

Solving Linear Systems + ~~Echelon Forms~~

Lecture 2

Practice Problem

$$3x_1 - 2x_2 + 6x_3 - 4x_4 = 0$$

Write down four distinct points in \mathbb{R}^4 which are in the point set defined by the above linear equation.

Answer

$$(0, 0, 0, 0)$$

$$(-2, -2, 1, 1)$$

$$(2, 3, 4, 6)$$

$$(4, 1, -1, 1)$$

$$3x_1 - 2x_2 + 6x_3 - 4x_4 = 0 \quad \checkmark$$

$$3(-2) - 2(-2) + 6 - 4 = 0$$

$$3(2) - 2(3) + 6(4) - 4(6) = 0 \quad \checkmark$$

$$3(4) - 2 - 6 - 4 = 0 \quad \checkmark$$

Objectives

1. Solve linear systems by elimination method
2. Solve linear systems by row operations
3. Introduce echelon forms as a kind of matrix which "represents" solutions
4. Learn how to "read off" a solution from an echelon form matrix.

Keywords

substitution method

elimination method

forward elimination

back-substitution

elementary row operations

scaling, replacement, interchange

Sympy operations

echelon form

row-reduced echelon form (RREF)

general form solution

Recap

Recall: Linear Equations

Definition. A *linear equation* in the variables x_1, x_2, \dots, x_n is an equation of the form

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b$$

where a_1, a_2, \dots, a_n, b are real numbers (\mathbb{R})

Recall: Linear Equations

Definition. A *linear equation* in the variables x_1, x_2, \dots, x_n is an equation of the form

coefficients

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b$$

where a_1, a_2, \dots, a_n, b are real numbers (\mathbb{R})

Recall: Linear Equations

Definition. A *linear equation* in the variables x_1, x_2, \dots, x_n is an equation of the form

unknowns

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b$$

where a_1, a_2, \dots, a_n, b are real numbers (\mathbb{R})

Recall: Linear Equations (Point sets)

Linear equations describe *point sets*:

$$\{(s_1, s_2, \dots, s_n) \in \mathbb{R}^n : a_1s_1 + a_2s_2 + \dots + a_ns_n = b\}$$

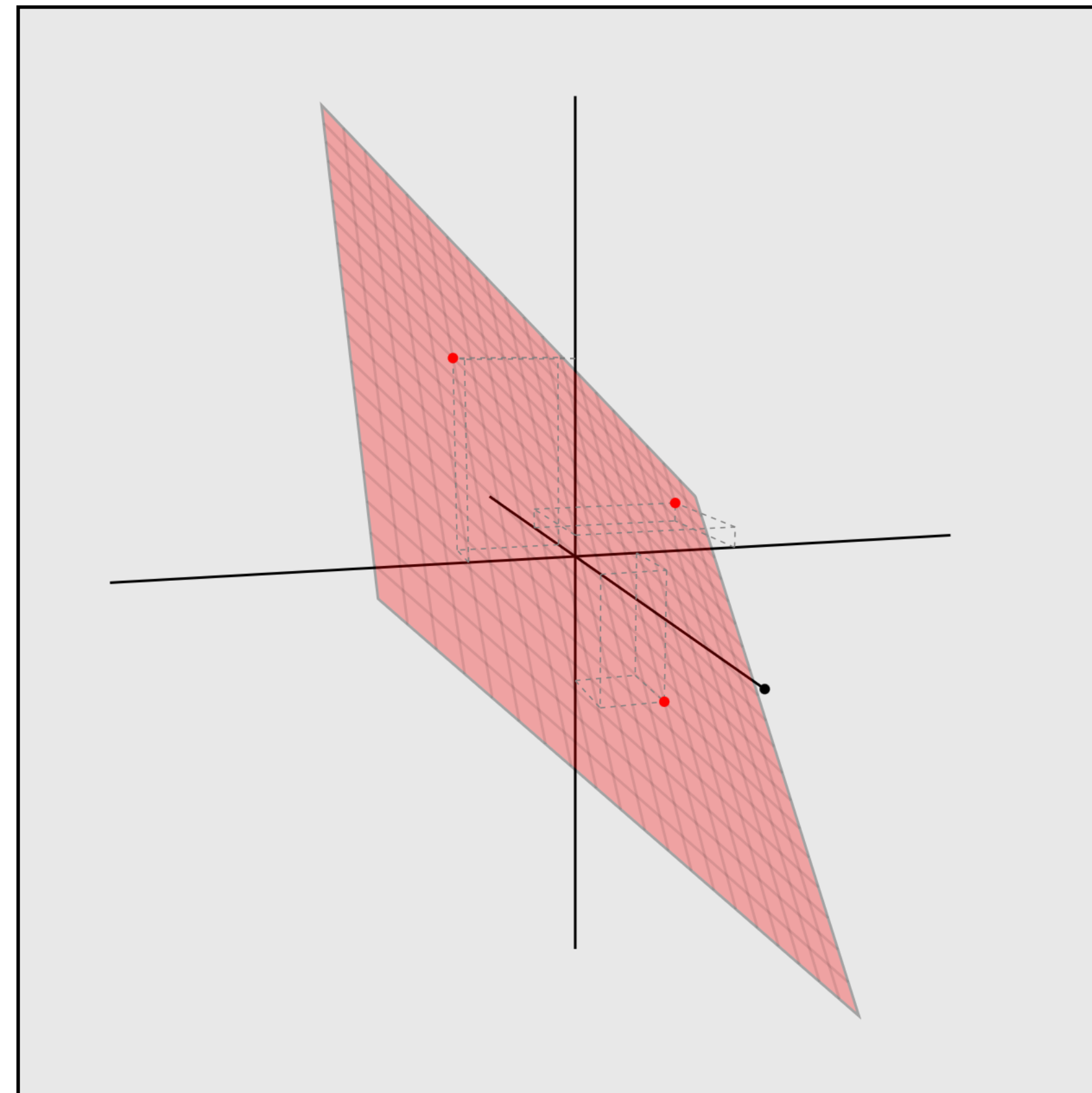
Recall: Linear Equations (Point sets)

Linear equations describe *point sets*:

$$\{(s_1, s_2, \dots, s_n) \in \mathbb{R}^n : a_1s_1 + a_2s_2 + \dots + a_ns_n = b\}$$

The collections of numbers such that the equation holds.

Recall: Linear Equations (Pictorially)



Recall: Linear Systems (General-form)

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

Recall: Linear Systems (General-form)

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

⋮

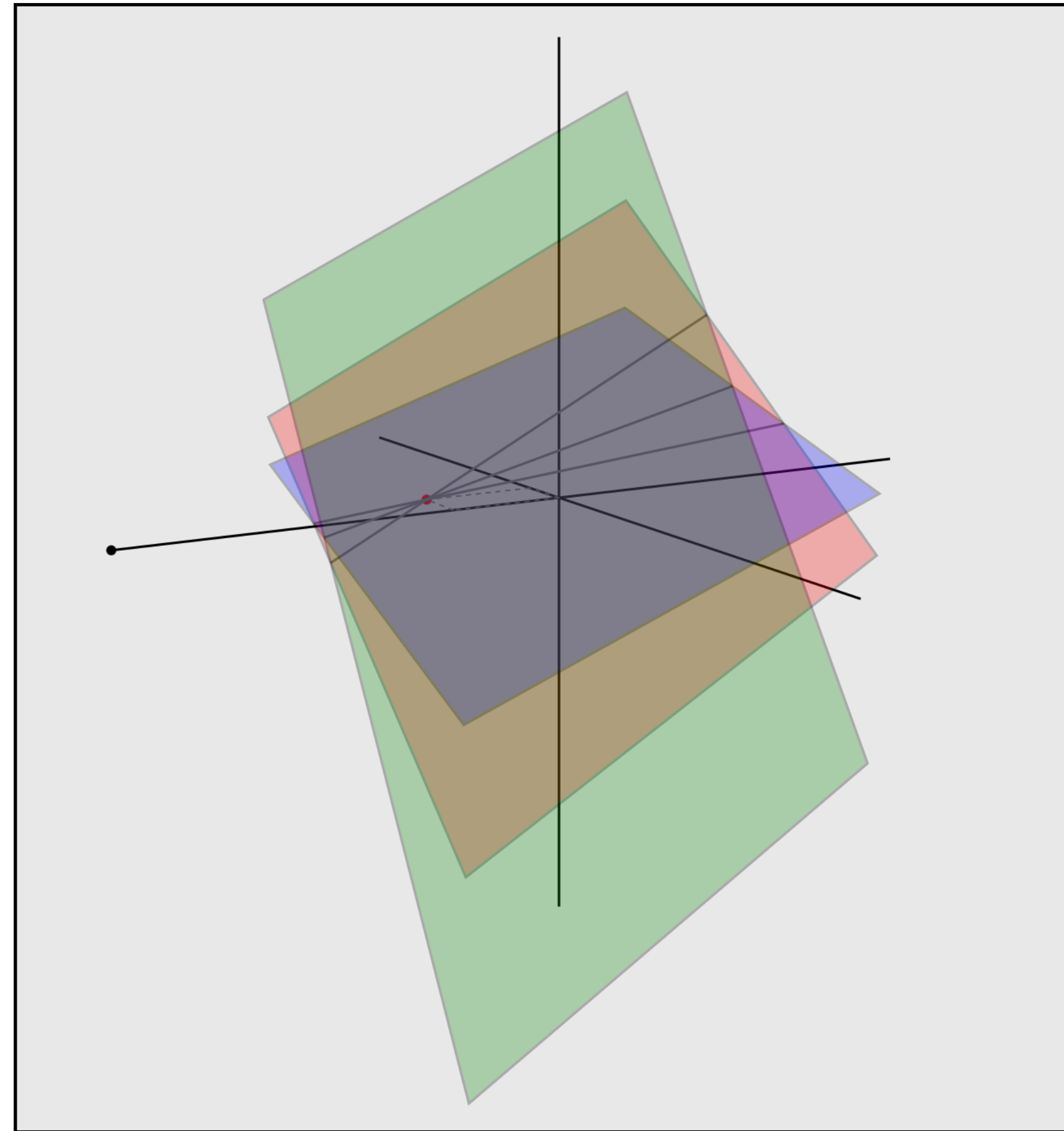
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

Does a system have a solution?

How many solutions are there?

What are its solutions?

Recall: Linear Systems (Pictorially)



Recall: Number of Solutions

zero the system is inconsistent

one the system has a unique solution

many the system has infinity solutions

Recall: Number of Solutions

zero the system is inconsistent

one the system has a unique solution

many the system has infinity solutions

These are the **only** options

Recall: Matrix Representations

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

⋮

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

Recall: Matrix Representations

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

Recall: Matrix Representations

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augmented matrix

Recall: Matrix Representations

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

coefficient matrix

Solving Linear Systems (Elimination Method)

Line Intersection Problem

$$2x = -6 - 3y$$

$$x = -3 - \frac{3}{2}y$$

$$4(-3 - \frac{3}{2}y) - 5y =$$

$$-12 - 6y - 5y = 10$$

$$-11y = 22$$

$$y = -2$$

$$2x + 3y = -6$$

$$4x - 5y = 10$$

$$2x + 3(-2) = -6$$

$$2x = 0$$

$$x = 0$$

$$x = 0$$

$$y = -2$$

Solving Systems with Two Variables

$$2x + 3y = -6$$

$$4x - 5y = 10$$

Solving Systems with Two Variables

$$2x + 3y = -6$$

$$4x - 5y = 10$$

The Approach

Solving Systems with Two Variables

$$2x + 3y = -6$$

$$4x - 5y = 10$$

The Approach

Solve for x in terms of y in EQ1

Solving Systems with Two Variables

$$2x + 3y = -6$$

$$4x - 5y = 10$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Solving Systems with Two Variables

$$2x + 3y = -6$$

$$4x - 5y = 10$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$2x = (-3)y - 6$$

$$4x - 5y = 10$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = (-3/2)y - 3$$

$$4x - 5y = 10$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = (-3/2)y - 3$$

$$4((-3/2)y - 3) - 5y = 10$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = (-3/2)y - 3$$

$$-6y - 12 - 5y = 10$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = (-3/2)y - 3$$

$$-11y = 22$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = (-3/2)y - 3$$

$$y = -2$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = (-3/2)(-2) - 3$$

$$y = -2$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = 3 - 3$$

$$y = -2$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = 0$$

$$y = -2$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

another perspective...

Solving Systems with Two Variables

$$2x + 3y = -6$$

$$4x - 5y = 10$$

The Approach

Eliminate x from the EQ2 and solve for y

Substitute y into EQ1 and solve for x

Let's work through it again...

$$E_1 \quad 2x + 3y = -6$$

$$E_2 \quad 4x - 5y = 10$$

$$E_2 - 2E_1$$

$$4x - 5y - (4x + 6y) = 10 + 12$$

$$2x + 3y = -6$$

$$-11y = 22$$

$$E_2 / -11$$

$$2x + 3y = -6$$

$$y = -2$$

$$E_1 - 3E_2$$



$$2x = 0$$

$$x = 0$$

$$E_1 / 2$$

$$x = 0$$

$$y = -2$$

Solving Systems of Linear Equations

1. ~~Some simple examples~~
2. **Elimination and Back-Substitution**
3. Row Equivalence

Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$6x + 5y + 9z = -4$$

Solving Systems with Three Variables

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The Approach

Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$6x + 5y + 9z = -4$$

The Approach

Eliminate x from the EQ2 and EQ3

Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$6x + 5y + 9z = -4$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Solving Systems with Three Variables

$$x - 2y + z = 5$$

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$$6x + 5y + 9z = -4$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$6x + 5y + 9z = -4$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from from EQ1

Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$6x + 5y + 9z = -4$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Elimination

Back-Substitution

Let's work through it

$$E_3 - 6E_1$$

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$17y + 3z = -34$$

$$E_2 / 2$$

$$x - 2y + z = 5$$

$$y - 4z = -2$$

$$17y + 3z = -34$$

$$-17y + 68z + 34$$

$$E_1 \quad x - 2y + z = 5$$

$$E_2 \quad 2y - 8z = -4$$

$$E_3 \quad 6x + 5y + 9z = -4$$

$$-6x + 12y - 6z - 30$$

$$E_3 - 17E_2$$

$$x - 2y + z = 5$$

$$y - 4z = -2$$

$$7z = 0$$

$$E_3 / 7$$

$$x - 2y + z = 5$$

$$y - 4z = -2$$

$$z = 0$$

$$E_2 + 4E_3 \quad E_1 - E_3$$

$$x - 2y = 5$$

$$y = -2$$

$$z = 0$$

$$E_1 + 2E_2$$

$$x = 1$$

$$y = -2$$

$$z = 0$$

Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$6(5 + 2y - z) + 5y + 9z = -4$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$30 + 12y - 6z + 5y + 9z = -4$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$17y + 3z = -34$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$17(8z - 4)/2 + 3z = -34$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$17(4z - 2) - 3z = -34$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$68z - 34 - 3z = 26$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$71z = 0$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$x - 2y + 0 = 5$$

$$2y - 8(0) = -4$$

$$z = 0$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$x - 2y = 5$$

$$2y = -4$$

$$z = 0$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$x - 2(-2) = 5$$

$$y = -2$$

$$z = 0$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$\begin{aligned}x &= 1 \\y &= -2 \\z &= 0\end{aligned}$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$\begin{aligned}x &= 1 \\y &= -2 \\z &= 0\end{aligned}$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Elimination

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Back-Substitution

Verifying the Solution

$$1 - 2(-2) + 0 = 5 \quad \checkmark$$

$$2(-2) - 8(0) = -4 \quad \checkmark$$

$$6(1) + 5(-2) + 0 = -4 \quad \checkmark$$

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$6x + 5y + 9z = -4$$

$$x = 1$$

$$y = -2$$

$$z = 0$$

Verifying the Solution

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$6x + 5y + 9z = -4$$

$$x = 1$$

$$y = -2$$

$$z = 0$$

Verifying the Solution

$$(1) - 2(-2) + (0) = 5$$

$$2(-2) - 8(0) = -4$$

$$6(1) + 5(-2) + 9(0) = -4$$

$$x = 1$$

$$y = -2$$

$$z = 0$$

Verifying the Solution

$$1 + 4 + 0 = 5$$

$$-4 + 0 = -4$$

$$6 - 10 + 0 = -4$$

$$x = 1$$

$$y = -2$$

$$z = 0$$

Verifying the Solution

$$5 = 5$$

$$-4 = -4$$

$$-4 = -4$$

The solution simultaneously satisfies the equations

$$x = 1$$

$$y = -2$$

$$z = 0$$

Solving Systems as Matrices

How does this look with matrices?

Observation. Each intermediate step of elimination and back-substitution gives us a new linear system with the same solutions

Solving Systems as Matrices

How does this look with matrices?

Observation. Each intermediate step of elimination and back-substitution gives us a new linear system with the same solutions

Can we represent these intermediate steps as operations on matrices?

Row Equivalence

Let's look back at this...

$$2x + 3y = -6$$

$$4x - 5y = 10$$

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \\ -4 & -6 & +12 \end{bmatrix} \quad R_2 \leftarrow R_2 - 2R_1$$

$$R_2 \leftarrow R_2 / 11$$

$$\begin{bmatrix} 2 & 3 & -6 \\ 0 & -11 & 22 \end{bmatrix}$$

$$R_1 \leftarrow R_1 - 3R_2$$

$$\begin{bmatrix} 2 & 3 & -6 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_1 \leftarrow R_1 / 2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix}$$

$$\begin{aligned} 2x + 3y &= -6 \\ -11y &= 22 \end{aligned}$$

$$\begin{aligned} x &= 0 \\ y &= -2 \end{aligned}$$

Elementary Row Operations

scaling

multiply a row by a number

replacement

add a multiple of one row to another

interchange

switch two rows

Elementary Row Operations

scaling

multiply a row by a number

replacement

add a multiple of one row to another

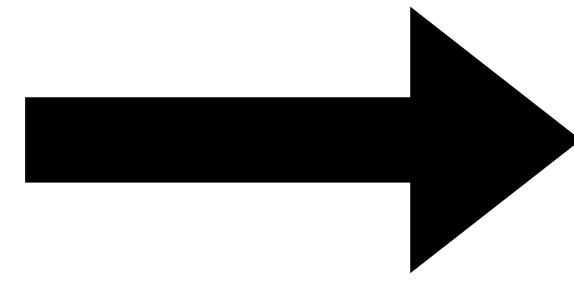
interchange

switch two rows

These operations don't change the solutions

Scaling Example

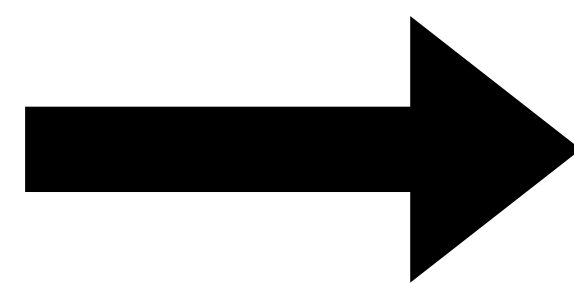
$$\begin{aligned} 2x + 3y &= -6 \\ 4x - 5y &= 10 \end{aligned}$$



$$\begin{aligned} 4x + 6y &= -12 \\ 4x - 5y &= 10 \end{aligned}$$

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix}$$

$R_1 \leftarrow 2R_1$



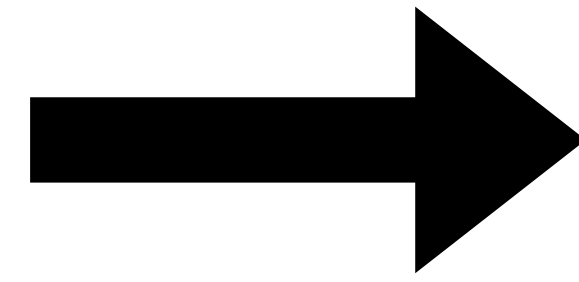
$2(2) \quad 2(3) \quad 2(-6)$

$$\begin{bmatrix} 4 & 6 & -12 \\ 4 & -5 & 10 \end{bmatrix}$$

Replacement Example

$$2x + 3y = -6$$

$$4x - 5y = 10$$

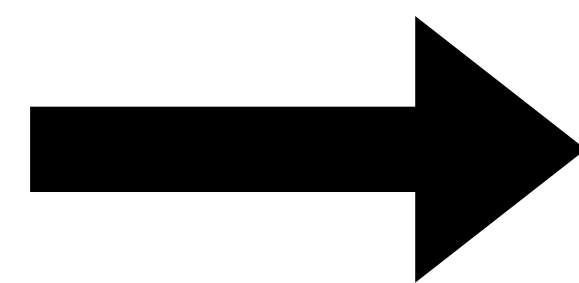


$$2x + 3y = -6$$

$$6x - 2y = 4$$

$$R_2 \leftarrow R_1 + R_2$$

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix}$$

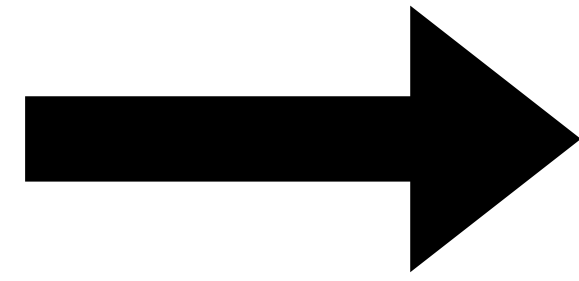


$$\begin{bmatrix} 2 & 3 & -6 \\ 6 & -2 & 4 \end{bmatrix}$$

$$(2+4) \quad (3-5) \quad (-6+10)$$

Interchange Example

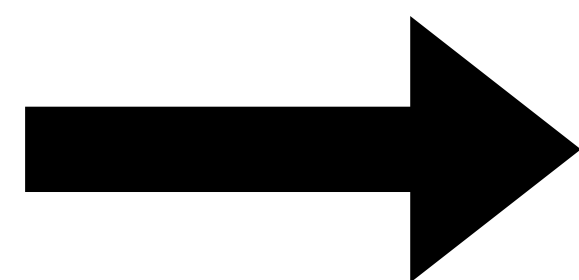
$$\begin{aligned}2x + 3y &= -6 \\4x - 5y &= 10\end{aligned}$$




$$\begin{aligned}4x - 5y &= 10 \\2x + 3y &= -6\end{aligned}$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix}$$




$$\begin{bmatrix} 4 & -5 & 10 \\ 2 & 3 & -6 \end{bmatrix}$$

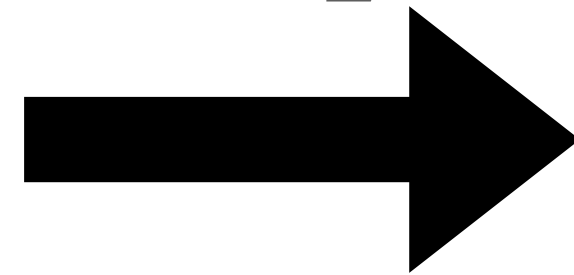
Example: Row Reductions

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 2 & 3 & -6 \\ 0 & -11 & 22 \end{bmatrix}$$

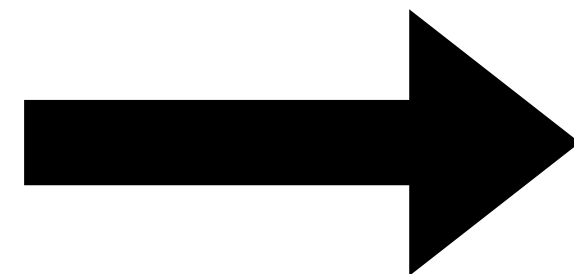
Example: Row Reductions

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix}$$

$$R_2 \leftarrow R_2 - 2R_1$$



$$R_2 \leftarrow R_2 / (-11)$$



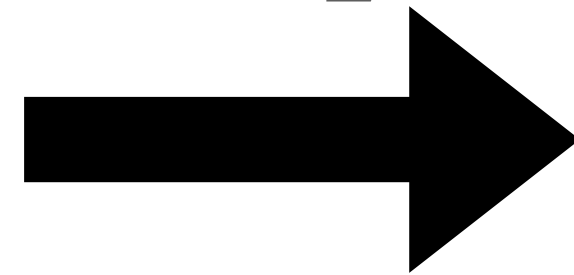
$$\begin{bmatrix} 2 & 3 & -6 \\ 0 & -11 & 22 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & -6 \\ 0 & 1 & -2 \end{bmatrix}$$

Example: Row Reductions

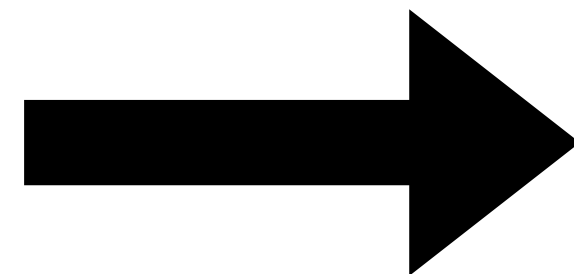
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$$R_2 \leftarrow R_2 - 2R_1$$



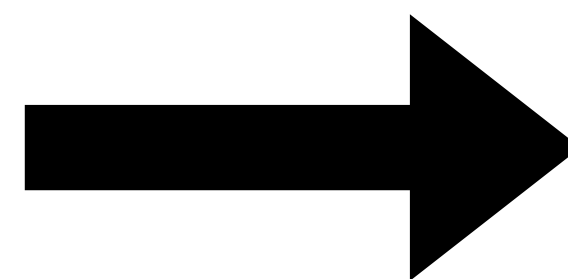
$$\begin{bmatrix} 2 & 3 & -6 \\ 0 & -11 & 22 \end{bmatrix}$$

$$R_2 \leftarrow R_2 / (-11)$$



$$\begin{bmatrix} 2 & 3 & -6 \\ 0 & 1 & -2 \end{bmatrix}$$

$$R_1 \leftarrow R_1 - 3R_2$$



$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix}$$

Example: Row Reductions

$$\begin{array}{ccc} \begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix} & \begin{array}{c} R_2 \leftarrow R_2 - 2R_1 \\ \longrightarrow \\ R_2 \leftarrow R_2 / (-11) \\ \longrightarrow \\ R_1 \leftarrow R_1 - 3R_2 \\ \longrightarrow \\ R_1 \leftarrow R_1 / 2 \\ \longrightarrow \end{array} & \begin{array}{c} \begin{bmatrix} 2 & 3 & -6 \\ 0 & -11 & 22 \end{bmatrix} \\ \\ \begin{bmatrix} 2 & 3 & -6 \\ 0 & 1 & -2 \end{bmatrix} \\ \\ \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix} \\ \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix} \end{array} \end{array} \quad \begin{array}{l} x = 0 \\ y = -2 \end{array}$$

Example: Row Reductions

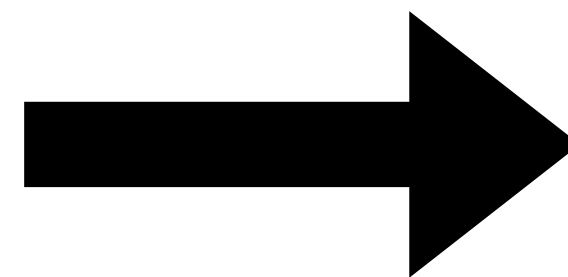
$$R_2 \leftarrow R_2 - 2R_1$$

$$R_2 \leftarrow R_2 / (-11)$$

$$R_1 \leftarrow R_1 - 3R_2$$

$$R_1 \leftarrow R_1 / 2$$

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

Example: Row Reductions

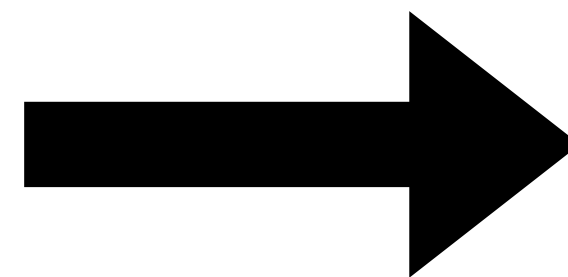
$$R_2 \leftarrow R_2 - 2R_1$$
$$R_2 \leftarrow R_2 / (-11)$$

elimination

$$R_1 \leftarrow R_1 - 3R_2$$
$$R_1 \leftarrow R_1 / 2$$

substitution

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix}$$

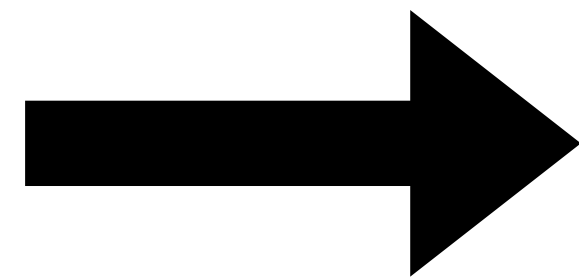


$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

Row Equivalence

Definition. Two matrices are *row equivalent* if one can be transformed into the other by a sequence of row operations

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix}$$

Row Equivalence

Definition. Two matrices are *row equivalent* if one can be transformed into the other by a sequence of row operations

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix}$$

We can compute solutions by sequence of row operations

Question

$$\begin{bmatrix} 1 & 2 & 4 \\ 1 & 6 & 2 \\ 1 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 & 2 \\ 0 & -4 & 2 \\ 0 & -2 & 3 \end{bmatrix}$$

Write a sequence of row operations that converts the matrix on the left to the matrix on the right.

Answer

$$R_1 \leftrightarrow R_2$$
$$\begin{bmatrix} 1 & 6 & 2 \\ 1 & 2 & 4 \\ 1 & 4 & 5 \end{bmatrix}$$

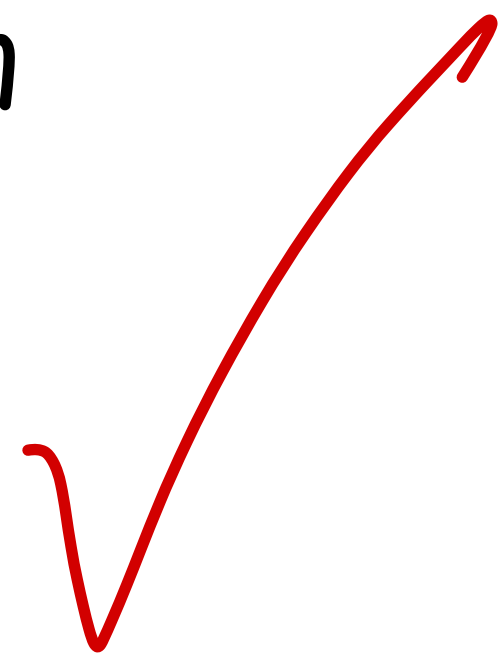
$$R_2 \leftarrow R_2 - R_1$$

$$\begin{bmatrix} 1 & 6 & 2 \\ 0 & -4 & 2 \\ 1 & 4 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 1 & 6 & 2 \\ 1 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 & 2 \\ 0 & -4 & 2 \\ 0 & -2 & 3 \end{bmatrix}$$

$$R_3 \leftarrow R_3 - R_1$$

$$\begin{bmatrix} 1 & 6 & 2 \\ 0 & -4 & 2 \\ 0 & -2 & 3 \end{bmatrix}$$



first, a demo
(SymPy)

Taking Stock

Taking Stock

Observation. Solutions look like simple systems of linear equations

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said another way: it's easy to "read off" the solutions of some systems

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Solving a system of linear equations is the same as row reducing its augmented matrix to a matrix which "represents a solution".

What matrices "represent solutions"?

Motivating Questions

What matrices "represent solutions"? (which have solutions that are easy to "read off"?)

How does the number of solutions affect the shape of these matrix?

How do we use row operations to get to those matrices?

Motivating Questions

Let's consider these first

What matrices "represent solutions"? (which have solutions that are easy to "read off"?)

How does the number of solutions affect the shape of these matrix?

How do we use row operations to get to those matrices?

Unique Solution Case

Unique Solution Case

$$\begin{bmatrix} 2 & -3 & 5 & 11 \\ 2 & -1 & 13 & 39 \\ 1 & -1 & 5 & 14 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$x = 1$$

$$y = 2$$

$$z = 3$$

Unique Solution Case

$$\begin{bmatrix} 2 & -3 & 5 & 11 \\ 2 & -1 & 13 & 39 \\ 1 & -1 & 5 & 14 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$x = 1$$

$$y = 2$$

$$z = 3$$

Like all the
examples we've seen
so far

The Identity Matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The Identity Matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1s along the diagonal

0s elsewhere

Unique Solution Case

coefficient matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

a system of linear equations whose **coefficient matrix** is the identity matrix represents a unique solution

No Solution Case

Example

$$R_1 \leftarrow R_1 - R_2$$

$$\begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

$$\begin{aligned} 0x + 0y + 0z &= -1 \\ 0 &= -1 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

$$x + y + z = 1$$

$$x + y + z = 2$$

No Solution Case

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

No Solution Case

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

two parallel
planes

No Solution Case

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

two parallel
planes

\sim

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

row representing $0 = 1$

No Solution Case

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

row representing $0 = 1$

a system with no solutions can be reduced to a matrix with the row

$$0 \ 0 \ \dots \ 0 \ 1$$

Infinite Solution Case

Example

$$\begin{bmatrix} 2 & 4 & 2 & 14 \\ 1 & 7 & 1 & 12 \end{bmatrix}$$

Infinite Solution Case

$$\begin{bmatrix} 2 & 4 & 2 & 14 \\ 1 & 7 & 1 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Infinite Solution Case

$$\begin{bmatrix} 2 & 4 & 2 & 14 \\ 1 & 7 & 1 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$x_1 + x_3 = 2$$

$$x_2 = 1$$

Infinite Solution Case

$$\begin{bmatrix} 2 & 4 & 2 & 14 \\ 1 & 7 & 1 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$x_1 + x_3 = 2$$

$$x_2 = 1$$

a system with infinity solutions can be reduced to a system which leaves a variable unrestricted

Infinite Solution Case

$$x_1 + x_3 = 2$$

$$x_2 = 1$$

it doesn't matter
what x_3 is if we
want to satisfy
this system of
equations

$$x_1 = 2$$

$$x_2 = 1$$

$$x_3 = 0$$

Infinite Solution Case

$$x_1 + x_3 = 2$$

$$x_2 = 1$$

it doesn't matter
what x_3 is if we
want to satisfy
this system of
equations

$$x_1 = 1.5$$

$$x_2 = 1$$

$$x_3 = 0.5$$

Infinite Solution Case

$$x_1 + x_3 = 2$$

$$x_2 = 1$$

it doesn't matter
what x_3 is if we
want to satisfy
this system of
equations

$$x_1 = 20$$

$$x_2 = 1$$

$$x_3 = -18$$

Infinite Solution Case

$$x_1 + x_3 = 2$$

$$x_2 = 1$$

it doesn't matter
what x_3 is if we
want to satisfy
this system of
equations

$$x_1 = 2 - x_3$$

$$x_2 = 1$$

x_3 is free

Infinite Solution Case

$$\begin{aligned}x_1 + x_3 &= 2 \\x_2 &= 1\end{aligned}$$

it doesn't matter
what x_3 is if we
want to satisfy
this system of
equations

$$\begin{aligned}x_1 &= 2 - x_3 \\x_2 &= 1 \\x_3 &\text{ is free}\end{aligned}$$

general form

In Sum

none

reduces to a system with the equation $0 = 1$

one

reduces to a system whose coefficient matrix is the identity matrix

infinity

reduces to a system which leaves a variable unrestricted

In Sum

- none** reduces to a system with the equation $0 = 1$
- one** reduces to a system whose coefficient matrix is the identity matrix
- infinity** reduces to a system which leaves a variable unrestricted

Ideally, we want one *form* that handles all three cases

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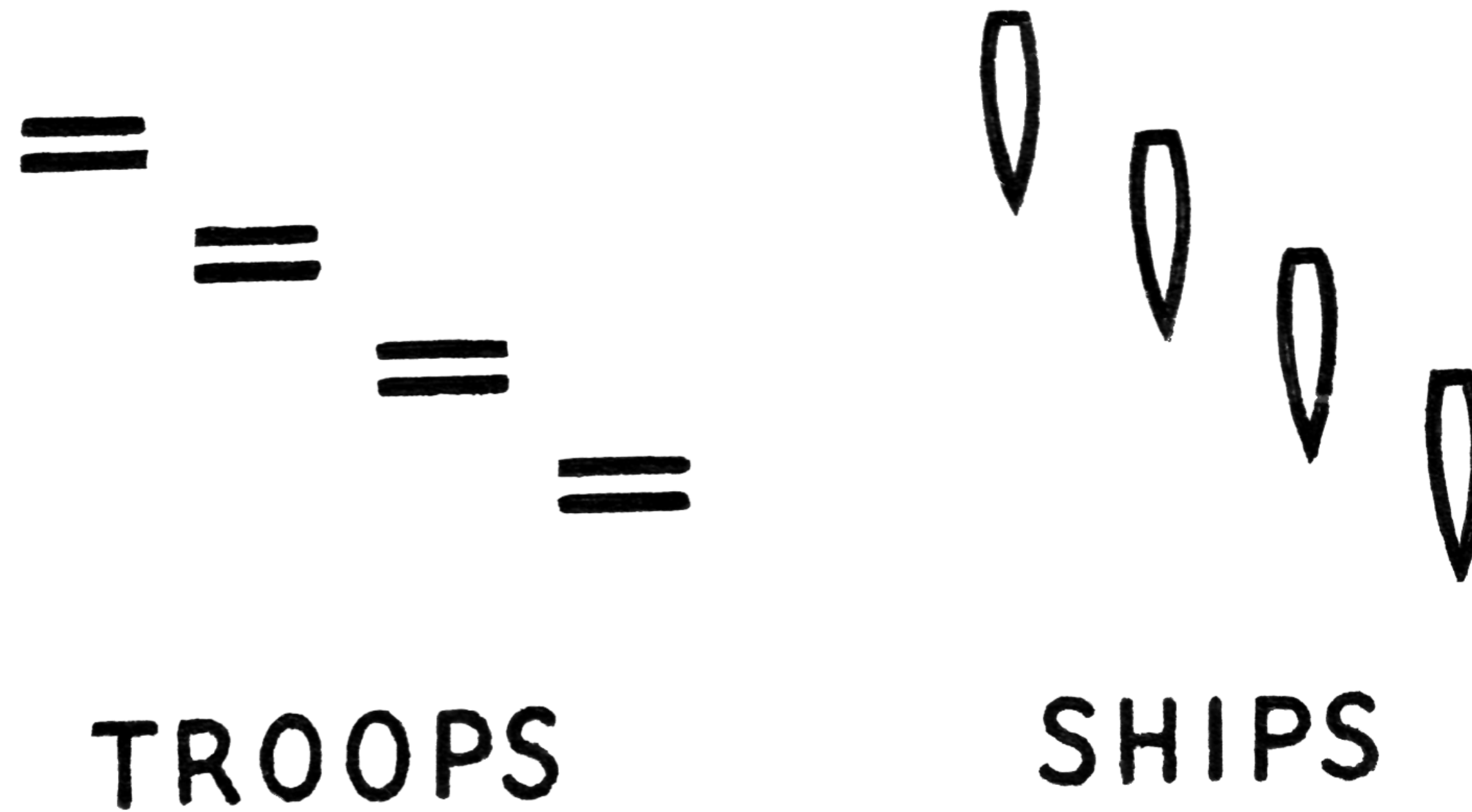
How does the number of solutions affect the shape of these matrix?

How do we use row operations to get to those matrices?

this is Gaussian elimination (next lecture)

Echelon Form

The Picture (and a bit of history)



Echelon Form (Pictorially)

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\blacksquare = \text{nonzero}, \quad * = \text{anything}$

Leading Entries

Definition. the *leading entry* of a row is the first nonzero value

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \\ 1 & -1 & 10 \end{bmatrix}$$

← no leading entry

Echelon Form

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Definition. A matrix is in *echelon form* if

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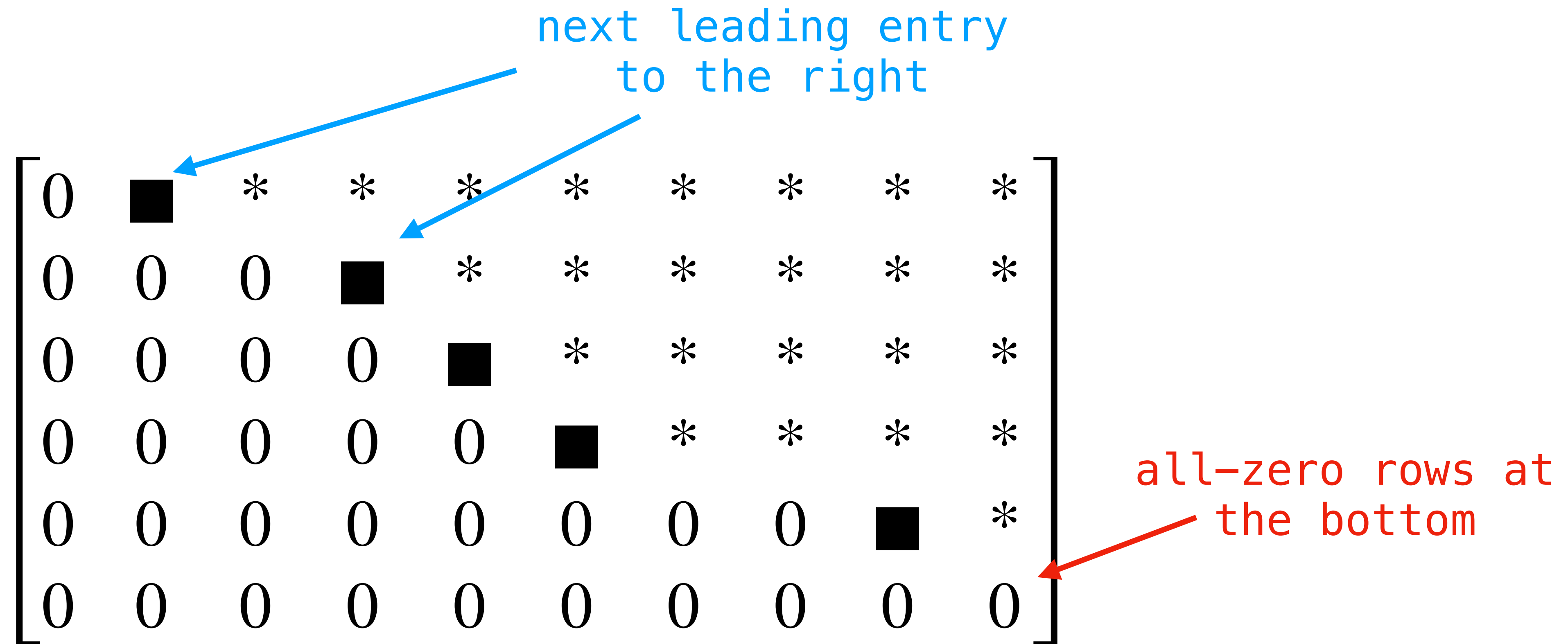
1. The leading entry of each row appears to the right of the leading entry above it
2. Every all-zeros row appears below any non-zero rows

Echelon Form (Pictorially)

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\blacksquare = \text{nonzero}, * = \text{anything}$

Echelon Form (Pictorially)



$\blacksquare = \text{nonzero}, * = \text{anything}$

Question

Is the identity matrix in echelon form?

Answer: Yes

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

the leading entries of each row appears to the right of the leading entry above it

it has no all-zero rows

Question

Is this matrix in echelon form?

$$\begin{bmatrix} 2 & 3 & -8 \\ 0 & 1 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

Answer: No

$$\begin{bmatrix} 2 & 3 & -8 \\ 0 & 1 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

The leading entry of the least row is not to the right of the leading entry of the second row

What's special about Echelon forms?

Theorem. Let A be the augmented matrix of an inconsistent linear system. If $A \sim B$ and B is in echelon form then B has the row

$$[0 \ 0 \ \dots \ 0 \ 0 \ \blacksquare]$$

Example

The Problem with Echelon Forms

If our system *is* consistent, we can't get a solution quite yet.

The Problem with Echelon Forms

If our system *is* consistent, we can't get a solution quite yet.

We need to simplify our matrix a bit more until it
"represents" a solution

Reduced Echelon Form

Row-Reduced Echelon Form (RREF)

Definition. A matrix is in *(row-)reduced echelon form* if

1. The leading entry of each row appears to the right of the leading entry above it
2. Every all-zeros row appears below any non-zero rows
3. The leading entries of non-zero rows are 1
4. the leading entries are the only non-zero entries of their columns

Reduced Echelon Form (Pictorially)

leading entries are 1

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

other column entries are 0

Reduced Echelon Form (A Simple Example)

$$\begin{aligned}x_1 + x_3 &= 2 \\x_2 &= 1\end{aligned}$$

Reduced Echelon Form (A Simple Example)

$$x_1 + x_3 = 2$$

$$x_2 = 1$$

$$x_1 = 2 - x_3$$

$$x_2 = 1$$

x_3 is free

What's special about RREF?

Every leading variable can be written in terms of every non-leading variable.

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

The Fundamental Points

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Point 1. we can "read off" the solutions of a system of linear equations from its RREF

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Point 1. we can "read off" the solutions of a system of linear equations from its RREF

Point 2. *every* matrix is row equivalent to a unique matrix in reduced echelon form

How-To: Solving a System of Linear Equations

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1. Write your system as an augmented matrix

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2. Find the RREF of that matrix

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1. Write your system as an augmented matrix

2. Find the RREF of that matrix

3. Read off the solution from the RREF

Our next topic

General-Form Solutions

Basic and Free Variables

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Definition. a *pivot position* (i,j) in a matrix is the position of a leading entry in it's reduced echelon form

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$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Basic and Free Variables

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Definition. A variable is *basic* if its column has a pivot position (this is called a *pivot column*). It is *free* otherwise.

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

x_1 is basic

x_2 is basic

x_3 is free

Solutions of Reduced Echelon Forms

the row of a pivot position in row i describes the value of x_i in a solution to the system, in terms of the free variables

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

General Form Solution

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$x_1 = 2 - x_3$$

$$x_2 = 1$$

x_3 is free

for each pivot position (i,j) , isolate x_i in the equation in row j

if x_i does not have a pivot position, write

x_i is free

Example

$$\begin{bmatrix} 1 & 2 & 0 & -2 & 4 \\ 0 & 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Why we care about Reduced Echelon Forms?

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*the goal of back-substitution is to reduce an echelon form matrix to a **reduced** echelon form*

*the goal of Gaussian elimination is to reduce an **augmented** matrix to a **reduced** echelon form*

***reduced echelon forms describe solutions to
Linear equations***

Question

write down a solution in general form for this reduced echelon form matrix

$$\begin{bmatrix} 1 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Answer

$$\begin{bmatrix} 1 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

demo
(a.rref())