Solving Linear Systems

Geometric Algorithms
Lecture 2

CAS CS 132

Practice Problem

$$3x_1 - 2x_2 + 6x_3 - 4x_4 = 0$$

Write down four distinct points in $\mathbb{R}^4$ which are in the point set defined by the above linear equation.

$$3x_1 - 2x_2 + 6x_3 - 4x_4 = 0$$

Answer

Objectives

1. Solve linear systems by elimination method
2. Solve linear systems by row operations
3. Introduce echelon forms as as a kind of matrix which "represents" solutions
4. Learn how to "read off" a solution from an echelon form matrix.

Keywords

substitution method

elimination method

forward elimination

back-substitution

elementary row operations

scaling, replacement, interchange

Sympy operations

echelon form

row-reduced echelon form (RREF)

general form solution

Recap

Recall: Linear Equations

Definition. A ***linear equation*** in the variables $x_1, x_2, \ldots, x_n$ is an equation of the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

where $a_1, a_2, \ldots, a_n, b$ are real numbers ($\mathbb{R}$)

Recall: Linear Equations

Definition. A ***linear equation*** in the variables $x_1, x_2, \ldots, x_n$ is an equation of the form

coefficients

$$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = b$$

where $a_1, a_2, \ldots, a_n, b$ are real numbers ($\mathbb{R}$)

Recall: Linear Equations

Definition. A ***linear equation*** in the variables $x_1, x_2, \ldots, x_n$ is an equation of the form

unknowns

$$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = b$$

where $a_1, a_2, \ldots, a_n, b$ are real numbers ($\mathbb{R}$)

Recall: Linear Equations (Point sets)

`Linear equations describe` ***`point sets`***`:`

$$\{(s_1, s_2, \ldots, s_n) \in \mathbb{R}^n : a_1 s_1 + a_2 s_2 + \ldots + a_n s_n = b\}$$

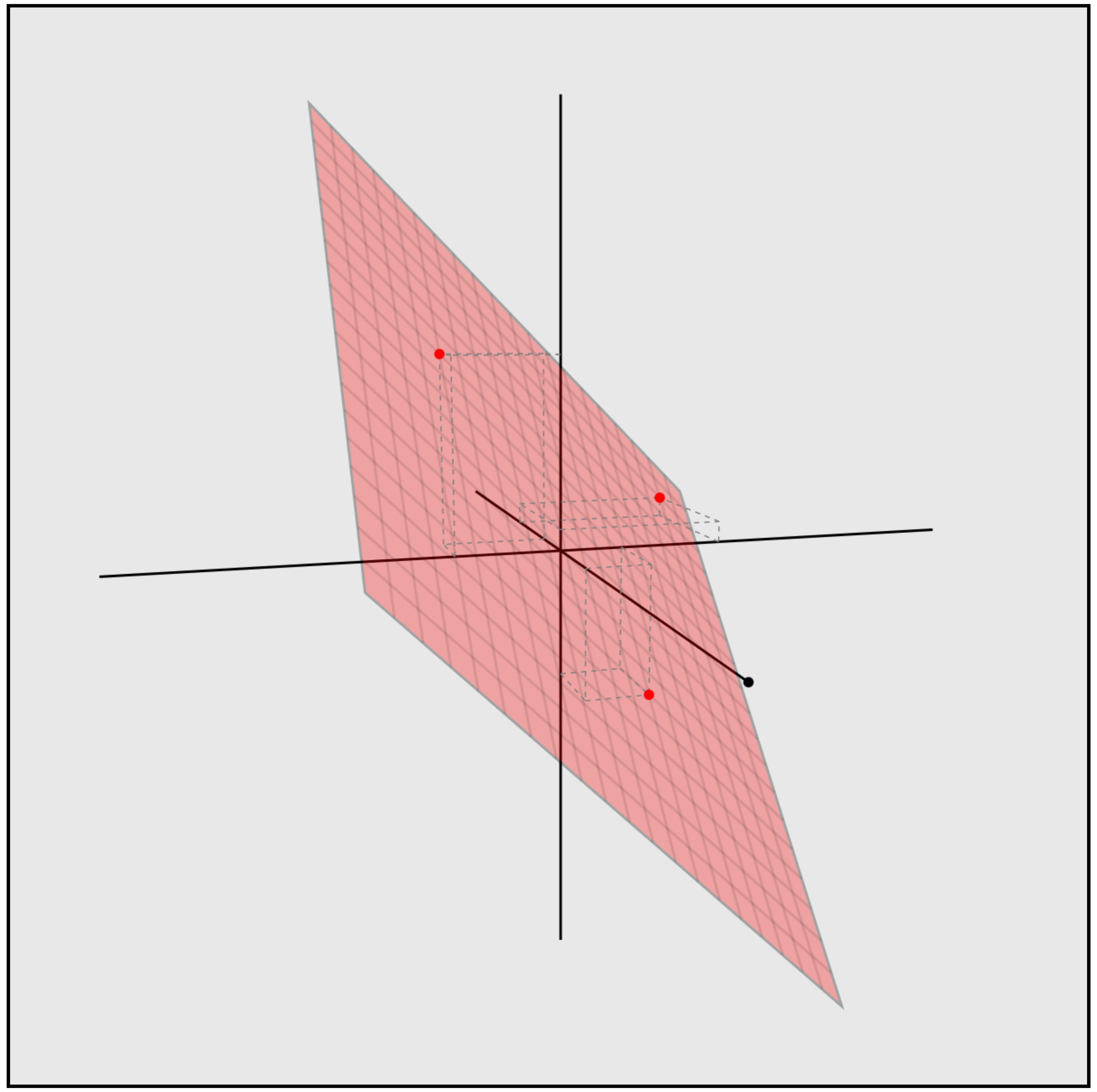
Recall: Linear Equations (Point sets)

Linear equations describe ***point sets***:

$$\{(s_1, s_2, \ldots, s_n) \in \mathbb{R}^n : a_1 s_1 + a_2 s_2 + \ldots + a_n s_n = b\}$$

The collections of numbers such that the equation holds.

Recall: Linear Equations (Pictorially)



Recall: Linear Systems (General-form)

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}
$$

Recall: Linear Systems (General-form)

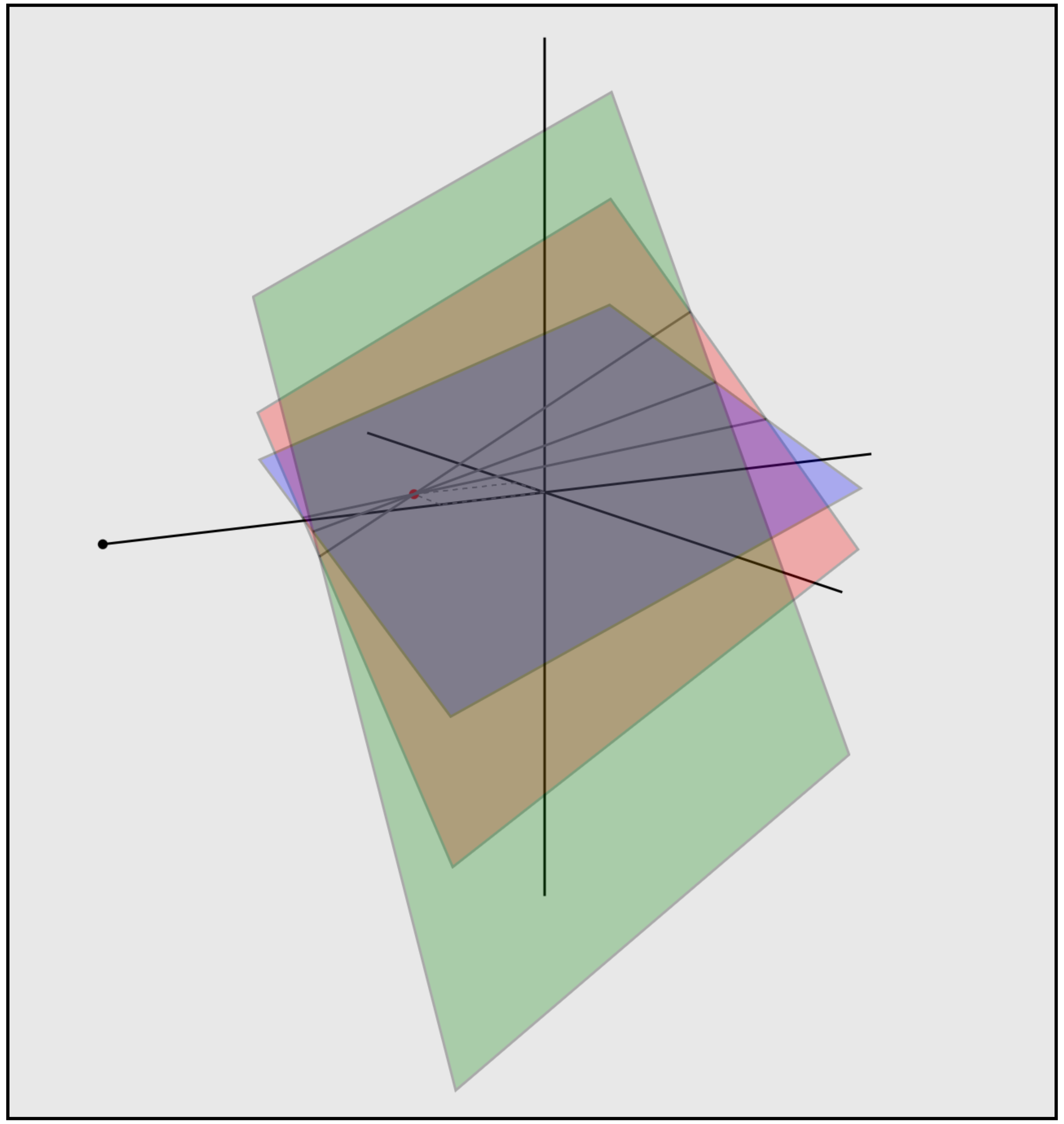
$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}
$$

Does a system have a solution?

How many solutions are there?

What are its solutions?

Recall: Linear Systems (Pictorially)



Recall: Number of Solutions

```
zero the system is inconsistent

one  the system has a unique solution

many the system has infinity solutions
```

Recall: Number of Solutions

zero the system is inconsistent

one the system has a unique solution

many the system has infinity solutions

These are the **only** options

Recall: Matrix Representations

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}
$$

Recall: Matrix Representations

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

Recall: Matrix Representations

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

augmented matrix

Recall: Matrix Representations

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

`coefficient matrix`

Solving Linear Systems (Elimination Method)

Line Intersection Problem

$$2x + 3y = -6$$
$$4x - 5y = 10$$

Solving Systems with Two Variables

$$2x + 3y = -6$$

$$4x - 5y = 10$$

Solving Systems with Two Variables

$$2x + 3y = -6$$
$$4x - 5y = 10$$

The Approach

Solving Systems with Two Variables

$$2x + 3y = -6$$

$$4x - 5y = 10$$

The Approach

Solve for x in terms of y in EQ1

Solving Systems with Two Variables

$$2x + 3y = -6$$

$$4x - 5y = 10$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Solving Systems with Two Variables

$$2x + 3y = -6$$

$$4x - 5y = 10$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$2x = (-3)y - 6$$

$$4x - 5y = 10$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = (-3/2)y - 3$$
$$4x - 5y = 10$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = (-3/2)y - 3$$

$$4((-3/2)y - 3) - 5y = 10$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = (-3/2)y - 3$$

$$-6y - 12 - 5y = 10$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = (-3/2)y - 3$$

$$-11y = 22$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = (-3/2)y - 3$$
$$y = -2$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = (-3/2)(-2) - 3$$
$$y = -2$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = 3 - 3$$
$$y = -2$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = 0$$
$$y = -2$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

another perspective...

Solving Systems with Two Variables

$$2x + 3y = -6$$
$$4x - 5y = 10$$

The Approach

Eliminate x *from the EQ2 and solve for* y

Substitute y *into EQ1 and solve for* x

Let's work through it again...

$$2x + 3y = -6$$
$$4x - 5y = 10$$

Solving Systems of Linear Equations

1. ~~Some simple examples~~
2. **Elimination and Back-Substitution**
3. Row Equivalence

Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$6x + 5y + 9z = -4$$

Solving Systems with Three Variables

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The Approach

Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$6x + 5y + 9z = -4$$

The Approach

Eliminate x from the EQ2 and EQ3

Solving Systems with Three Variables

$$x - 2y + z = 5$$
$$2y - 8z = -4$$
$$6x + 5y + 9z = -4$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Solving Systems with Three Variables

$$\begin{aligned} x - 2y + z &= 5 \\ 2y - 8z &= -4 \\ 6x + 5y + 9z &= -4 \end{aligned}$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Solving Systems with Three Variables

$$x - 2y + z = 5$$
$$2y - 8z = -4$$
$$6x + 5y + 9z = -4$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from from EQ1

Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$6x + 5y + 9z = -4$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Elimination

Eliminate z from EQ2 and EQ1

Eliminate y from from EQ1

Back–Substitution

Let's work through it

$$x - 2y + z = 5$$
$$2y - 8z = -4$$
$$6x + 5y + 9z = -4$$

Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$6(5 + 2y - z) + 5y + 9z = -4$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$30 + 12y - 6z + 5y + 9z = -4$$

The Approach

- **Eliminate x from the EQ2 and EQ3**
- Eliminate y from EQ3
- Eliminate z from EQ2 and EQ1
- Eliminate y from EQ1

Solving Systems with Three Variables

$$x - 2y + z = 5$$
$$2y - 8z = -4$$
$$17y + 3z = -34$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$17(8z - 4)/2 + 3z = -34$$

The Approach

- Eliminate x from the EQ2 and EQ3
- **Eliminate y from EQ3**
- Eliminate z from EQ2 and EQ1
- Eliminate y from EQ1

Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$17(4z - 2) - 3z = -34$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$68z - 34 - 3z = 26$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$x - 2y + z = 5$$
$$2y - 8z = -4$$
$$71z = 0$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$x - 2y + 0 = 5$$

$$2y - 8(0) = -4$$

$$z = 0$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$\begin{aligned} x - 2y &= 5 \\ 2y &= -4 \\ z &= 0 \end{aligned}$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$x - 2(-2) = 5$$

$$y = -2$$

$$z = 0$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$x = 1$$
$$y = -2$$
$$z = 0$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$x = 1$$
$$y = -2$$
$$z = 0$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Elimination

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Back–Substitution

Verifying the Solution

$$x - 2y + z = 5$$
$$2y - 8z = -4$$
$$6x + 5y + 9z = -4$$

$$x = 1$$
$$y = -2$$
$$z = 0$$

Verifying the Solution

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$6x + 5y + 9z = -4$$

$$x = 1$$

$$y = -2$$

$$z = 0$$

Verifying the Solution

$$(1) - 2(-2) + (0) = 5$$

$$2(-2) - 8(0) = -4$$

$$6(1) + 5(-2) + 9(0) = -4$$

$$x = 1$$

$$y = -2$$

$$z = 0$$

Verifying the Solution

$$1 + 4 + 0 = 5$$
$$-4 + 0 = -4$$
$$6 - 10 + 0 = -4$$

$$x = 1$$
$$y = -2$$
$$z = 0$$

Verifying the Solution

$$5 = 5$$

$$-4 = -4$$

$$-4 = -4$$

The solution simultaneously satisfies the equations

$$x = 1$$

$$y = -2$$

$$z = 0$$

Solving Systems as Matrices

How does this look with matrices?

Observation. Each intermediate step of elimination and back-substitution gives us a new linear system with the <u>same solutions</u>

Solving Systems as Matrices

How does this look with matrices?

Observation. Each intermediate step of elimination and back-substitution gives us a new linear system with the same solutions

Can we represent these intermediate steps as operations on matrices?

Row Equivalence

Let's look back at this...

$$2x + 3y = -6$$
$$4x - 5y = 10$$

Elementary Row Operations

scaling	multiply a row by a number
replacement	add a multiple of one row to another
interchange	switch two rows

Elementary Row Operations

scaling	multiply a row by a number
replacement	add a multiple of one row to another
interchange	switch two rows

These operations don't change the solutions

Scaling Example

$$\begin{aligned} 2x + 3y &= -6 \\ 4x - 5y &= 10 \end{aligned} \quad \longrightarrow \quad \begin{aligned} 4x + 6y &= -12 \\ 4x - 5y &= 10 \end{aligned}$$

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix} \quad \longrightarrow \quad \begin{bmatrix} 4 & 6 & -12 \\ 4 & -5 & 10 \end{bmatrix}$$

Replacement Example

$$\begin{aligned} 2x + 3y &= -6 \\ 4x - 5y &= 10 \end{aligned} \quad \rightarrow \quad \begin{aligned} 2x + 3y &= -6 \\ 6x - 2y &= 4 \end{aligned}$$

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 2 & 3 & -6 \\ 6 & -2 & 4 \end{bmatrix}$$

Interchange Example

$$2x + 3y = -6$$
$$4x - 5y = 10$$

→

$$4x - 5y = 10$$
$$2x + 3y = -6$$

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix}$$

→

$$\begin{bmatrix} 4 & -5 & 10 \\ 2 & 3 & -6 \end{bmatrix}$$

Example: Row Reductions

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 2 & 3 & -6 \\ 0 & -11 & 22 \end{bmatrix}$$

Example: Row Reductions

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$$\xrightarrow{R_2 \leftarrow R_2/(-11)} \begin{bmatrix} 2 & 3 & -6 \\ 0 & 1 & -2 \end{bmatrix}$$

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$$\xrightarrow{R_1 \leftarrow R_1 - 3R_2} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix}$$

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$$\xrightarrow{R_1 \leftarrow R_1/2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix}$$

Example: Row Reductions

$$R_2 \leftarrow R_2 - 2R_1$$
$$R_2 \leftarrow R_2/(-11)$$
$$R_1 \leftarrow R_1 - 3R_2$$
$$R_1 \leftarrow R_1/2$$

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

Example: Row Reductions

$R_2 \leftarrow R_2 - 2R_1$

$R_2 \leftarrow R_2/(-11)$

elimination

$R_1 \leftarrow R_1 - 3R_2$

$R_1 \leftarrow R_1/2$

substitution

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

Row Equivalence

Definition. Two matrices are ***row equivalent*** if one can be transformed into the other by a sequence of row operations

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix}$$

Row Equivalence

Definition. Two matrices are ***row equivalent*** if one can be transformed into the other by a sequence of row operations

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix}$$

We can compute solutions by sequence of row operations

Question

$$\begin{bmatrix} 1 & 2 & 4 \\ 1 & 6 & 2 \\ 1 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 & 2 \\ 0 & -4 & 2 \\ 0 & -2 & 3 \end{bmatrix}$$

Write a sequence of row operations that converts the matrix on the left to the matrix on the right.

Answer

$$\begin{bmatrix} 1 & 2 & 4 \\ 1 & 6 & 2 \\ 1 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 & 2 \\ 0 & -4 & 2 \\ 0 & -2 & 3 \end{bmatrix}$$

first, a demo (SymPy)

Taking Stock

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Observation. Solutions look like simple systems of linear equations

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said another way: it's easy to "read off" the solutions of some systems

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<u>Solving a system of linear equations</u> is the same as <u>row reducing its augmented matrix</u> to a matrix which "represents a solution".

Taking Stock

Observation. Solutions look like simple systems of linear equations

said another way: it's easy to "read off" the solutions of some systems

<u>Solving a system of linear equations</u> is the same as <u>row reducing its augmented matrix</u> to a matrix which "represents a solution".

What matrices "represent solutions"?

Motivating Questions

What matrices "represent solutions"? (which have solutions that are easy to "read off"?)

How does the number of solutions affect the shape of these matrix?

How do we use row operations to get to those matrices?

Motivating Questions

Let's consider these first

What matrices "represent solutions"? (which have solutions that are easy to "read off"?)

How does the number of solutions affect the shape of these matrix?

How do we use row operations to get to those matrices?

Unique Solution Case

Unique Solution Case

$$\begin{bmatrix} 2 & -3 & 5 & 11 \\ 2 & -1 & 13 & 39 \\ 1 & -1 & 5 & 14 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$x = 1$$
$$y = 2$$
$$z = 3$$

Unique Solution Case

$$\begin{bmatrix} 2 & -3 & 5 & 11 \\ 2 & -1 & 13 & 39 \\ 1 & -1 & 5 & 14 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$x = 1$$
$$y = 2$$
$$z = 3$$

Like all the examples we've seen so far

The Identity Matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The Identity Matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1s along the diagonal

0s elsewhere

Unique Solution Case

coefficient matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

a system of linear equations whose **coefficient matrix** is the identity matrix represents a unique solution

No Solution Case

Example

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

No Solution Case

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

No Solution Case

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

two parallel planes

No Solution Case

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

two parallel planes

row representing $0 = 1$

No Solution Case

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

row representing $0 = 1$

a system with no solutions can be reduced to a matrix with the row

$$0\ 0\ \ldots\ 0\ 1$$

Infinite Solution Case

Example

$$\begin{bmatrix} 2 & 4 & 2 & 14 \\ 1 & 7 & 1 & 12 \end{bmatrix}$$

Infinite Solution Case

$$\begin{bmatrix} 2 & 4 & 2 & 14 \\ 1 & 7 & 1 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Infinite Solution Case

$$\begin{bmatrix} 2 & 4 & 2 & 14 \\ 1 & 7 & 1 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$x_1 + x_3 = 2$$
$$x_2 = 1$$

Infinite Solution Case

$$\begin{bmatrix} 2 & 4 & 2 & 14 \\ 1 & 7 & 1 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$x_1 + x_3 = 2$$

$$x_2 = 1$$

a system with infinity solutions can be reduced to a system which leaves a variable <u>unrestricted</u>

Infinite Solution Case

$$x_1 + x_3 = 2$$
$$x_2 = 1$$

it doesn't matter what x_3 is if we want to satisfy this system of equations

$$x_1 = 2$$
$$x_2 = 1$$
$$x_3 = 0$$

Infinite Solution Case

$$x_1 + x_3 = 2$$
$$x_2 = 1$$

it doesn't matter what x_3 is if we want to satisfy this system of equations

$$x_1 = 1.5$$
$$x_2 = 1$$
$$x_3 = 0.5$$

Infinite Solution Case

$$x_1 + x_3 = 2$$
$$x_2 = 1$$

it doesn't matter what x_3 is if we want to satisfy this system of equations

$$x_1 = 20$$
$$x_2 = 1$$
$$x_3 = -18$$

Infinite Solution Case

$$x_1 + x_3 = 2$$
$$x_2 = 1$$

it doesn't matter what x_3 is if we want to satisfy this system of equations

$x_1 = 2 - x_3$

$x_2 = 1$

x_3 is free

Infinite Solution Case

$$x_1 + x_3 = 2$$
$$x_2 = 1$$

it doesn't matter what x_3 is if we want to satisfy this system of equations

$x_1 = 2 - x_3$
$x_2 = 1$
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general form

In Sum

none	reduces to a system with the equation $0 = 1$
one	reduces to a system whose coefficient matrix is the identity matrix
infinity	reduces to a system which leaves a variable unrestricted

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Ideally, we want one *form* that handles all three cases

Motivating Questions

What matrices represent solutions? (which have solutions that are easy to read off?)

How does the number of solutions affect the shape of these matrix?

How do we use row operations to get to those matrices?

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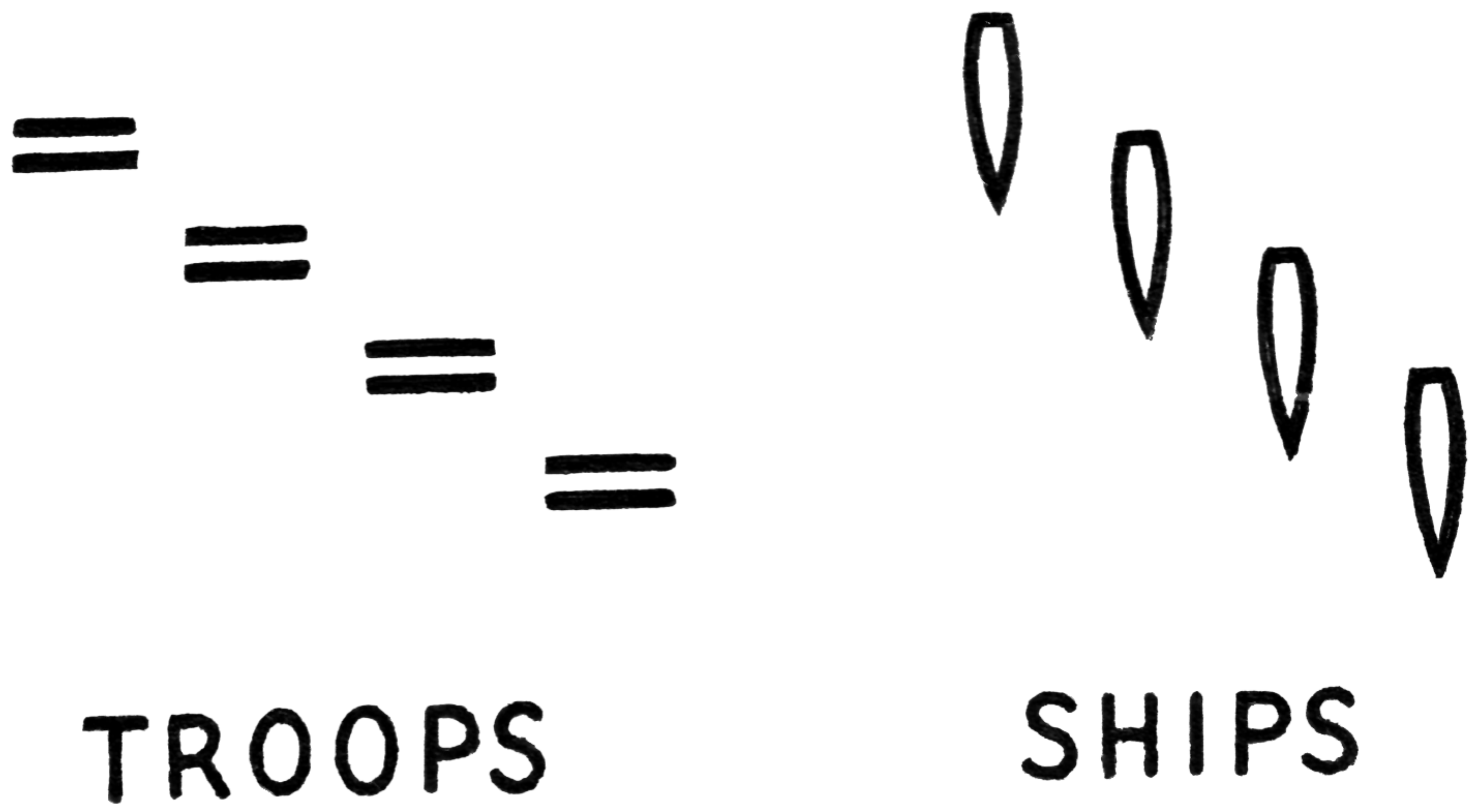
this is Gaussian elimination (next lecture)

Echelon Form

The Picture (and a bit of history)



https://commons.wikimedia.org/wiki/File:Echelon_1_(PSF).png

Echelon Form (Pictorially)

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

■ = nonzero, * = anything

Leading Entries

Definition. the ***leading entry*** of a row is the first nonzero value

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \\ 1 & -1 & 10 \end{bmatrix}$$

no leading entry

Echelon Form

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Definition. A matrix is in ***echelon form*** if

1. The leading entry of each row appears to the right of the leading entry above it
2. Every all-zeros row appears below any non-zero rows

Echelon Form (Pictorially)

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

■ = nonzero, * = anything

Echelon Form (Pictorially)

next leading entry to the right

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

all-zero rows at the bottom

■ = nonzero, * = anything

Question

Is the identity matrix in echelon form?

Answer: Yes

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

the leading entries of each row appears to the right of the leading entry above it

it has no all–zero rows

Question

Is this matrix in echelon form?

$$\begin{bmatrix} 2 & 3 & -8 \\ 0 & 1 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

Answer: No

$$\begin{bmatrix} 2 & 3 & -8 \\ 0 & 1 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

```
The leading entry of the least row is not to
the right of the leading entry of the second
row
```

What's special about Echelon forms?

Theorem. Let A be the augmented matrix of an inconsistent linear system. If $A \sim B$ and B is in echelon form then B has the row

$$[0\ 0\ \ldots\ 0\ 0\ \blacksquare]$$

Example

The Problem with Echelon Forms

If our system *is* consistent, we can't get a solution quite yet.

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If our system *is* consistent, we can't get a solution quite yet.

We need to simplify our matrix a bit more until it "represents" a solution

Reduced Echelon Form

Row-Reduced Echelon Form (RREF)

Definition. A matrix is in ***(row-)reduced echelon form*** if

1. The leading entry of each row appears to the right of the leading entry above it
2. Every all-zeros row appears below any non-zero rows
3. The leading entries of non-zero rows are 1
4. the leading entries are the only non-zero entries of their columns

Reduced Echelon Form (Pictorially)

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduced Echelon Form (Pictorially)

leading entries are 1

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

other column entries are 0

Reduced Echelon Form (A Simple Example)

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x_3 `is free`

What's special about RREF?

Every leading variable can be written in terms of every non-leading variable.

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

The Fundamental Points

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Point 1. we can "read off" the solutions of a system of linear equations from its RREF

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Point 2. *every* matrix is row equivalent to a unique matrix in reduced echelon form

How-To: Solving a System of Linear Equations

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Our next topic

General-Form Solutions

Basic and Free Variables

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Definition. a ***pivot position*** (i, j) in a matrix is the position of a leading entry in it's reduced echelon form

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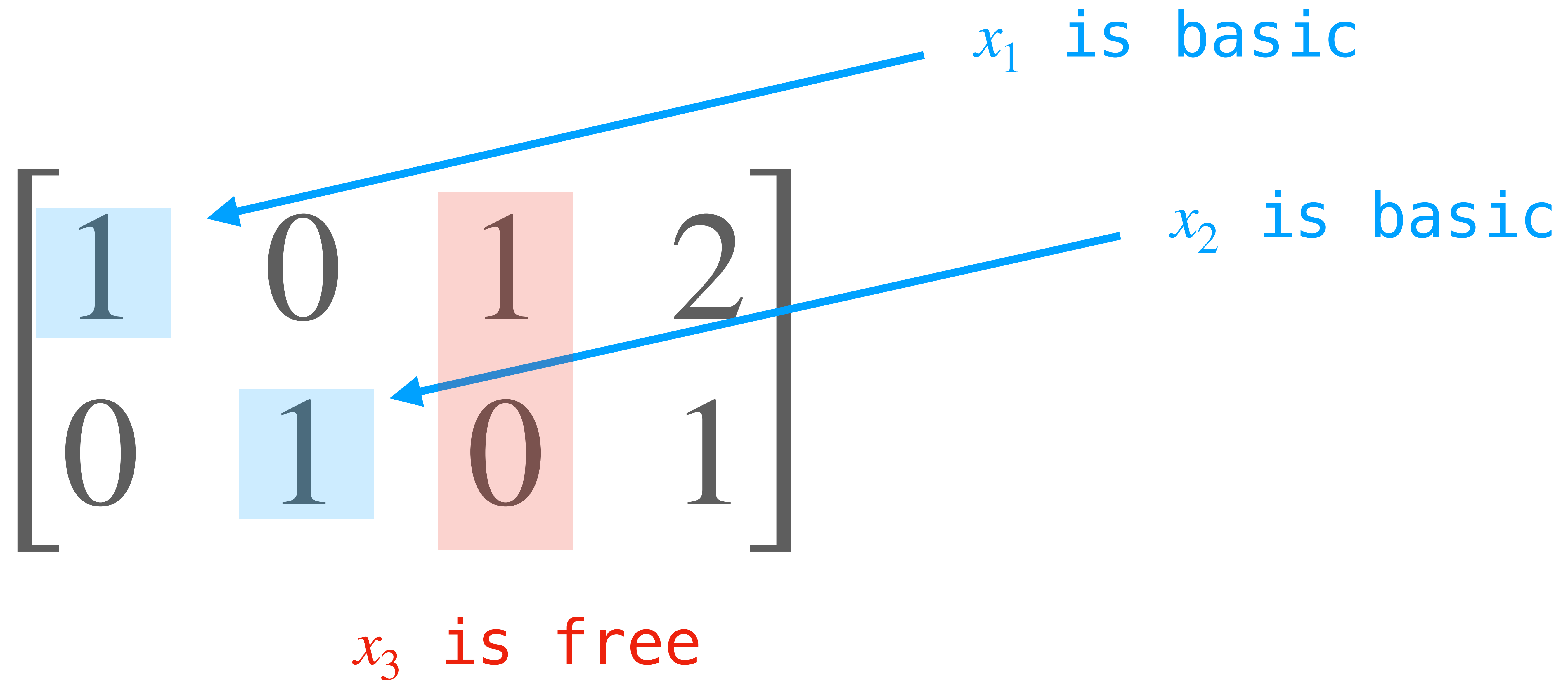

Solutions of Reduced Echelon Forms

the row of a pivot position in row i describes the value of x_i in a solution to the system, in terms of the free variables

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

General Form Solution

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$x_1 = 2 - x_3$

$x_2 = 1$

x_3 is free

for each pivot position (i, j), isolate x_i in the equation in row j

if x_i does not have a pivot position, write

x_i is free

Example

$$\begin{bmatrix} 1 & 2 & 0 & -2 & 4 \\ 0 & 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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the goal of <u>back–substitution</u> is to reduce an echelon form matrix to a ***reduced*** *echelon form*

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*the goal of <u>back–substitution</u> is to reduce an echelon form matrix to a **reduced** echelon form*

*the goal of <u>Gaussian elimination</u> is to reduce an **augmented** matrix to a **reduced** echelon form*

reduced echelon forms describe solutions to linear equations

Question

write down a solution in general form for this reduced echelon form matrix

$$\begin{bmatrix} 1 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Answer

$$\begin{bmatrix} 1 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

demo
(a.rref())