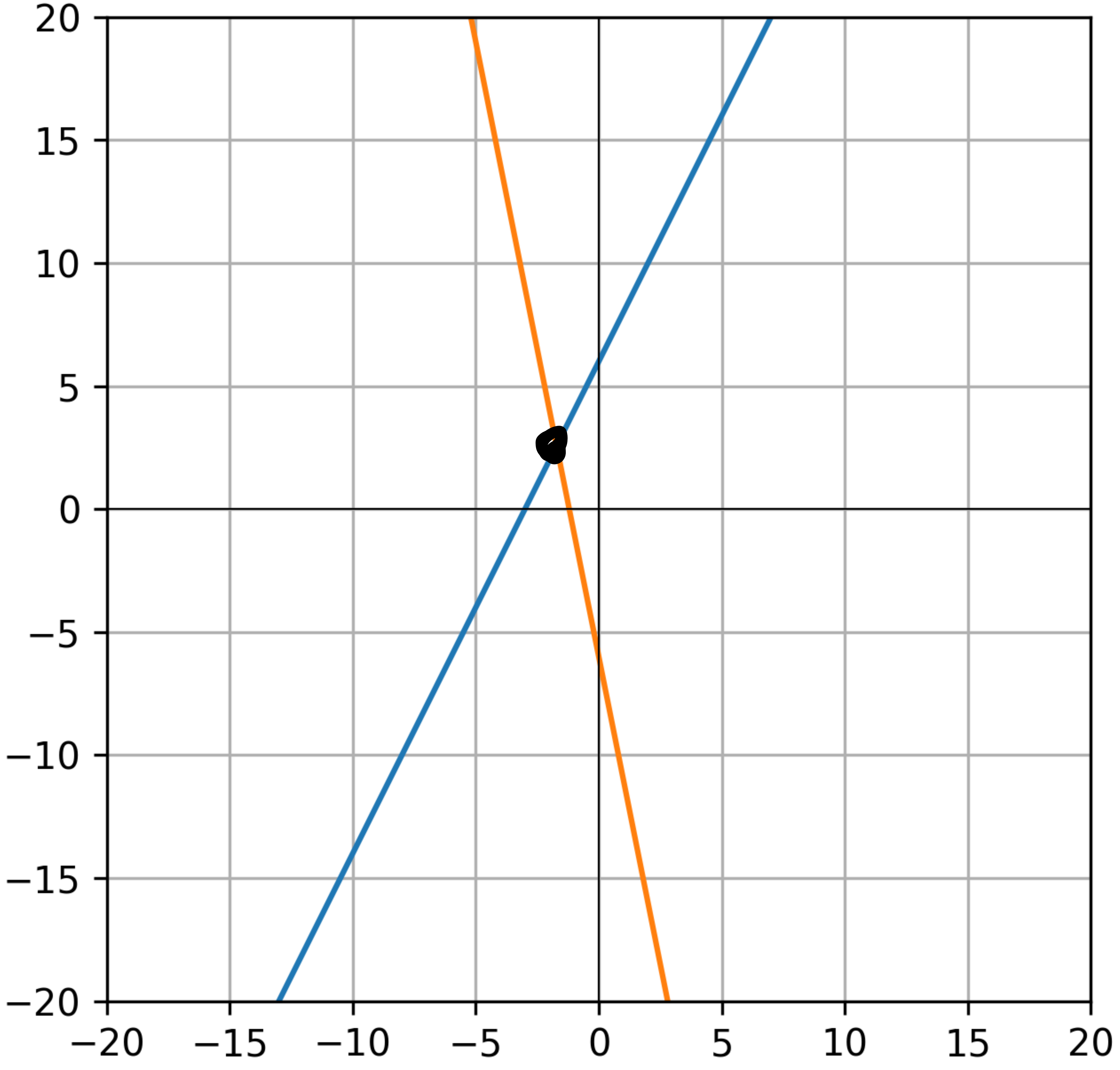
$$y = \left(\frac{-a}{b}\right)x + \frac{c}{b}$$

Line Intersection

$$y = m_1 x + b_1$$
$$y = m_2 x + b_2$$

Question. Given two lines, where do they intersect?

Line Intersection (Graph)



Line Intersection (Alternative)

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

Question. Given two (general form) lines, what values of x and y satisfy ***both*** equations?

Line Intersection (Alternative)

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

Question. Given two (general form) lines, what values of x and y satisfy ***both*** equations?

This is the same question

Motivation

1. ~~Lines and line intersections~~
2. **An example from chemistry**

Example: Balancing Chemical Equations

$$C_6H_{12}O_6 \rightarrow C_2H_5OH + CO_2$$

Glucose Ethanol

Example: Balancing Chemical Equations

$$C_6H_{12}O_6 \rightarrow C_2H_5OH + CO_2$$

Glucose Ethanol

```
We want to know how much ethanol is produced by
fermentation (for science)
```

Example: Balancing Chemical Equations

$$\alpha\, C_6H_{12}O_6 \rightarrow \beta\, C_2H_5OH + \gamma\, CO_2$$

Glucose Ethanol

We want to know how much ethanol is produced by fermentation (for science)

The number of atoms has to be preserved on each side of the equation

Balancing Chemical Equations

$$\alpha C_6H_{12}O_6 \rightarrow \beta C_2H_5OH + \gamma CO_2$$

Glucose Ethanol

Balancing Chemical Equations

$$\alpha C_6H_{12}O_6 \rightarrow \beta C_2H_5OH + \gamma CO_2$$

Glucose   Ethanol

$$6\alpha = 2\beta + \gamma \qquad \text{(C)}$$

$$12\alpha = 6\beta \qquad \text{(H)}$$

$$6\alpha = \beta + 2\gamma \qquad \text{(O)}$$

Balancing Chemical Equations

$$\alpha C_6H_{12}O_6 \rightarrow \beta C_2H_5OH + \gamma CO_2$$

Glucose Ethanol

$$6\alpha - 2\beta - \gamma = 0 \quad \text{(C)}$$

$$12\alpha - 6\beta = 0 \quad \text{(H)}$$

$$6\alpha - \beta - 2\gamma = 0 \quad \text{(O)}$$

Objectives

1. ~~Motivation~~
2. **Definitions**
3. Solve systems of linear equations

Defining Systems of Linear Equations

1. Linear equations
2. Systems of linear equations
3. Consistency
4. Matrix representations

Defining Systems of Linear Equations

1. **Linear equations**
2. Systems of linear equations
3. Consistency
4. Matrix representations

Linear Equations

Definition. A ***linear equation*** in the variables $x_1, x_2, \ldots, x_n$ is an equation ~~of~~ that can be expressed in the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

where $a_1, a_2, \ldots, a_n, b$ are real numbers ($\mathbb{R}$)

Linear Equations

Definition. A ***linear equation*** in the variables $x_1, x_2, \ldots, x_n$ is an equation of the form

coefficients

$$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = b$$

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Linear Equations

Definition. A ***linear equation*** in the variables $x_1, x_2, \ldots, x_n$ is an equation of the form

unknowns

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

where $a_1, a_2, \ldots, a_n, b$ are real numbers ($\mathbb{R}$)

Examples

$$3x + 2y = 6$$ ✓

$$\sqrt{x} = 5$$ ✗

$$x^2 + 3y + \log z = 2$$ ✗

$$3x = 6 - 2y$$ ✓

$$\pi x + \sqrt{2} y + 0.1 z = e$$ ✓

Linear Equations (Point sets)

Linear equations describe ***point sets***:

$$\{(s_1, s_2, \ldots, s_n) \in \mathbb{R}^n : a_1 s_1 + a_2 s_2 + \ldots + a_n s_n = b\}$$

Linear Equations (Point sets)

$\mathbb{R}$: line

$\mathbb{R}^2$: plane

$\mathbb{R}^3$: space

$\mathbb{R}^4$: ? $\mathbb{R}^{100}$: ?

Linear equations describe ***point sets***:

$$\{(s_1, s_2, \ldots, s_n) \in \mathbb{R}^n : a_1 s_1 + a_2 s_2 + \ldots + a_n s_n = b\}$$

The collections of numbers such that the equation holds.

Examples

$$2x + 3y + 4z = 5$$

$(1/2, 0, 1)$

$\{(x, y, z) : 2x + 3y + 4z = 5\}$

$(1, 1, 0)$

$$2(1) + 3(1) + 4(0) = 5$$

$(0, 4/3, 1/4)$

$$2(0) + 3(4/3) + 4(1/4) = 5$$

Linear Equations (Geometrically)

If a 2D linear equation is a ***line*** then a 3D linear equation is...

$$2x + 3y + 4z = 5$$

Linear Equations (Geometrically)

If a 2D linear equation is a ***line*** then a 3D linear equation is...

Not a line...

Linear Equations (Geometrically)

If a 2D linear equation is a ***line*** then a 3D linear equation is...

Linear Equations (Geometrically)

If a 2D linear equation is a ***line*** then a 3D linear equation is...

A plane(!)

demo

Example 1

$$0x + 0y + z = 5$$

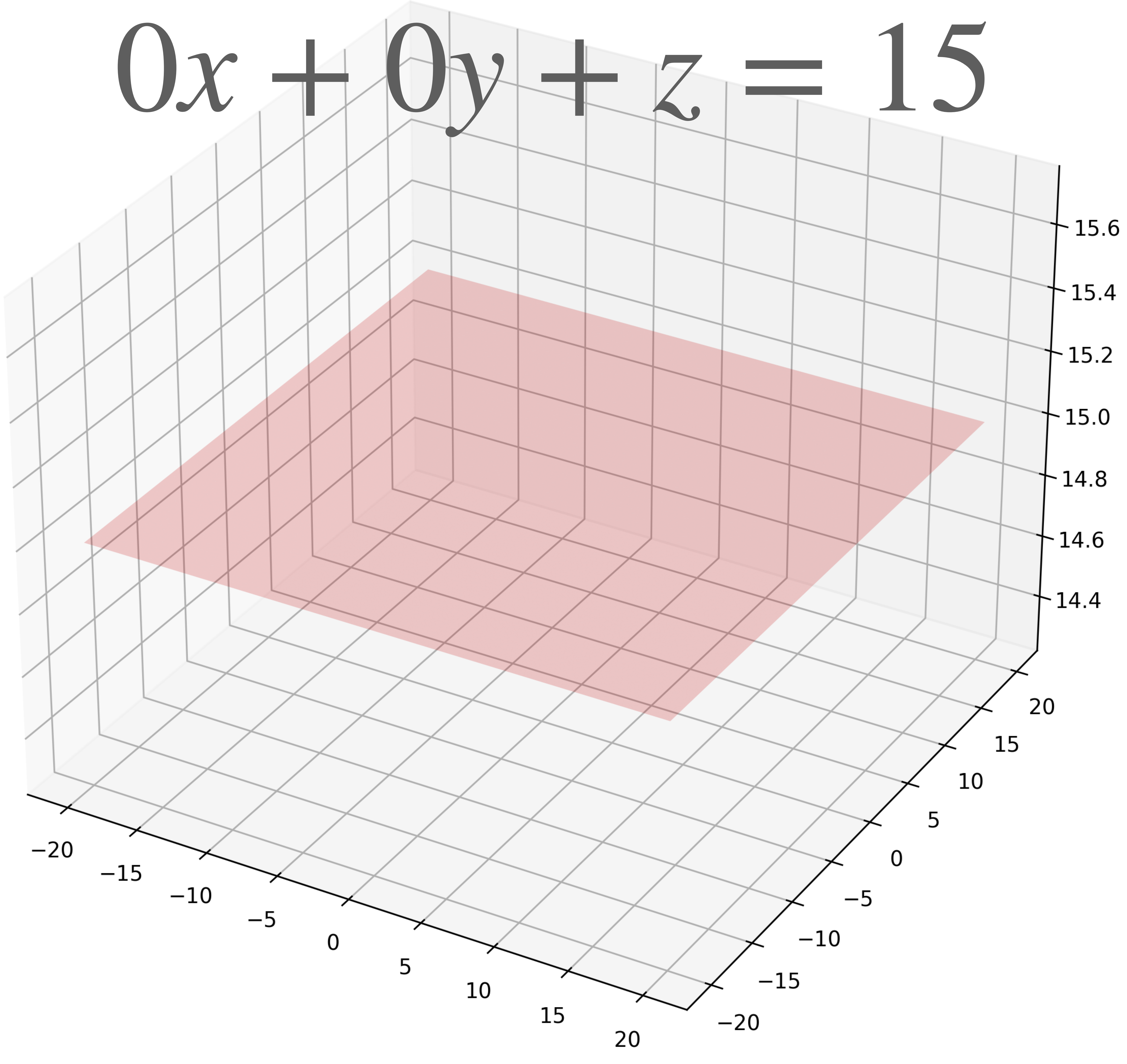
This equation describes the solution set

$$\{(x, y, z) : z = 5\}$$

so x and y can be whatever we want

Example 1

$$0x + 0y + z = 15$$



demo

Example 2

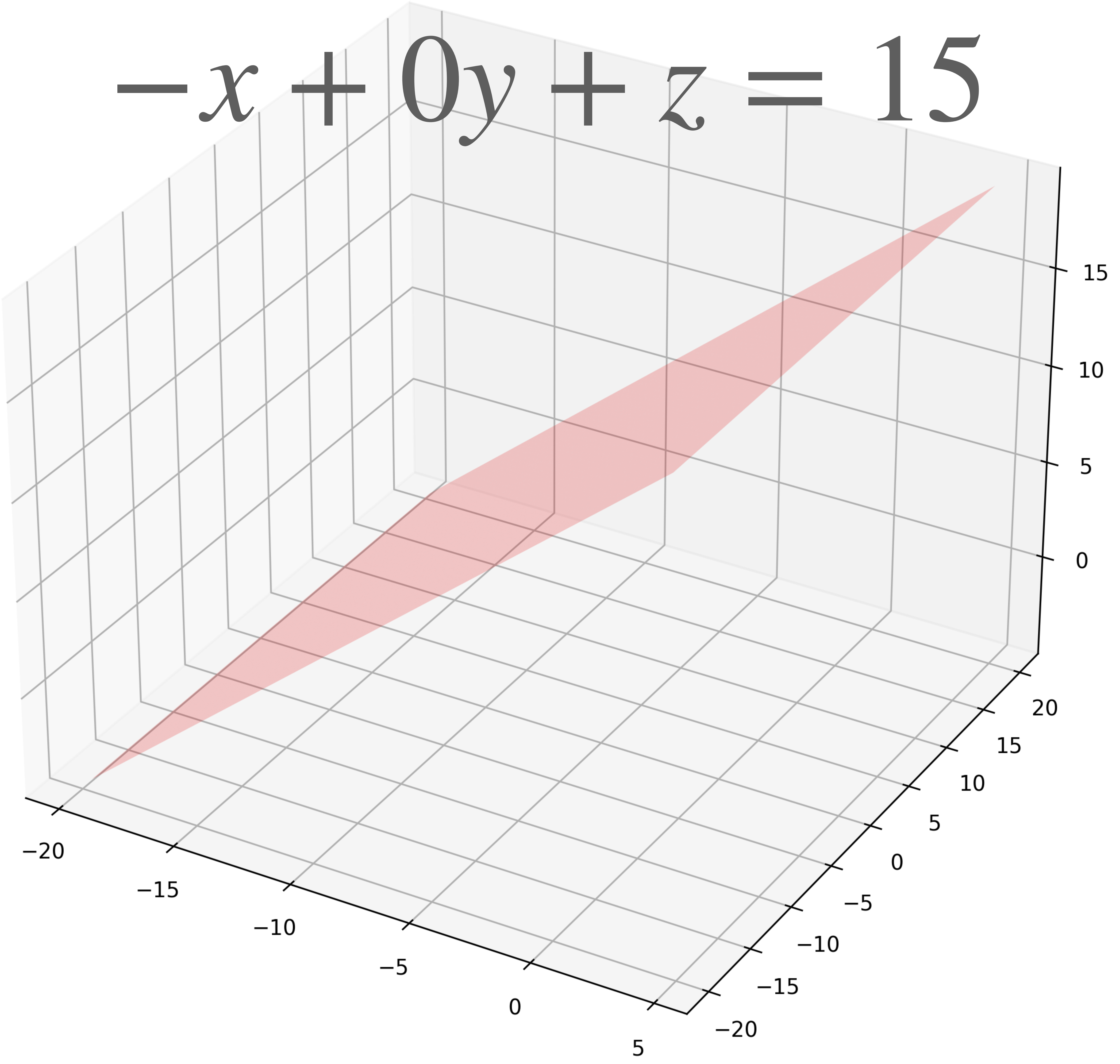
$$-x + 0y + z = 5$$

This equation describes the point set

$$\{(x, y, z) : z = x + 5\}$$

so y can be whatever we want

Example 2



demo

Example 3

$$-x + -y + z = 5$$

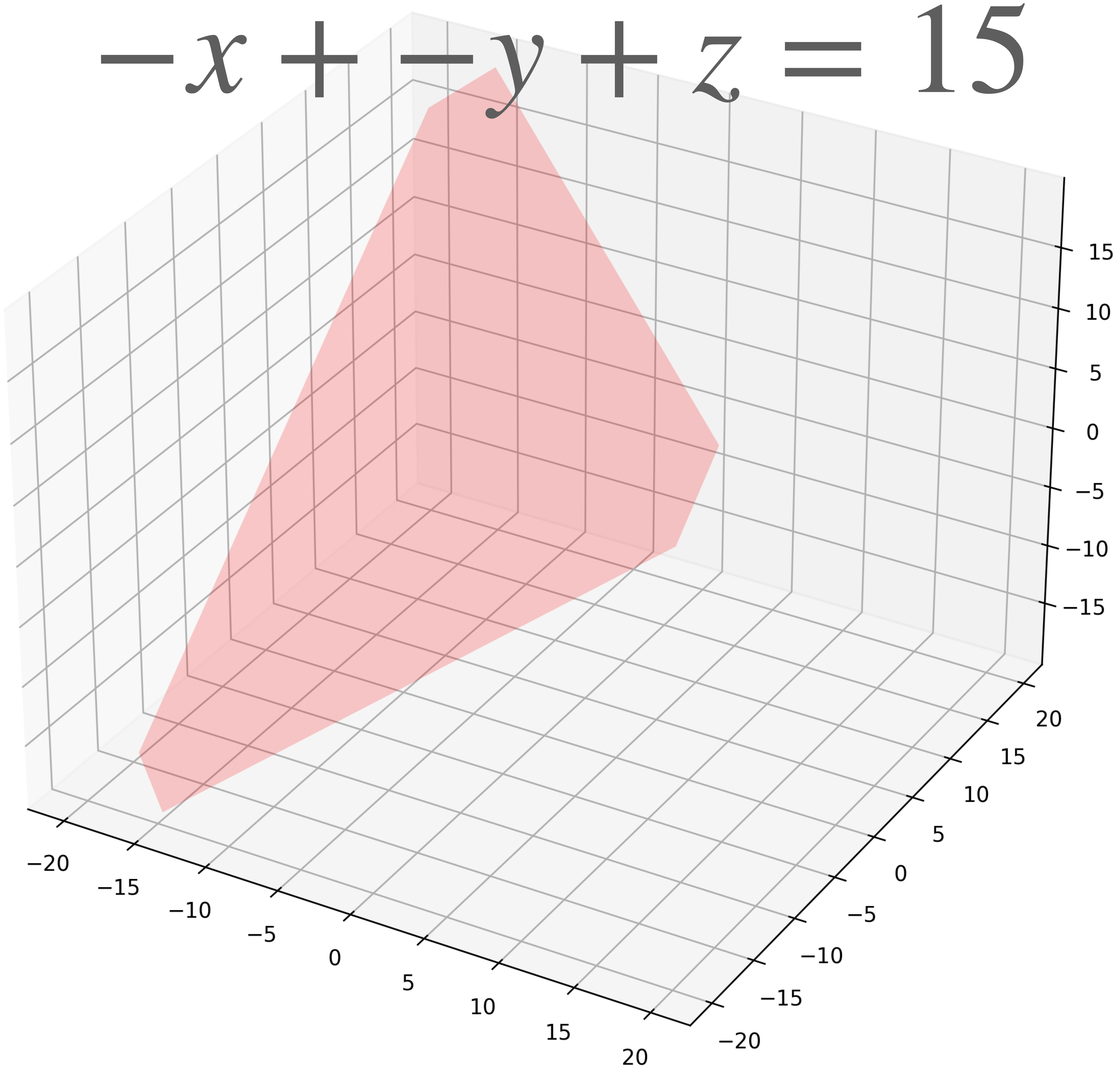
This equation describes the solution set

$$\{(x, y, z) : z = x + y + 5\}$$

so all variables depend on each other

Example 3

$$-x + -y + z = 15$$



demo

XYZ-intercepts

$$ax + by + cz = d$$

Just like with lines, we can define

x-intercept: $\frac{d}{a}$ y-intercept: $\frac{d}{b}$ z-intercept: $\frac{d}{c}$

These three points define the plane

Question

$5x + 3y + 0z = 5$

$0x + 0y = 5$

I just lied.

$0 = 5$

Give an example of a linear equation that defines a plane with an x-intercept and y-intercept but no z-intercept

$\{(x, y, z) : 5 = 5\}$

$0x + 0y + 5 = 5$

$5 = 5$

Answer

$$x + y = 1$$

z - coefficient must be 0.

Hyperplanes

Hyperplanes

after three dimensions, we can't visualize planes

Hyperplanes

after three dimensions, we can't visualize planes

the point set of a linear equation is called a ***hyperplane***

Hyperplanes

after three dimensions, we can't visualize planes

the point set of a linear equation is called a ***hyperplane***

Theme of the course: Hyperplanes "behave" like 3D planes in many respects

Defining Systems of Linear Equations

1. ~~Linear equations~~
2. **Systems of linear equations**
3. Consistency
4. Matrix representations

Systems of Linear Equations

$x + y = 5$
$x + z = 6$

Definition. A ***system of linear equations*** is just a collection of linear equations <u>over the same variables</u>.

$x + y + 0z = 5$
$x + 0y + z = 6$

Systems of Linear Equations

Definition. A ***system of linear equations*** is just a collection of linear equations <u>over the same variables</u>.

Definition. A ***solution*** to a system is a point that satisfies all its equations <u>simultaneously</u>

Example

linear system:

$$0z + x + 2y = 1$$
$$-x - y - z = -1$$
$$2x + 6y - z = 1$$

$3 + 2(-1) = 1$ ✓

$-3 + 1 + 1 = -1$ ✓

$2(3) + 6(-1) - (-1) = 1$ ✓

solution: $(3, -1, -1)$

System of Linear equations (General-form)

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}
$$

System of Linear equations (General-form)

$$
\begin{aligned}
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&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}
$$

Does a system have a solution?

How many solutions are there?

What are its solutions?

Defining Systems of Linear Equations

1. Linear equations
2. Systems of linear equations
3. **Consistency**
4. Matrix representations

Consistency

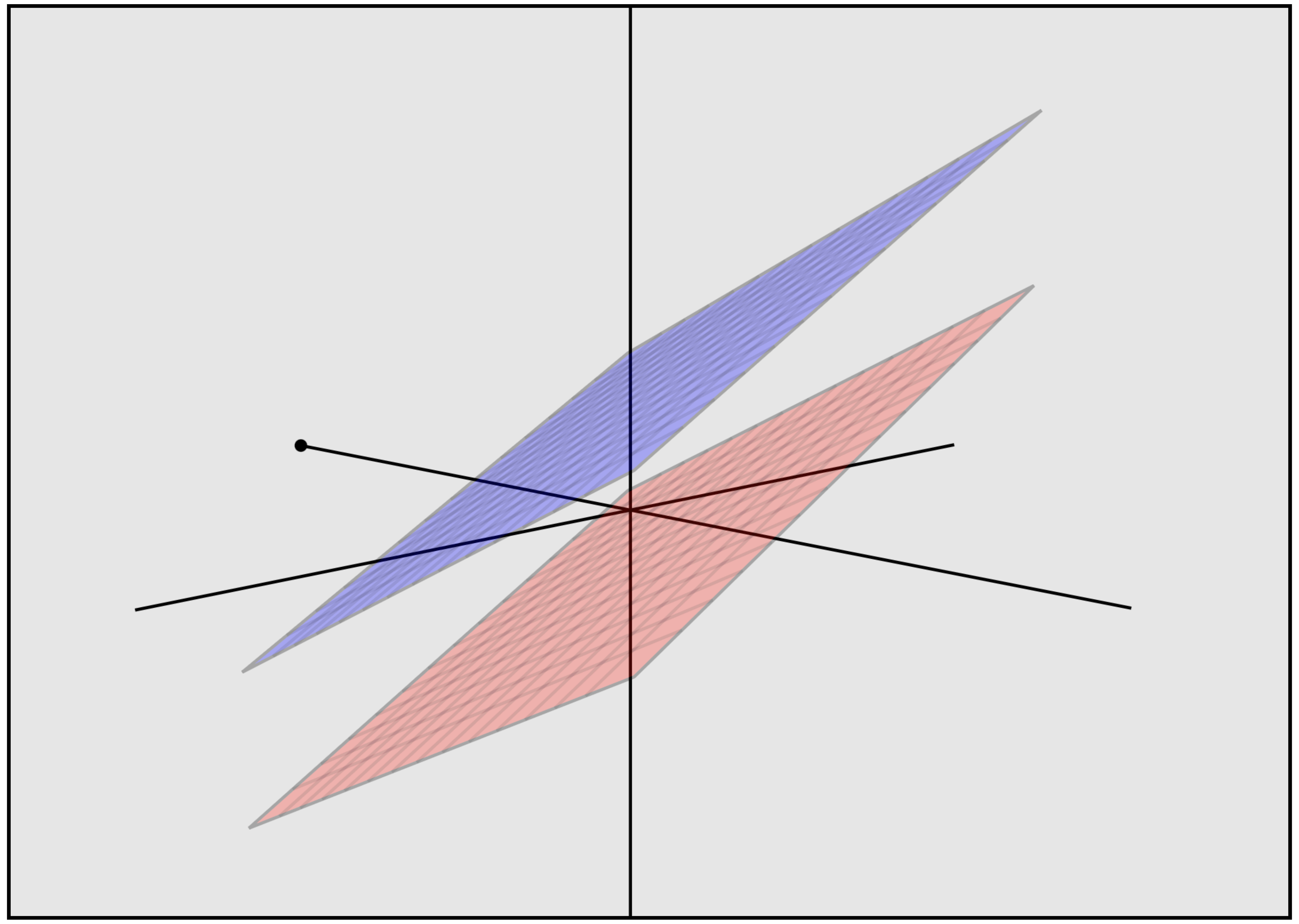
Definition. A system of linear equations is ***consistent*** if it has a solution

It is ***inconsistent*** if it has no solutions

Example

$$x + y + z = 1$$

$$x + y + z = -1$$



Number of Solutions

```
zero the system is inconsistent

one  the system has a unique solution

many the system has infinity solutions
```

Number of Solutions

`zero the system is inconsistent`

`one  the system has a unique solution`

`many the system has infinity solutions`

These are the **only** options

Defining Systems of Linear Equations

1. ~~Linear equations~~
2. ~~Systems of linear equations~~
3. ~~Consistency~~
4. **Matrix representations**

Matrix Representations

always writing down the unknowns is exhausting

we will write down linear systems as ***matrices***, which are just 2D grids of numbers with fixed width and height

Matrix Representations

always writing down the unknowns is exhausting

we will write down linear systems as ***matrices***, which are just 2D grids of numbers with fixed width and height

a matrix is just a representation

Matrix Representations

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}
$$

Matrix Representations

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

Matrix Representations

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

augmented matrix

Matrix Representations

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

`coefficient matrix`

Matrix Representations

$$6\alpha - 2\beta - \gamma = 0 \qquad \text{(C)}$$

$+0\gamma$

$$12\alpha - 6\beta = 0 \qquad \text{(H)}$$

$$6\alpha - \beta - 2\gamma = 0 \qquad \text{(O)}$$

Matrix Representations

$$\begin{bmatrix} 6 & -2 & -1 & 0 \\ 12 & -6 & 0 & 0 \\ 6 & -1 & -2 & 0 \end{bmatrix}$$

More Examples

$$\begin{aligned} 2x + 3y &= 4 \\ 5x &= 6 \end{aligned} \rightsquigarrow \begin{bmatrix} 2 & 3 & 4 \\ 5 & 0 & 6 \end{bmatrix}$$

Objectives

1. ~~Motivation~~
2. ~~Definitions~~
3. **Solve systems of linear equations**

Solving Systems of Linear Equations

1. Some simple examples
2. Elimination and Back-Substitution
3. Row Equivalence

Solving Systems of Linear Equations

1. **Some simple examples**
2. Elimination and Back-Substitution
3. Row Equivalence

Solving Systems of Linear Equations

1. Some simple examples
2. Elimination and Back-Substitution
3. Row Equivalence

We'll only consider systems with unique solutions for now.

Solving Systems with Two Variables

$$2x + 3y = -6$$
$$4x - 5y = 10$$

Solving Systems with Two Variables

$$2x + 3y = -6$$
$$4x - 5y = 10$$

The Approach

Solving Systems with Two Variables

$$2x + 3y = -6$$
$$4x - 5y = 10$$

The Approach

Solve for x in terms of y in EQ1

Solving Systems with Two Variables

$$2x + 3y = -6$$
$$4x - 5y = 10$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Solving Systems with Two Variables

$$2x + 3y = -6$$
$$4x - 5y = 10$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Let's work through it...

$$2x + 3y = -6$$
$$4x - 5y = 10$$

Solving Systems with Two Variables

$$2x = (-3)y - 6$$

$$4x - 5y = 10$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = (-3/2)y - 3$$

$$4x - 5y = 10$$

The Approach

- **Solve for x in terms of y in EQ1**
- Substitute result for x in EQ2 and solve for y
- Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = (-3/2)y - 3$$

$$4((-3/2)y - 3) - 5y = 10$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = (-3/2)y - 3$$

$$-6y - 12 - 5y = 10$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = (-3/2)y - 3$$

$$-11y = 22$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = (-3/2)y - 3$$
$$y = -2$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = (-3/2)(-2) - 3$$
$$y = -2$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = 3 - 3$$
$$y = -2$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = 0$$
$$y = -2$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

another perspective...

Solving Systems with Two Variables

$$2x + 3y = -6$$
$$4x - 5y = 10$$

The Approach

Eliminate x *from the EQ2 and solve for* y

Eliminate y *from EQ1 and solve for* x

Let's work through it again...

$$2x + 3y = -6$$
$$4x - 5y = 10$$

Solving Systems of Linear Equations

1. ~~Some simple examples~~
2. **Elimination and Back-Substitution**
3. Row Equivalence

Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$6x + 5y + 9z = -4$$

Solving Systems with Three Variables

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The Approach

Solving Systems with Three Variables

$$x - 2y + z = 5$$

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$$6x + 5y + 9z = -4$$

The Approach

Eliminate x from the EQ2 and EQ3

Solving Systems with Three Variables

$$x - 2y + z = 5$$
$$2y - 8z = -4$$
$$6x + 5y + 9z = -4$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Solving Systems with Three Variables

$$x - 2y + z = 5$$
$$2y - 8z = -4$$
$$6x + 5y + 9z = -4$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Solving Systems with Three Variables

$$x - 2y + z = 5$$
$$2y - 8z = -4$$
$$6x + 5y + 9z = -4$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$x - 2y + z = 5$$
$$2y - 8z = -4$$
$$6x + 5y + 9z = -4$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Elimination

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Back-Substitution

Let's work through it

$$x - 2y + z = 5$$
$$2y - 8z = -4$$
$$6x + 5y + 9z = -4$$

Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$6(5 + 2y - z) + 5y + 9z = -4$$

The Approach

- **Eliminate x from the EQ2 and EQ3**
- Eliminate y from EQ3
- Eliminate z from EQ2 and EQ1
- Eliminate y from EQ1

Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$30 + 12y - 6z + 5y + 9z = -4$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$x - 2y + z = 5$$
$$2y - 8z = -4$$
$$17y + 3z = -34$$

The Approach

- **Eliminate x from the EQ2 and EQ3**
- Eliminate y from EQ3
- Eliminate z from EQ2 and EQ1
- Eliminate y from EQ1

Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$17(8z - 4)/2 + 3z = -34$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$17(4z - 2) - 3z = -34$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$68z - 34 - 3z = 26$$

The Approach

- Eliminate x from the EQ2 and EQ3
- **Eliminate y from EQ3**
- Eliminate z from EQ2 and EQ1
- Eliminate y from EQ1

Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$71z = 0$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$x - 2y + 0 = 5$$

$$2y - 8(0) = -4$$

$$z = 0$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$x - 2y = 5$$
$$2y = -4$$
$$z = 0$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$x - 2(-2) = 5$$

$$y = -2$$

$$z = 0$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$x = 1$$
$$y = -2$$
$$z = 0$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$x = 1$$
$$y = -2$$
$$z = 0$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Elimination

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Back-Substitution

Verifying the Solution

$$x - 2y + z = 5$$
$$2y - 8z = -4$$
$$6x + 5y + 9z = -4$$

$$x = 1$$
$$y = -2$$
$$z = 0$$

Verifying the Solution

$$x - 2y + z = 5$$
$$2y - 8z = -4$$
$$6x + 5y + 9z = -4$$

$$x = 1$$
$$y = -2$$
$$z = 0$$

Verifying the Solution

$$(1) - 2(-2) + (0) = 5$$

$$2(-2) - 8(0) = -4$$

$$6(1) + 5(-2) + 9(0) = -4$$

$$x = 1$$

$$y = -2$$

$$z = 0$$

Verifying the Solution

$$1 + 4 + 0 = 5$$
$$-4 + 0 = -4$$
$$6 - 10 + 0 = -4$$

$$x = 1$$
$$y = -2$$
$$z = 0$$

Verifying the Solution

$$5 = 5$$
$$-4 = -4$$
$$-4 = -4$$

The solution simultaneously satisfies the equations

$$x = 1$$
$$y = -2$$
$$z = 0$$

Solving Systems of Linear Equations

1. ~~Some simple examples~~
2. ~~Elimination and Back-Substitution~~
3. **Row Equivalence**

Solving Systems as Matrices

How does this look with matrices?

Observation. Each intermediate step of elimination and back-substitution gives us a new linear system with the <u>same solutions</u>

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Can we represent these intermediate steps as operations on matrices?

Let's look back at this...

$$2x + 3y = -6$$
$$4x - 5y = 10$$

Elementary Row Operations

scaling	multiply a row by a number
replacement	add a multiple of one row to another
interchange	switch two rows

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These operations don't change the solutions

Scaling Example

$$\begin{aligned} 2x + 3y &= -6 \\ 4x - 5y &= 10 \end{aligned} \quad \longrightarrow \quad \begin{aligned} 4x + 6y &= -12 \\ 4x - 5y &= 10 \end{aligned}$$

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix} \quad \longrightarrow \quad \begin{bmatrix} 4 & 6 & -12 \\ 4 & -5 & 10 \end{bmatrix}$$

Replacement Example

$$\begin{aligned} 2x + 3y &= -6 \\ 4x - 5y &= 10 \end{aligned} \quad \longrightarrow \quad \begin{aligned} 2x + 3y &= -6 \\ 6x - 2y &= 4 \end{aligned}$$

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix} \quad \longrightarrow \quad \begin{bmatrix} 2 & 3 & -6 \\ 6 & -2 & 4 \end{bmatrix}$$

Interchange Example

$$\begin{aligned} 2x + 3y &= -6 \\ 4x - 5y &= 10 \end{aligned} \quad \rightarrow \quad \begin{aligned} 4x - 5y &= 10 \\ 2x + 3y &= -6 \end{aligned}$$

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 4 & -5 & 10 \\ 2 & 3 & -6 \end{bmatrix}$$

Example: Row Reductions

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 2 & 3 & -6 \\ 0 & -11 & 22 \end{bmatrix}$$

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Example: Row Reductions

$$R_2 \leftarrow R_2 - 2R_1$$
$$R_2 \leftarrow R_2/(-11)$$

elimination

$$R_1 \leftarrow R_1 - 3R_2$$
$$R_1 \leftarrow R_1/2$$

substitution

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

Row Equivalence

Definition. Two matrices are ***row equivalent*** if one can be transformed into the other by a sequence of row operations

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We can compute solutions by sequence of row operations

(Open-Ended) Question

How do we know when we're done? What is the "target" matrix?

We'll get to that next time...

demo
(SciPy)

Summary

Linear equations define hyperplanes

Systems of linear equations may or may not have solutions

Linear systems can be represented as matrices, which makes them more convenient to solve