Vector Equations

Geometric Algorithms
Lecture 5

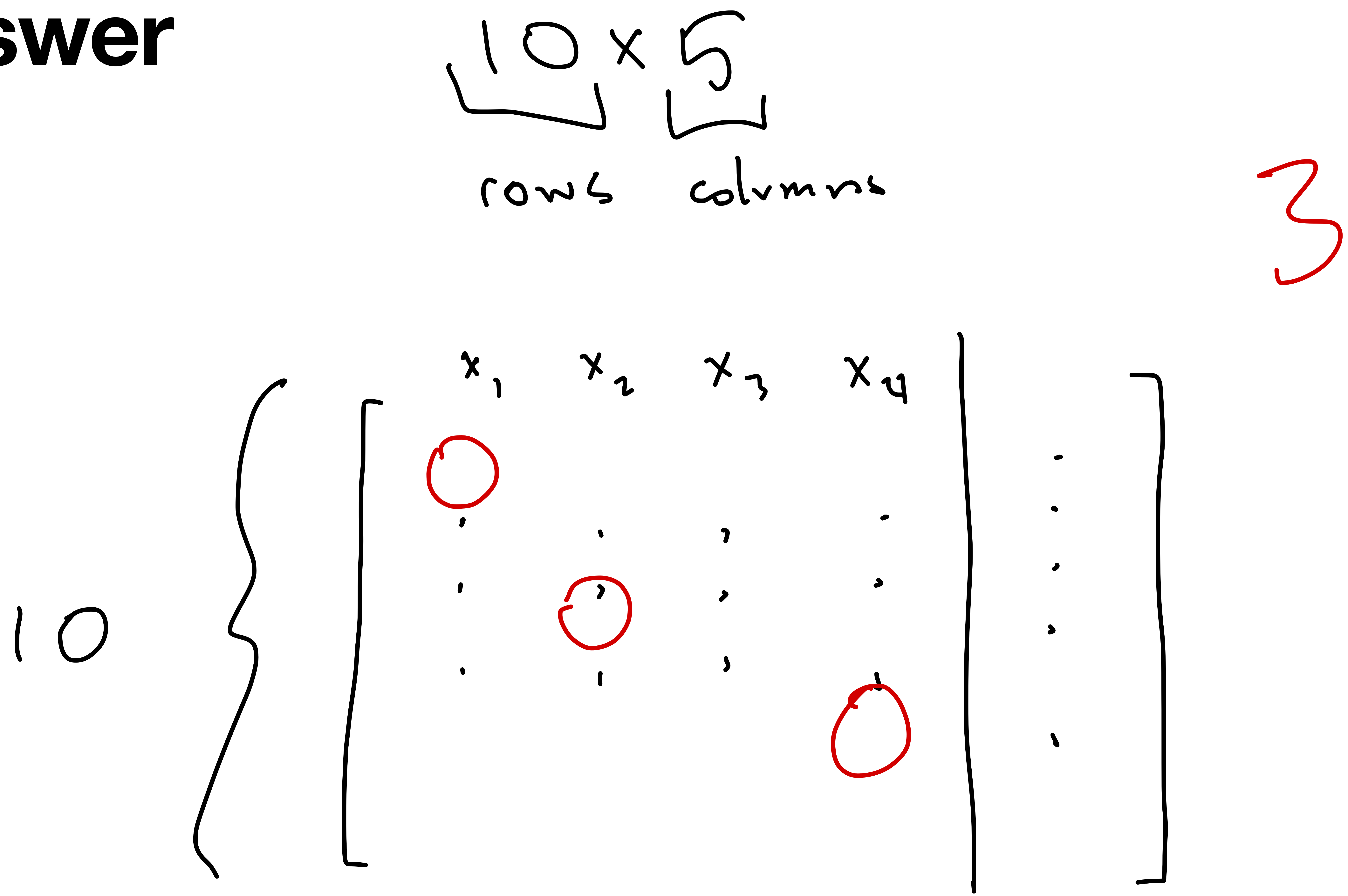
CAS CS 132

Practice Problem

Suppose that A is a 322×245 augmented matrix for a system with infinitely many solutions. What is the maximum number of pivot positions that A can have?

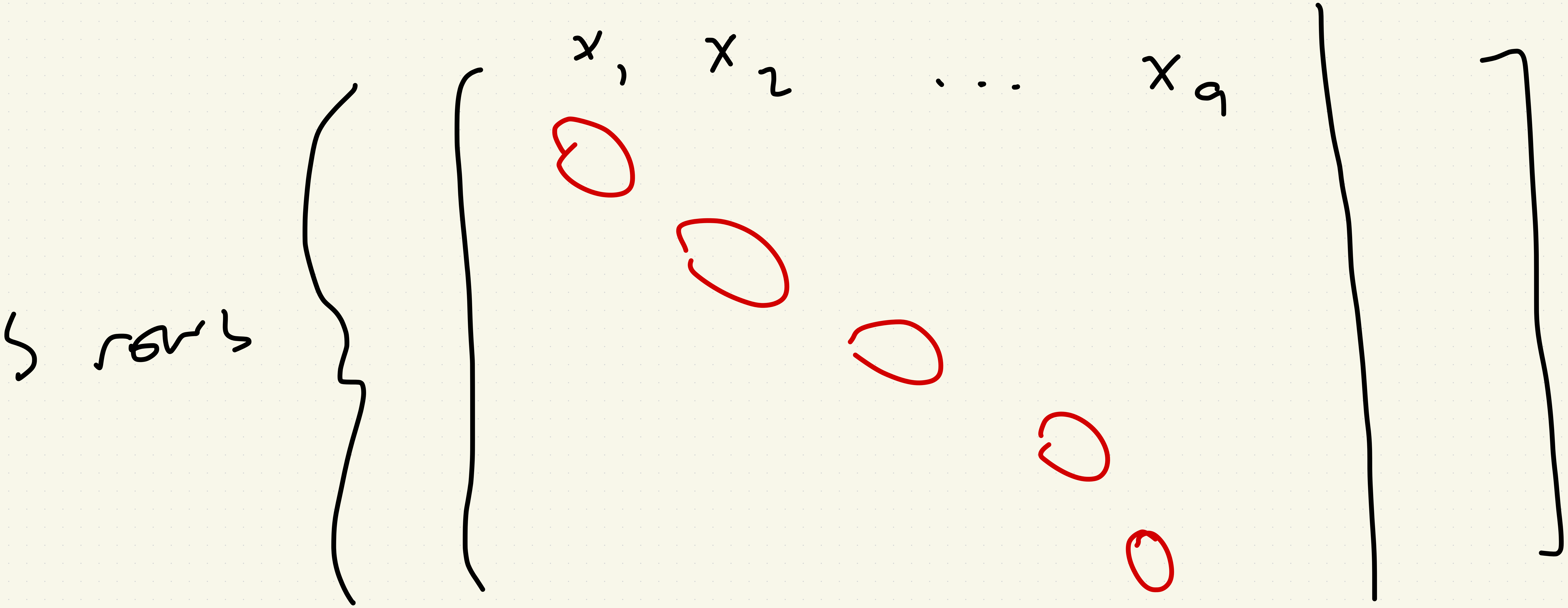
What about 245×322 ?

Answer



5×10

5



Objectives

1. Define vectors
2. Discuss vector operations and vector algebra
3. Draw the connection between vectors and systems of linear equations

Keywords

vector

vector addition

vector scaling/multiplication

the zero vector

vector equations

linear combinations

span

Motivation (An Aside)

Changing Perspective

$1+2+4+8+\cdots$

$$\sum_{i=0}^{n-1} 2^i = 2^n - 1$$

Show that this holds for all n

Changing Perspective

```
 0001
 0010
 0100
 1000 +
------
01111
```

$$\sum_{i=0}^{n-1} 2^i = 2^n - 1$$

$$100...000 - 000...001 = 011...111$$

show that this holds for all n

Changing Perspective

$$\sum_{i=0}^{n-1} 2^i = 2^n - 1$$

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show that this holds for all n

much easier in binary

Motivation?

vectors will be one of the most important shifts of perspective in this course

the insight is simple yet elegant

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the insight is simple yet elegant

maybe I'm reaching...

Big Data

a piece of data is a bunch of distinct values (numbers)

How can we tell if two piece of data are similar?

maybe if they are **close together** in a geometric sense

A Note on Algebra

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doing abstract algebra is like implementing an interface

we're defining an new thing called a "column vector"

we need to define what "equality" and "adding" and "multiplying by a number" means for column vectors

Vectors

What is a vector (in $\mathbb{R}^n$)?

A. an n–tuple of real numbers

B. a point in $\mathbb{R}^n$

C. a 1–column matrix with real values

D. all of the above

E. none of the above?

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it's common to conflate points and vectors

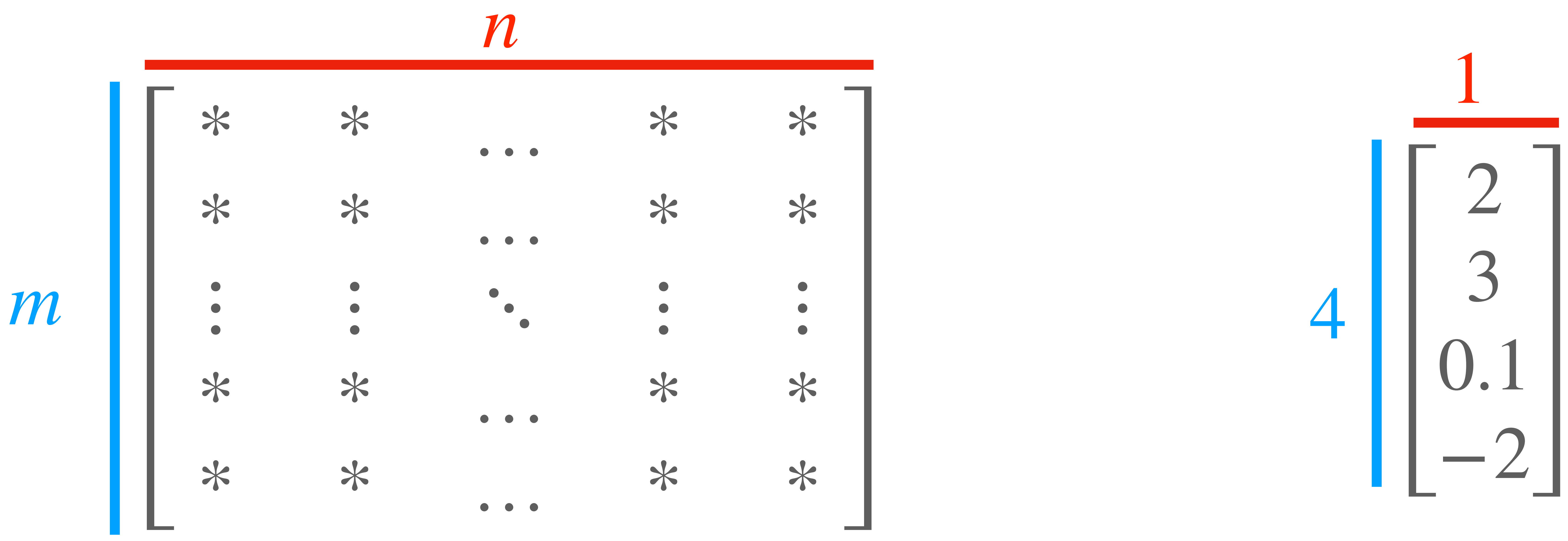
Column Vectors

Definition. a ***column vector*** is a matrix with a single column, e.g.,

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

A Note on Matrix Size

an $(m \times n)$ matrix is a matrix with m rows and n columns



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an $(m \times n)$ matrix is a matrix with m rows and n columns

$$\begin{bmatrix} * & * & \cdots & * & * \\ * & * & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & * & * \\ * & * & \cdots & * & * \end{bmatrix}$$

m rows, n columns

$$\begin{bmatrix} 2 \\ 3 \\ 0.1 \\ -2 \end{bmatrix}$$

4 rows, 1 column

$\mathbb{R}^{m \times n}$ is set of matrices with $\mathbb{R}$ entries

A Note on Matrix Size

an $(m \times n)$ matrix is a matrix with m rows and n columns

$\leftarrow \mathbb{R}^{4\times 1} \cong \mathbb{R}^{4}$

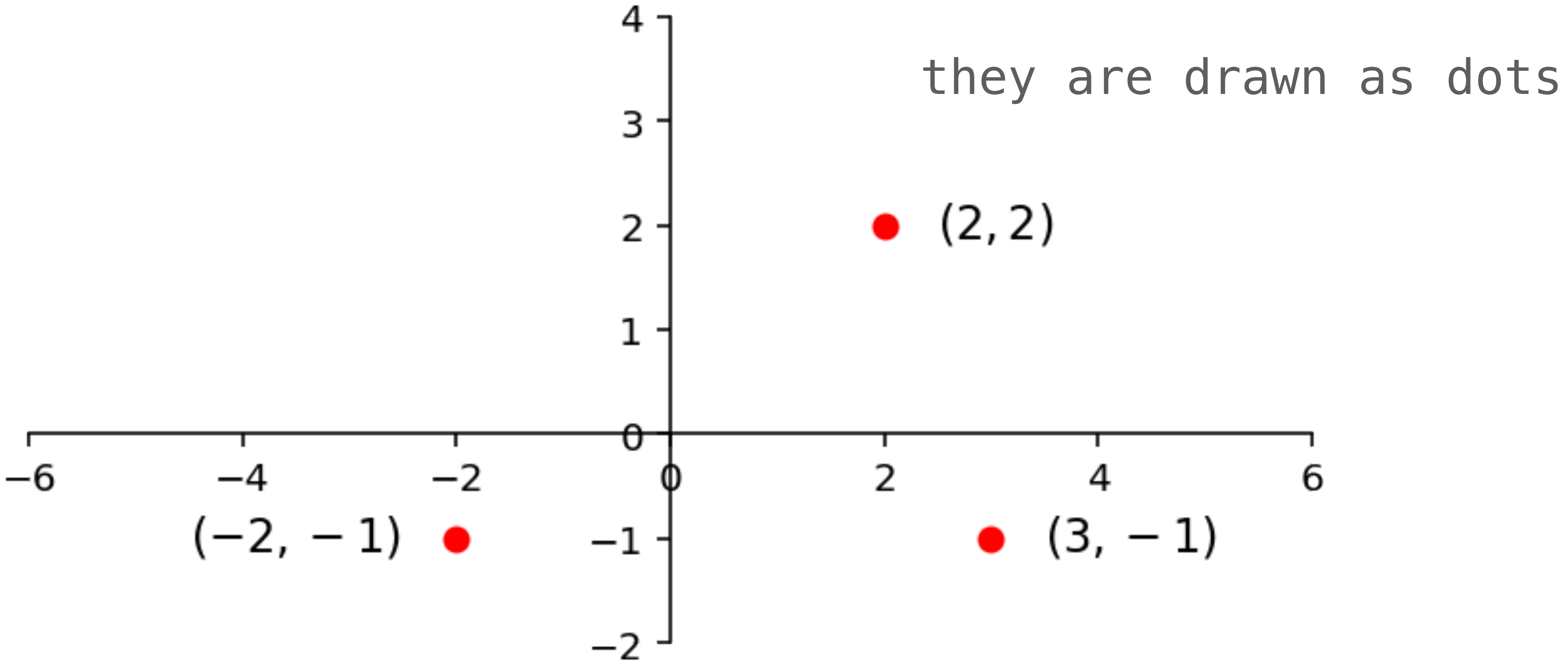
the number of rows of a vectors is called its **dimension**

$\mathbb{R}^{m\times n}$ is set of matrices with $\mathbb{R}$ entries

Examples

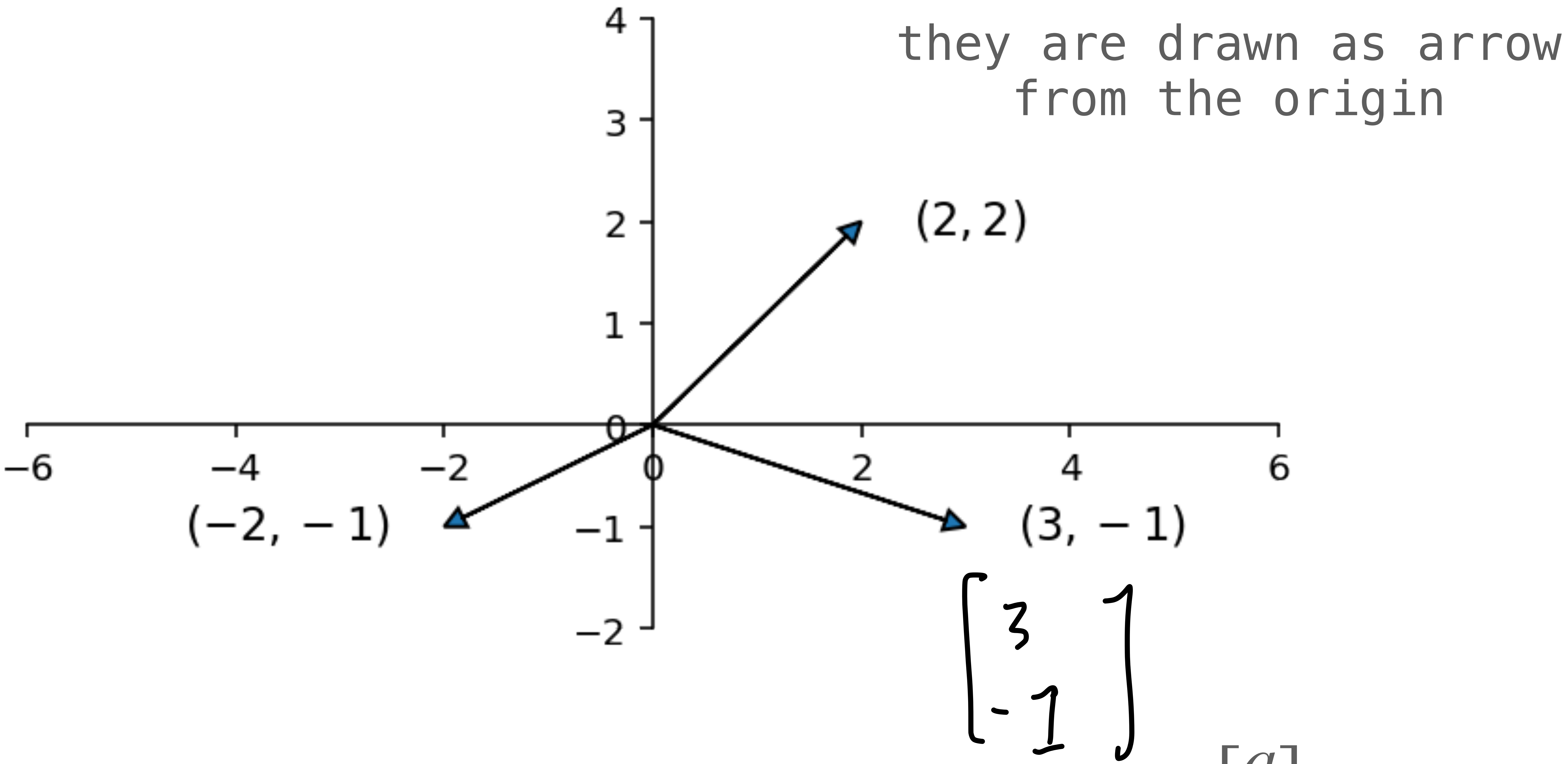
$$\begin{bmatrix} \pi \\ \sqrt{2} \\ 3 \\ 0 \end{bmatrix} \in \mathbb{R}^4$$

Notation (Points)



points in $\mathbb{R}^2$ are notated as (a, b)

Notation (Vectors)



vectors in $\mathbb{R}^2$ are notated as $\begin{bmatrix} a \\ b \end{bmatrix}$

Notation (Looking ahead)

we will often write $[a_1\ a_2\ \ldots\ a_n]^T$ for the vector

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^{n \times 1} \cong \mathbb{R}^n$$

!!IMPORTANT!!

$(a_1, a_2, \ldots, a_n)$ is not the same as $[a_1\ \ a_2\ \ \ldots\ \ a_n] \in \mathbb{R}^{1 \times n}$

Vector Operations

Vector "Interface"

Vector "Interface"

```
equality    what does it mean for two vectors
            to be equal?
```

Vector "Interface"

equality — what does it mean for two vectors to be equal?

addition — what does $\mathbf{u} + \mathbf{v}$ (adding two vectors mean?

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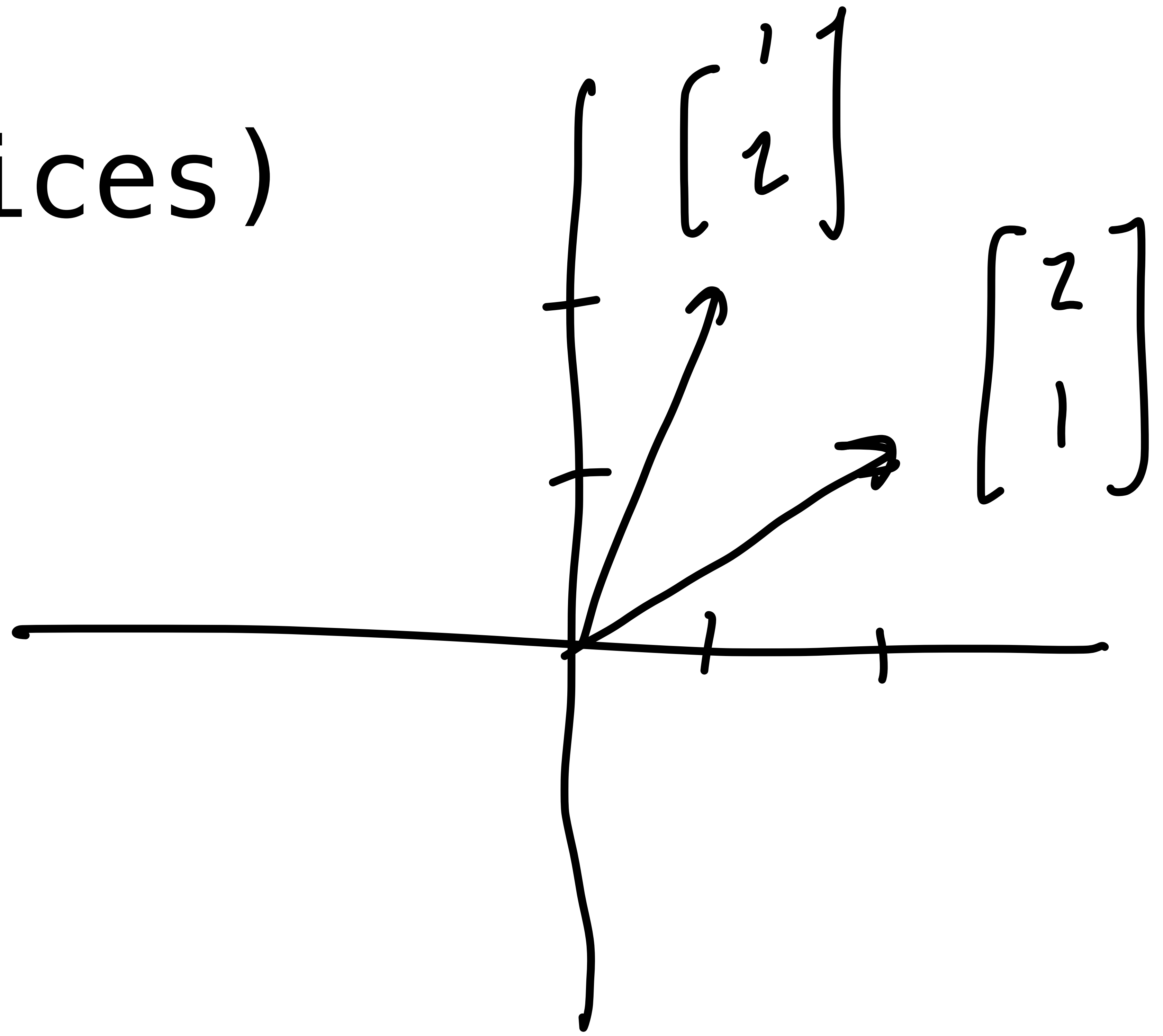
Vector Equality

two vectors are equal if their entries at each position are equal

(this is also the case for matrices)

!!IMPORTANT!!
ORDER MATTERS

$$\begin{bmatrix}1\\2\end{bmatrix} \neq \begin{bmatrix}2\\1\end{bmatrix}$$



Vector Equality

$$\begin{bmatrix}1\\2\\3\end{bmatrix} \stackrel{?}{=} \begin{bmatrix}1\\2\end{bmatrix}$$

$$\begin{bmatrix}a_1\\a_2\\\vdots\\a_n\end{bmatrix} = \begin{bmatrix}b_1\\b_2\\\vdots\\b_n\end{bmatrix}$$

is the same as

$$\begin{aligned}a_1 &= b_1\\a_2 &= b_2\\&\vdots\\a_n &= b_n\end{aligned}$$

Examples

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1+1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

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Vector Addition

adding two vectors means adding their entries

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix}$$

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!!IMPORTANT!!

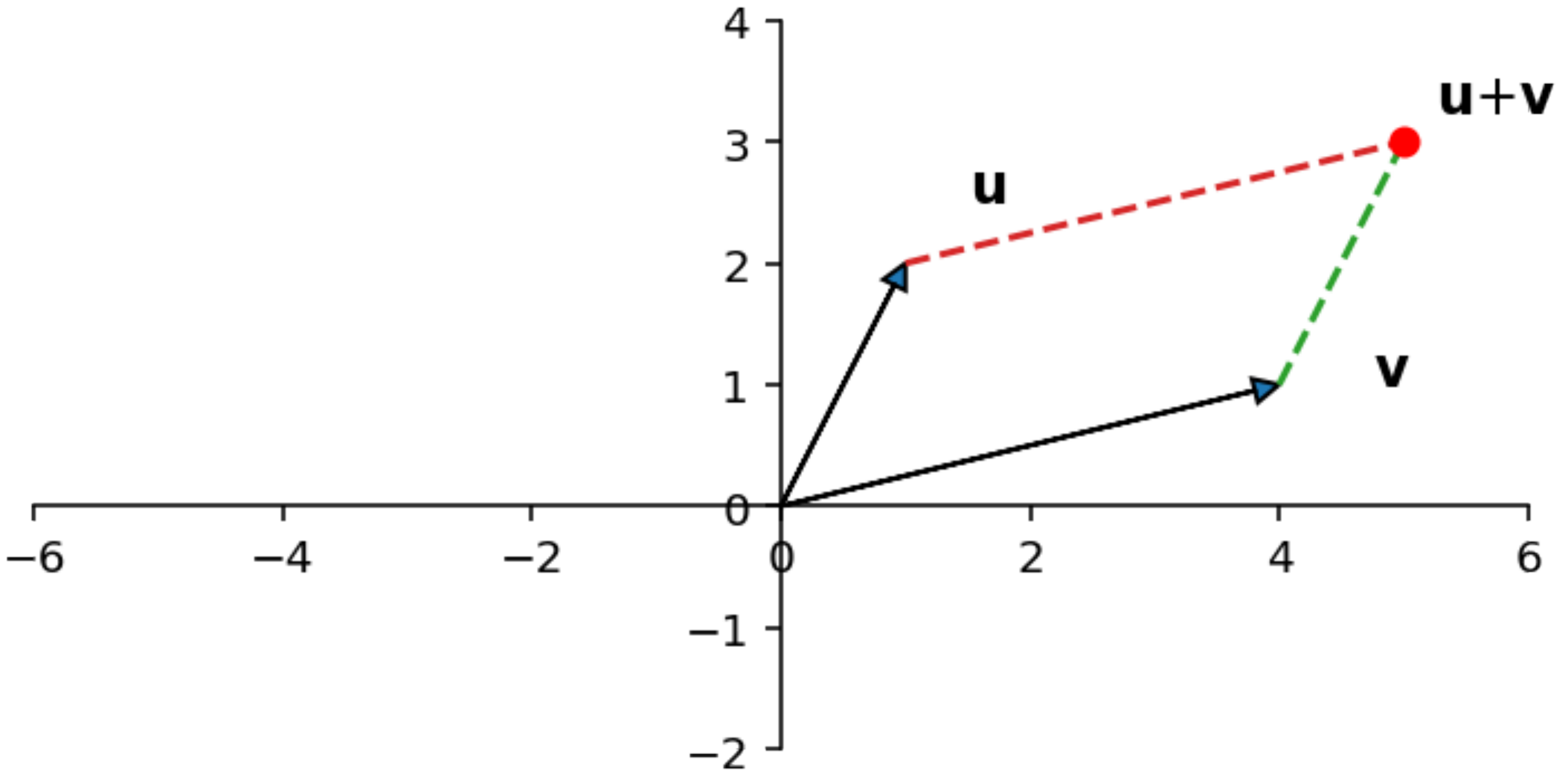
WE CAN ONLY ADD VECTORS OF THE SAME SIZE

Examples

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+1 \\ 2+0 \\ 3+1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$$

Vector Addition (Geometrically)

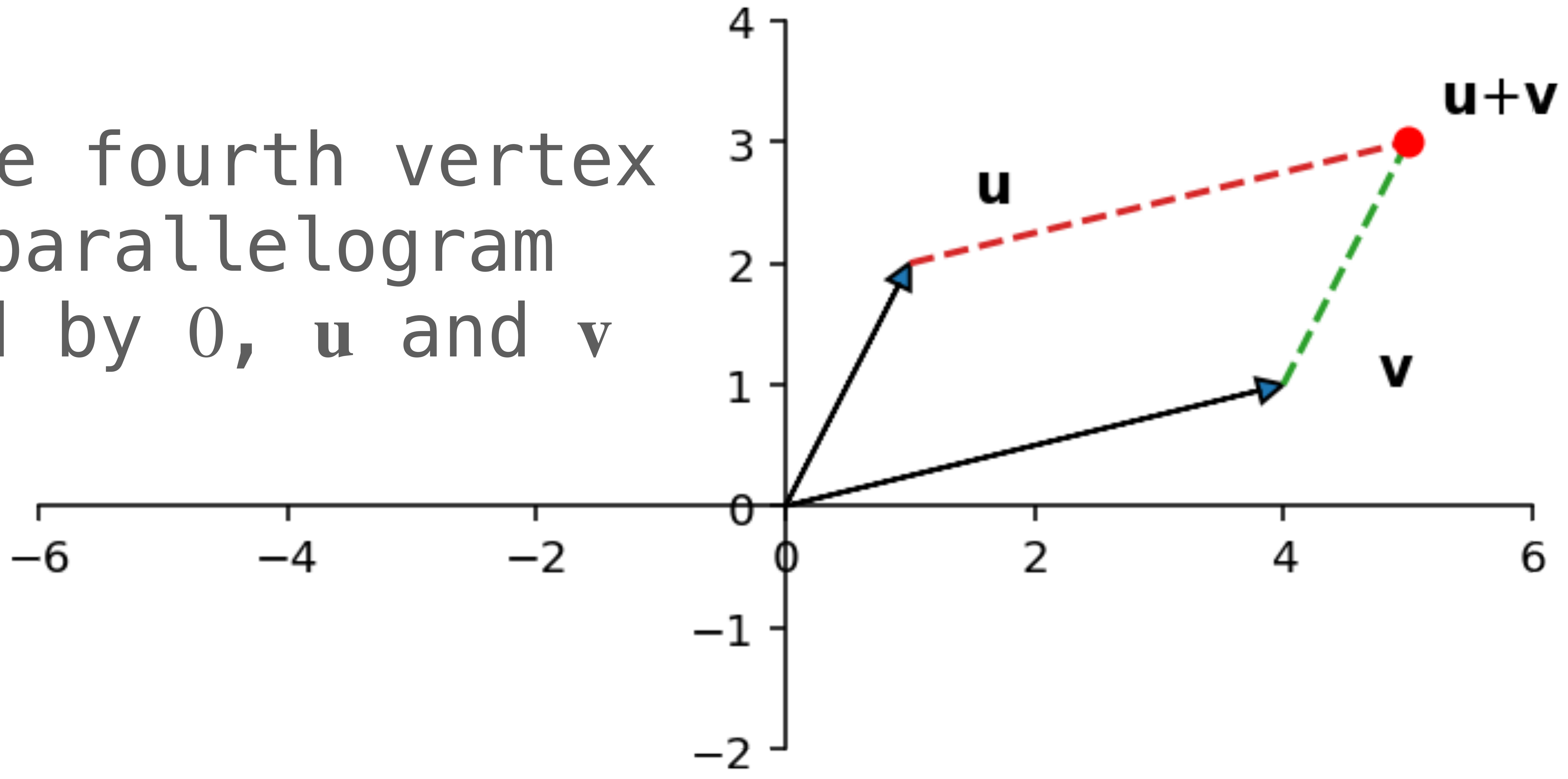
in $\mathbb{R}^2$ it's called the *parallelogram rule*



Vector Addition (Geometrically)

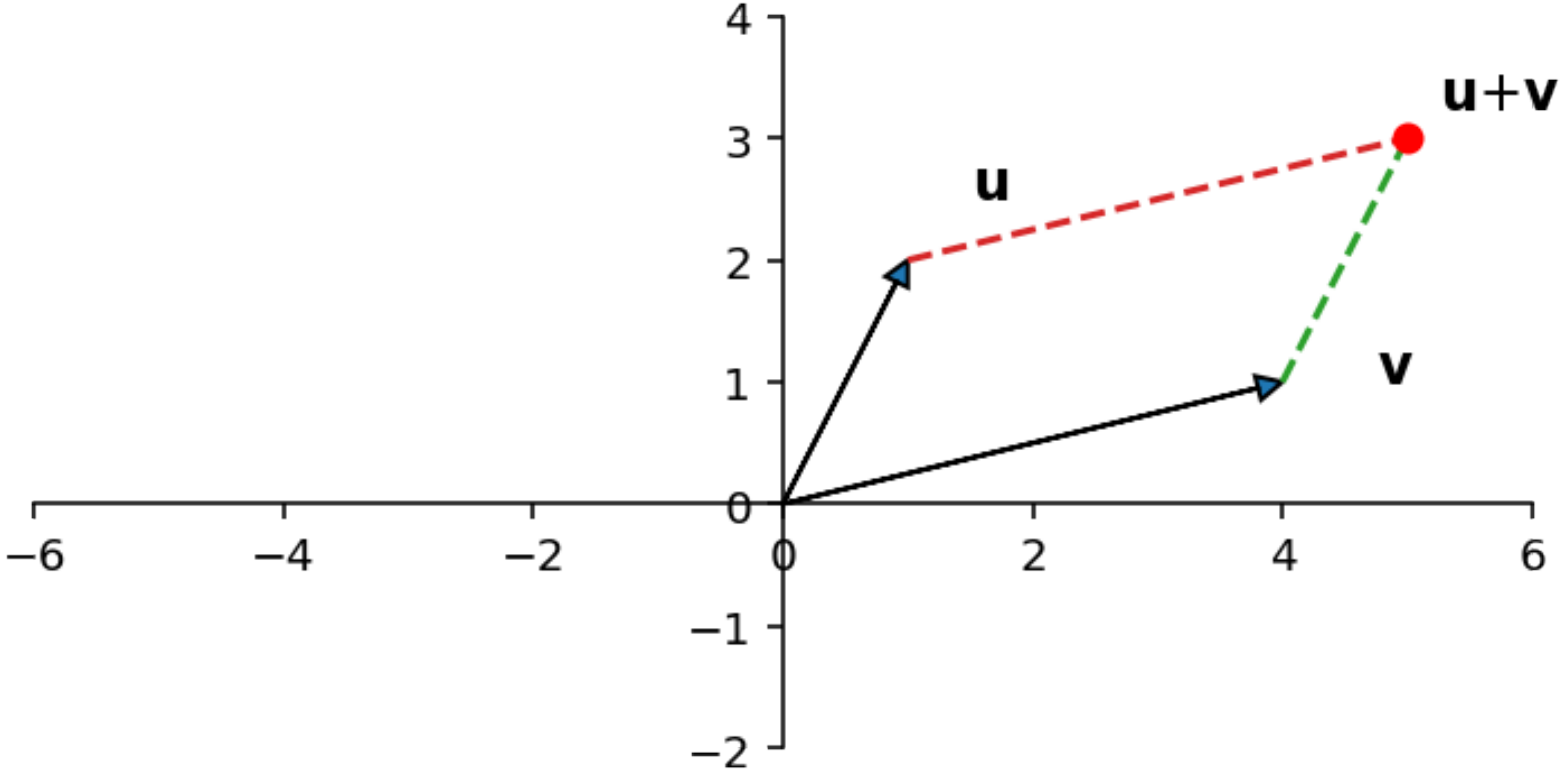
in $\mathbb{R}^2$ it's called the *parallelogram rule*

$\mathbf{u} + \mathbf{v}$ is the fourth vertex of the parallelogram generated by 0, $\mathbf{u}$ and $\mathbf{v}$



Vector Addition (Geometrically)

or the *tip–to–tail rule*



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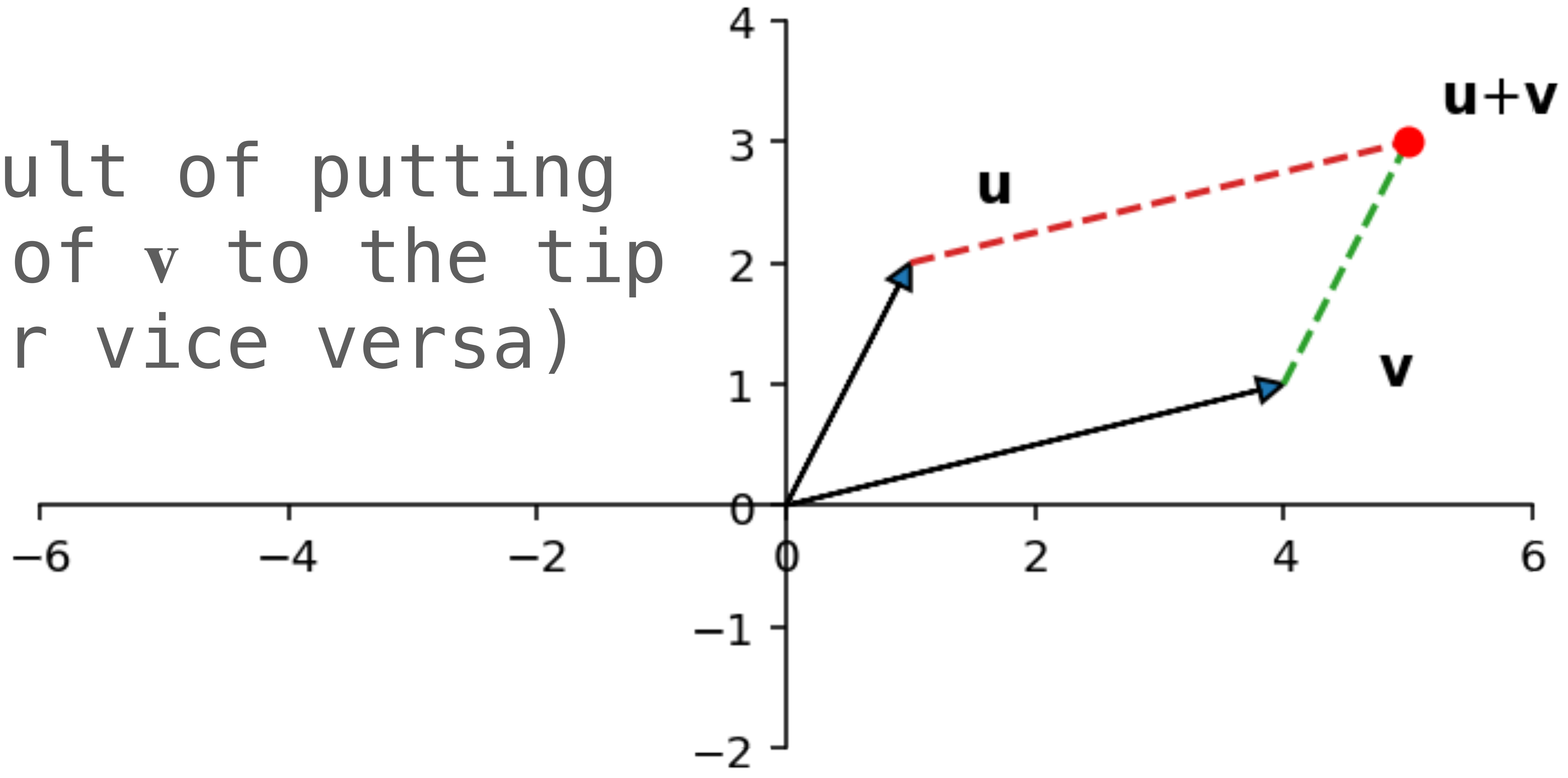
$\mathbf{u} + \mathbf{v}$ result of putting the tail of $\mathbf{v}$ to the tip of $\mathbf{u}$ (or vice versa)



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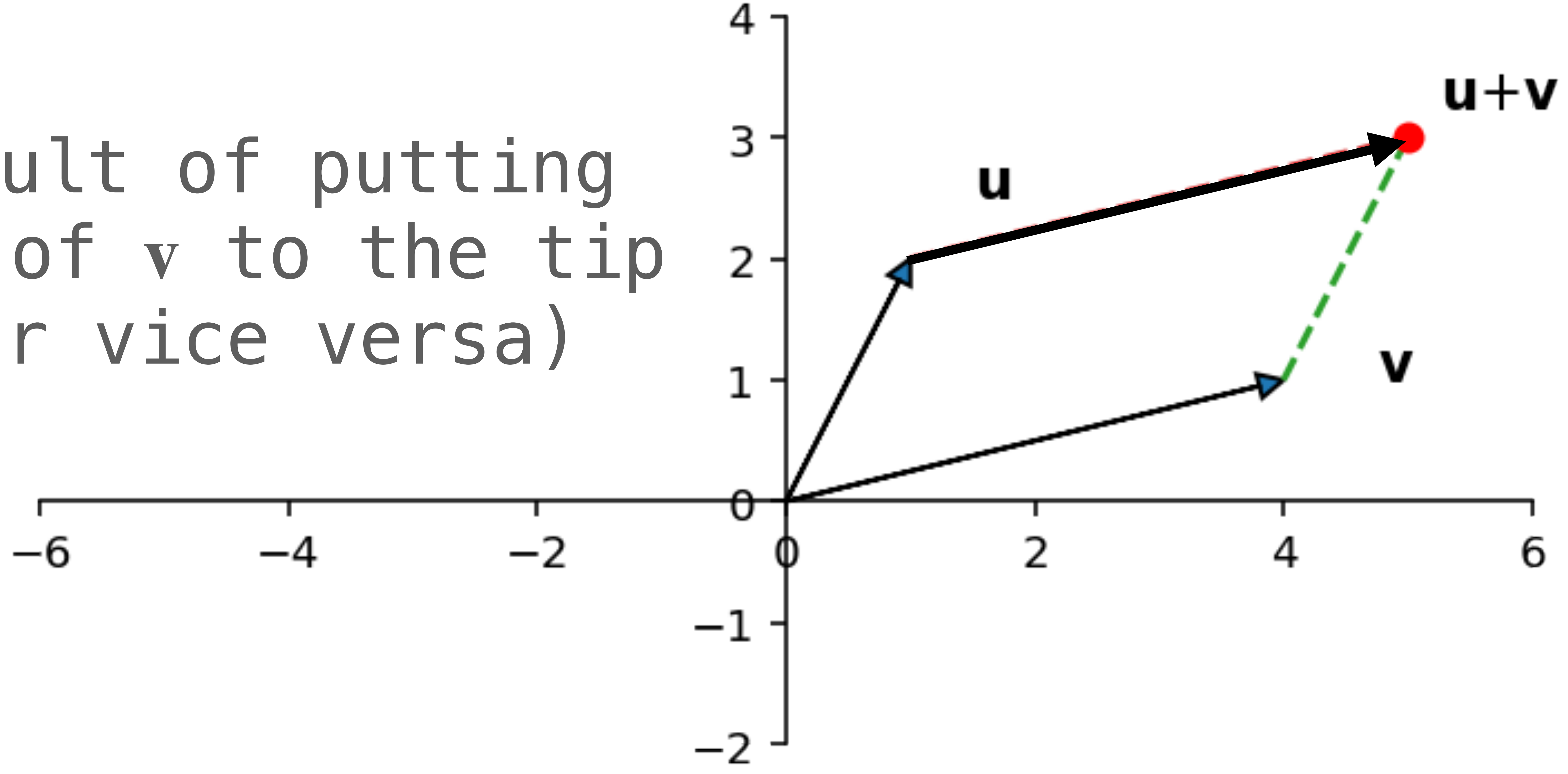

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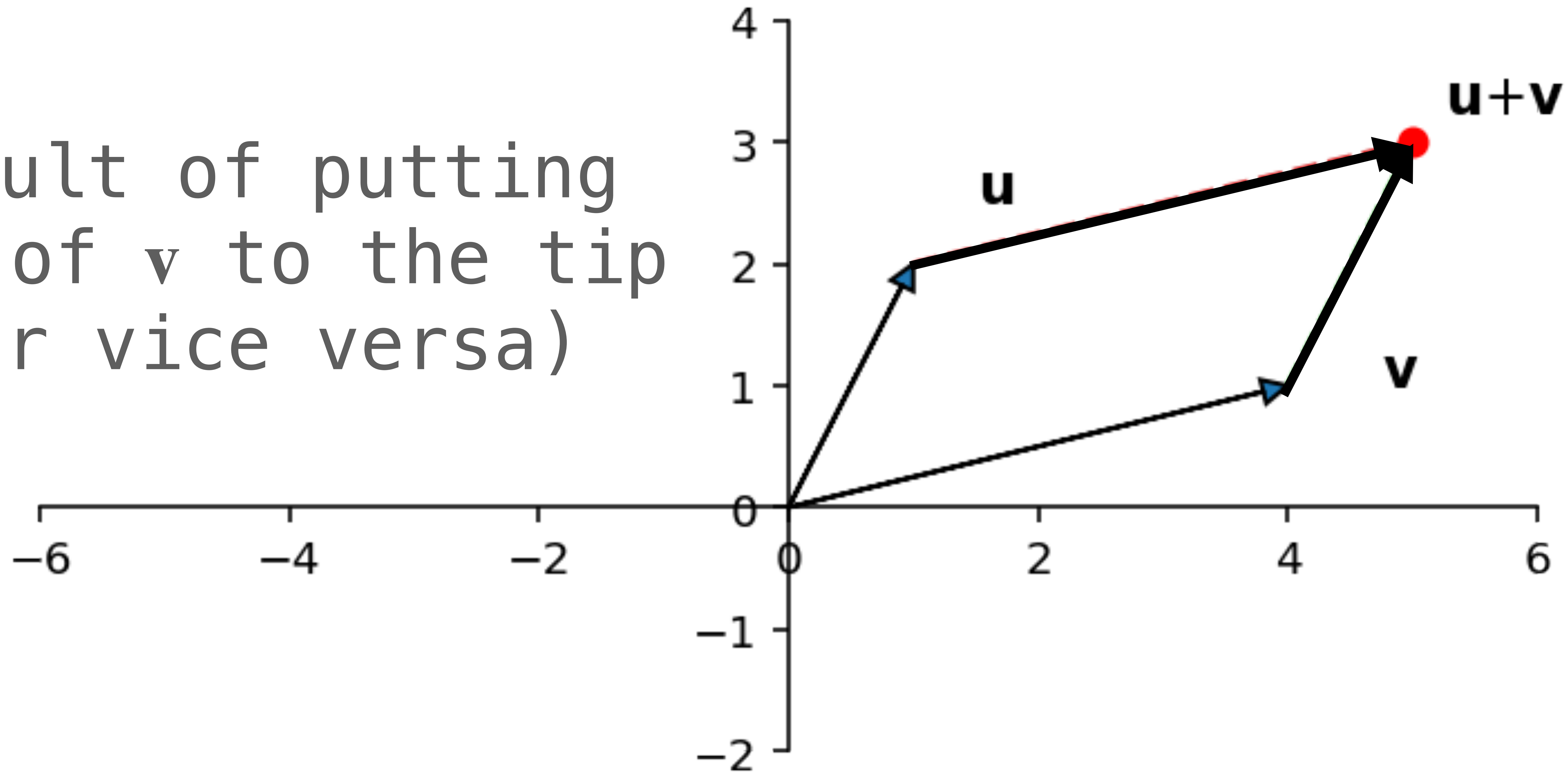

demo

(from ILA)

Vector "Interface"

equality	what does it mean for two vectors to be equal?
addition	what does $\mathbf{u} + \mathbf{v}$ (adding two vectors mean?
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What properties do they need to satisfy?

Vector Scaling/Multiplication

scaling/multiplying a vector by a number means multiplying each of it's elements

$$a \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} ab_1 \\ ab_2 \\ \vdots \\ ab_n \end{bmatrix}$$

Vector Scaling/Multiplication (Example)

$$3 \begin{bmatrix} 2 \\ 1 \\ 3.5 \\ 4 \end{bmatrix}$$

Vector Scaling/Multiplication (Example)

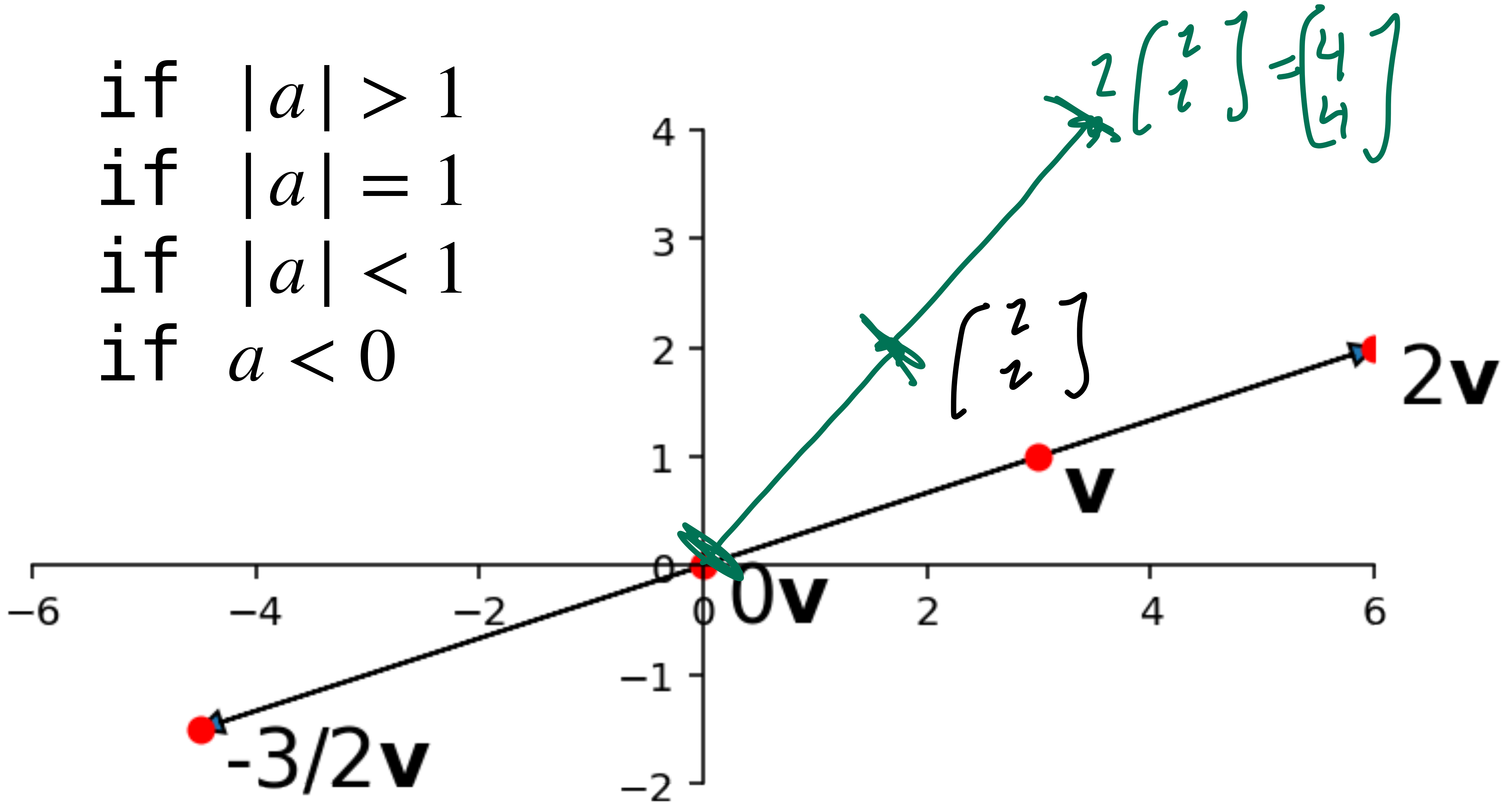
$$3 \begin{bmatrix} 2 \\ 1 \\ 3.5 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \cdot 2 \\ 3 \cdot 1 \\ 3 \cdot 3.5 \\ 3 \cdot 4 \end{bmatrix}$$

Vector Scaling/Multiplication (Example)

$$3 \begin{bmatrix} 2 \\ 1 \\ 3.5 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \cdot 2 \\ 3 \cdot 1 \\ 3 \cdot 3.5 \\ 3 \cdot 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 10.5 \\ 12 \end{bmatrix}$$

Vector Scaling (Geometrically)

longer	if $\|a\| > 1$
the same length	if $\|a\| = 1$
shorter	if $\|a\| < 1$
reversed	if $a < 0$



demo

(from ILA)

Algebraic Properties

For any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ and any real numbers c, d:

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$$

$$\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$$

$$\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$$

$$c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$$

$$(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$$

$$c(d\mathbf{u}) = (cd)\mathbf{u}$$

$$1\mathbf{u} = \mathbf{u}$$

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these are requirements for any **vector space**
they matter more for *bizarre* vector spaces

$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

Example "Proof"

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix} = \begin{bmatrix} v_1 + u_1 \\ v_2 + u_2 \\ v_3 + u_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Question (Practice)

$$3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 0 \\ 3 \\ -1 \end{bmatrix} - \begin{bmatrix} -3 \\ 4 \\ 2 \\ 0 \end{bmatrix}$$

Compute the value of the above vector.

Answer

$$3\begin{bmatrix}1\\1\\1\\1\end{bmatrix}+2\begin{bmatrix}2\\0\\3\\-1\end{bmatrix}-\begin{bmatrix}-3\\4\\2\\0\end{bmatrix}$$

$$\begin{bmatrix}3\\3\\3\\3\end{bmatrix}+\begin{bmatrix}4\\0\\6\\-2\end{bmatrix}-\begin{bmatrix}-3\\4\\2\\0\end{bmatrix}=\begin{bmatrix}10\\-1\\7\\1\end{bmatrix}$$

In Sum

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`we can add vectors`

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```
we can add vectors

we can scale vectors
```

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this gives us a way of generating new vectors from old ones

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What vectors can we make in this way?

Linear Combinations

Linear Combinations

Definition. a ***linear combination*** of vectors

$$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$$

is a vector of the form

$$\alpha_1 \mathbf{v}_1 + \alpha_1 \mathbf{v}_2 + \ldots + \alpha_n \mathbf{v}_n$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are in $\mathbb{R}$

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weights

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Looks suspiciously like a linear equation...

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are in $\mathbb{R}$

weights

Linear Combinations (Example)

$$3\begin{bmatrix}1\\1\\1\\1\end{bmatrix}+2\begin{bmatrix}2\\0\\3\\-1\end{bmatrix}-\begin{bmatrix}-3\\4\\2\\0\end{bmatrix}$$

Linear Combinations (Geometrically)

demo

(from ILA)

The Fundamental Concern

Can $\mathbf{u}$ be written as a linear combination of

$$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n?$$

That is, are there weights $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots \alpha_n\mathbf{v}_n = \mathbf{u}?$$

Why is this fundamental?

I'm going to ask that you suspend your disbelief...

Why is this fundamental?

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For now, how do we solve this problem?

Vector Equations and Linear Systems

The Fundamental Connection

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```
we don't know the weights, that's want we want
to find
```

The Fundamental Connection

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what if we write them as *unknowns*?

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$$x_1 \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

Some Symbol Pushing...

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$$\begin{bmatrix} x_1 \\ -2x_1 \\ -5x_1 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ 5x_2 \\ 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 + 2x_2 \\ -2x_1 + 5x_2 \\ -5x_1 + 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

$$\begin{aligned} x_1 + 2x_2 &= 7 \\ -2x_1 + 5x_2 &= 4 \\ -5x_1 + 6x_2 &= -3 \end{aligned}$$

The Fundamental Connection

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The Fundamental Connection

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$$x_1 + 2x_2 = 7$$

$$(-2)x_1 + 5x_2 = 4$$

$$-5x_1 + 6x_2 = -3$$

The Fundamental Connection

we don't know the weights, that's want we want to find

what if we write them as *unknowns*?

$$x_1 + 2x_2 = 7$$
$$(-2)x_1 + 5x_2 = 4$$
$$-5x_1 + 6x_2 = -3$$

we get a system of linear equations we know how to solve

The Fundamental Connection

More generally:

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{1m} \end{bmatrix} + x_2 \begin{bmatrix} a_{21} \\ a_{21} \\ \vdots \\ a_{2m} \end{bmatrix} + \ldots + x_n \begin{bmatrix} a_{n1} \\ a_{n2} \\ \vdots \\ a_{nm} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

The Fundamental Connection

$$\begin{bmatrix} a_{11}x_1 \\ a_{21}x_1 \\ \vdots \\ a_{1m}x_1 \end{bmatrix} + \begin{bmatrix} a_{21}x_2 \\ a_{21}x_2 \\ \vdots \\ a_{2m}x_1 \end{bmatrix} + \ldots + \begin{bmatrix} a_{n1}x_n \\ a_{n2}x_n \\ \vdots \\ a_{nm}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

`by vector scaling`

The Fundamental Connection

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

`by vector addition`

The Fundamental Connection

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}
$$

`by vector equality`

The Fundamental Connection

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$$

augmented matrix

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

system of linear equations

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vector equation

The Fundamental Connection

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$$

augmented matrix

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this is our big shift in perspective

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HOW TO: Linear Combination Problems

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this is notation for building a matrix out of column vectors

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Question

Can $\begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$ *be written as a linear combination of* $\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$ *and* $\begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$*?*

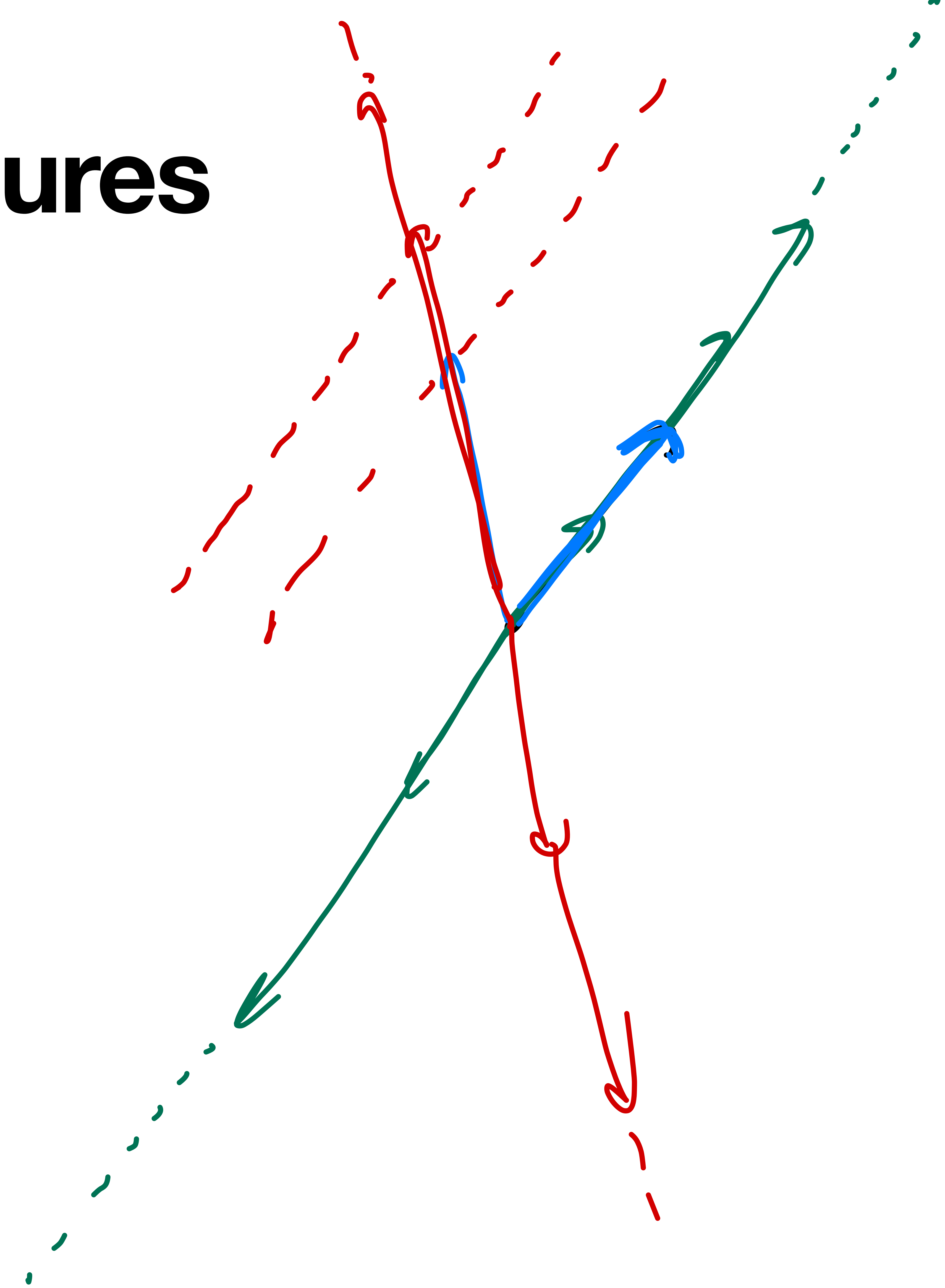
Answer

$$3\begin{bmatrix}1\\-2\\-5\end{bmatrix}+2\begin{bmatrix}2\\5\\6\end{bmatrix}=\begin{bmatrix}7\\4\\-3\end{bmatrix}$$

$$\begin{bmatrix}1 & 2 & 7\\-2 & 5 & 4\\5 & 6 & -3\end{bmatrix}\sim\begin{bmatrix}1 & 0 & 3\\0 & 1 & 2\\0 & 0 & 0\end{bmatrix}$$

Spans

Some Pictures



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$$\vec{0} \in \text{span}\{\vec{v}_1, \ldots, \vec{v}_n\} \qquad \vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

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read: $\mathbf{u}$ is an element of $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$

$\mathbf{u} \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ **exactly when $\mathbf{u}$ can be expressed as a linear combination of those vectors**

Linear Combinations and Spans (A Picture)

Spans (Geometrically)

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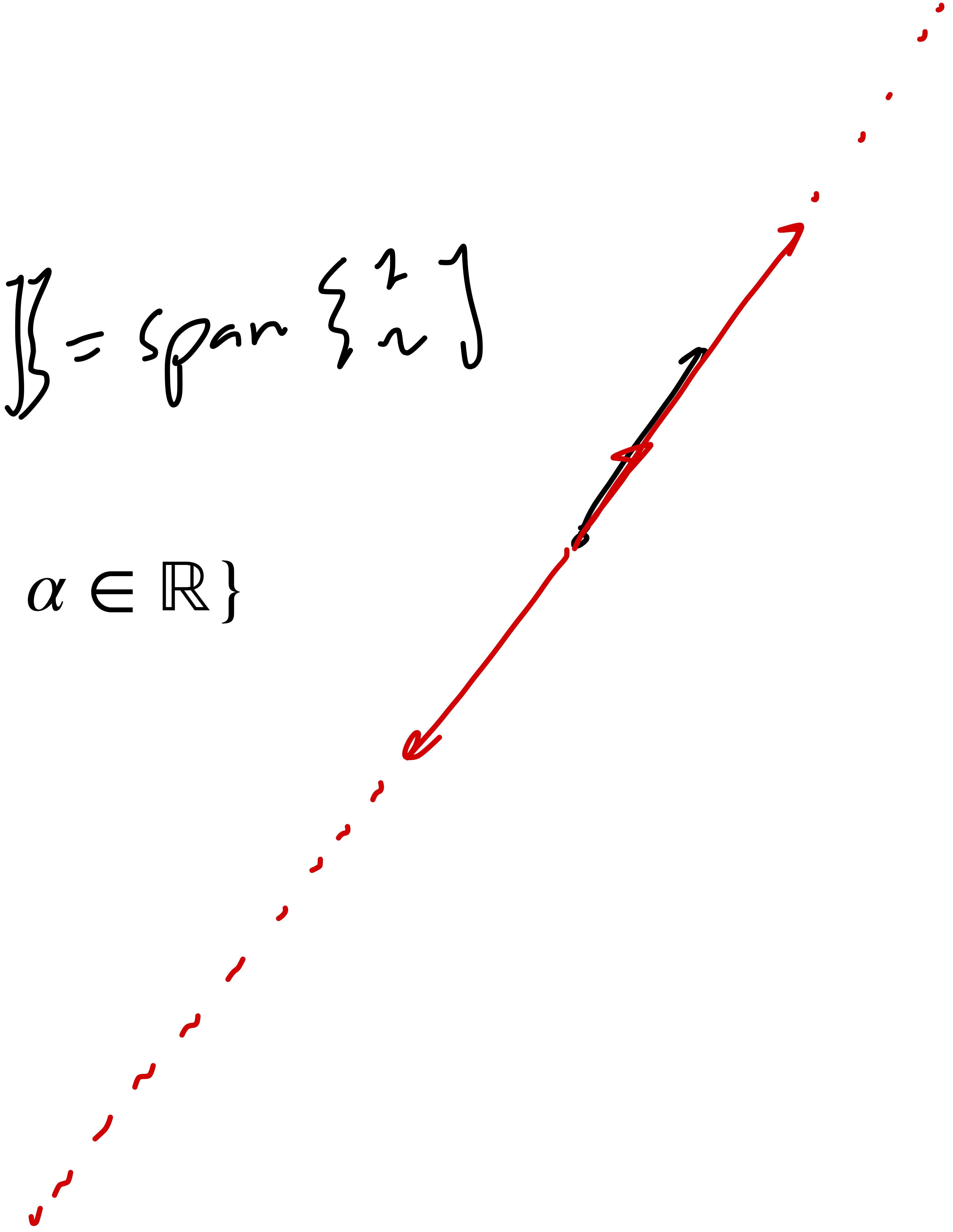
`for one vector`

Spans (Geometrically)

$$\text{span}\left\{\begin{bmatrix}4\\4\end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix}2\\2\end{bmatrix}\right\}$$

for one vector

$$\text{span}\{\mathbf{v}\} = \{\alpha\mathbf{v} : \alpha \in \mathbb{R}\}$$



$$\begin{bmatrix}2\\2\end{bmatrix} \neq \begin{bmatrix}4\\4\end{bmatrix}$$

$$\begin{bmatrix}2\\2\end{bmatrix} \in \text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$$

$$\begin{bmatrix}4\\4\end{bmatrix} \in \text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$$

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for one vector

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the span of one vector is a **line**

Spans (Geometrically)

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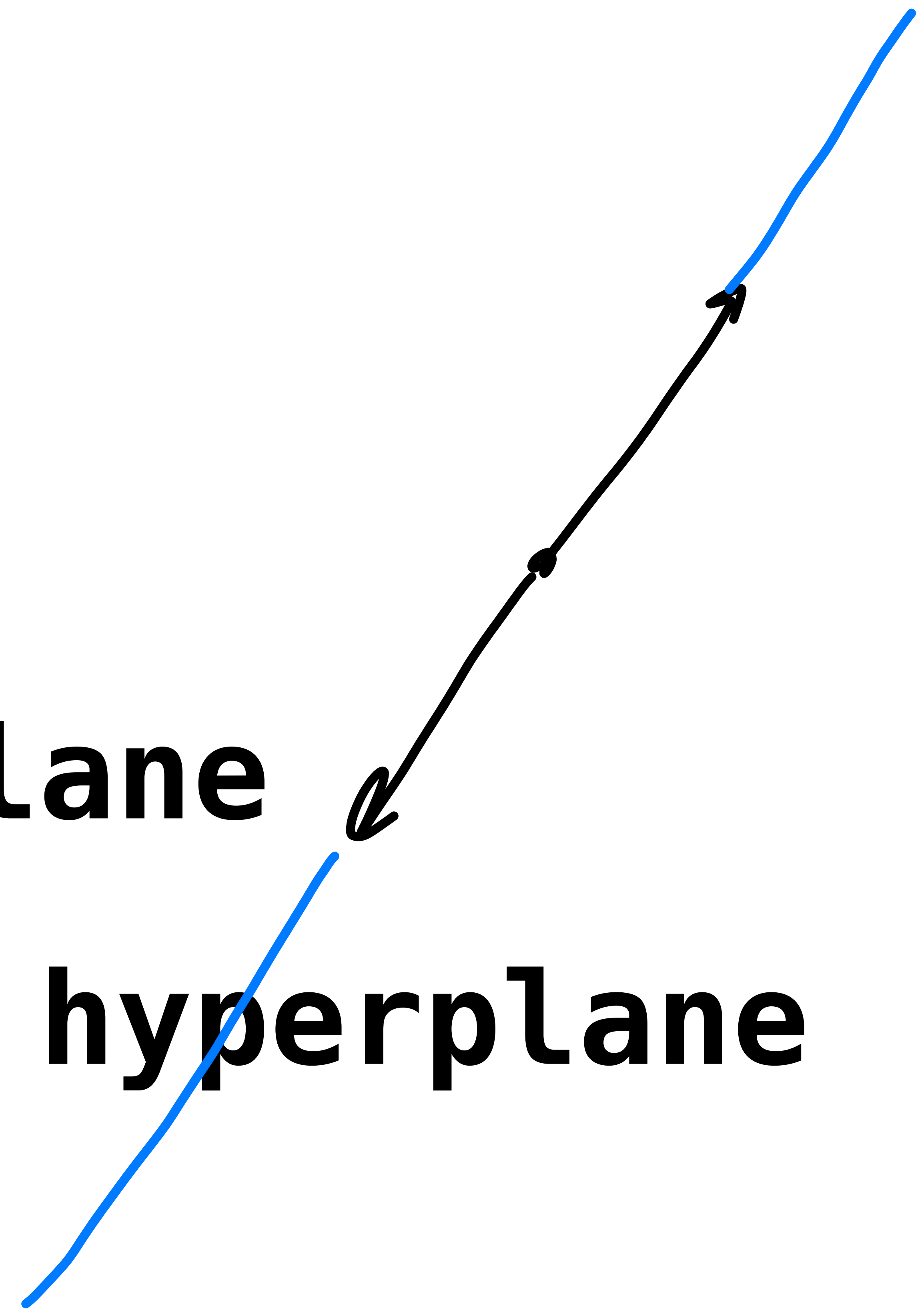
`the span of` **`two`** `vectors can be a` **`plane`**

Spans (Geometrically)

the span of **two** vectors can be a **plane**

the span of **three** vectors can be a **hyperplane**

Spans (Geometrically)



the span of **two** vectors can be a **plane**

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!!IMPORTANT!!
In all cases they pass through the origin

Spans (Geometrically)

demo

(from ILA)

HOW TO: Span Problems

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Question. Is $\mathbf{b} \in \mathsf{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$?

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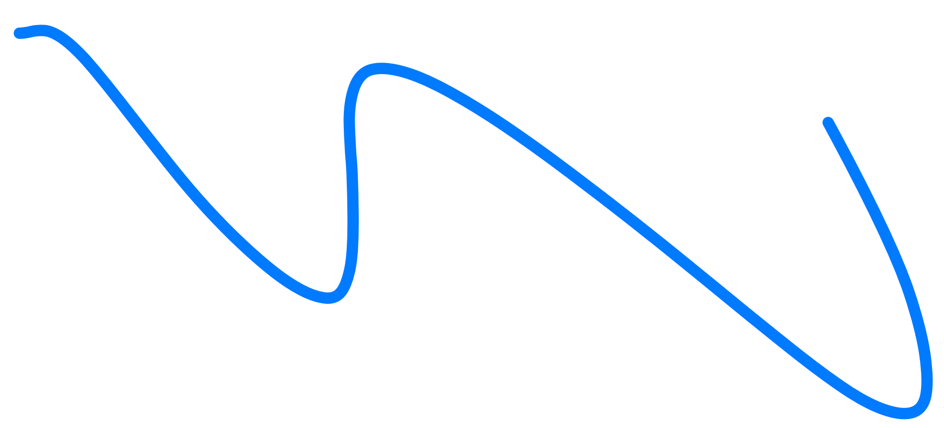
you know how to do this now

Example

$$Is \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix} \; in \; \text{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} \right\}?$$

Question (Conceptual)



What does it mean geometrically if $\mathbf{b} \notin \text{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$*?*

demo

(from ILA)

HOW TO: Inconsistency and Spans

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is the augmented matrix of an *inconsistent* system

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There is **no way** to write $\mathbf{b}$ as a linear combination

Example

Find a vector ***not*** *in* $\operatorname{span}\left\{\begin{bmatrix}2\\1\\1\end{bmatrix}, \begin{bmatrix}5\\3\\3\end{bmatrix}\right\}$***.***

Summary

vectors are fundamental objects

we can think of them as the columns of a linear system

we can scale them and add them together

they can span spaces which represent hyperplanes