Vector Equations

Geometric Algorithms
Lecture 5

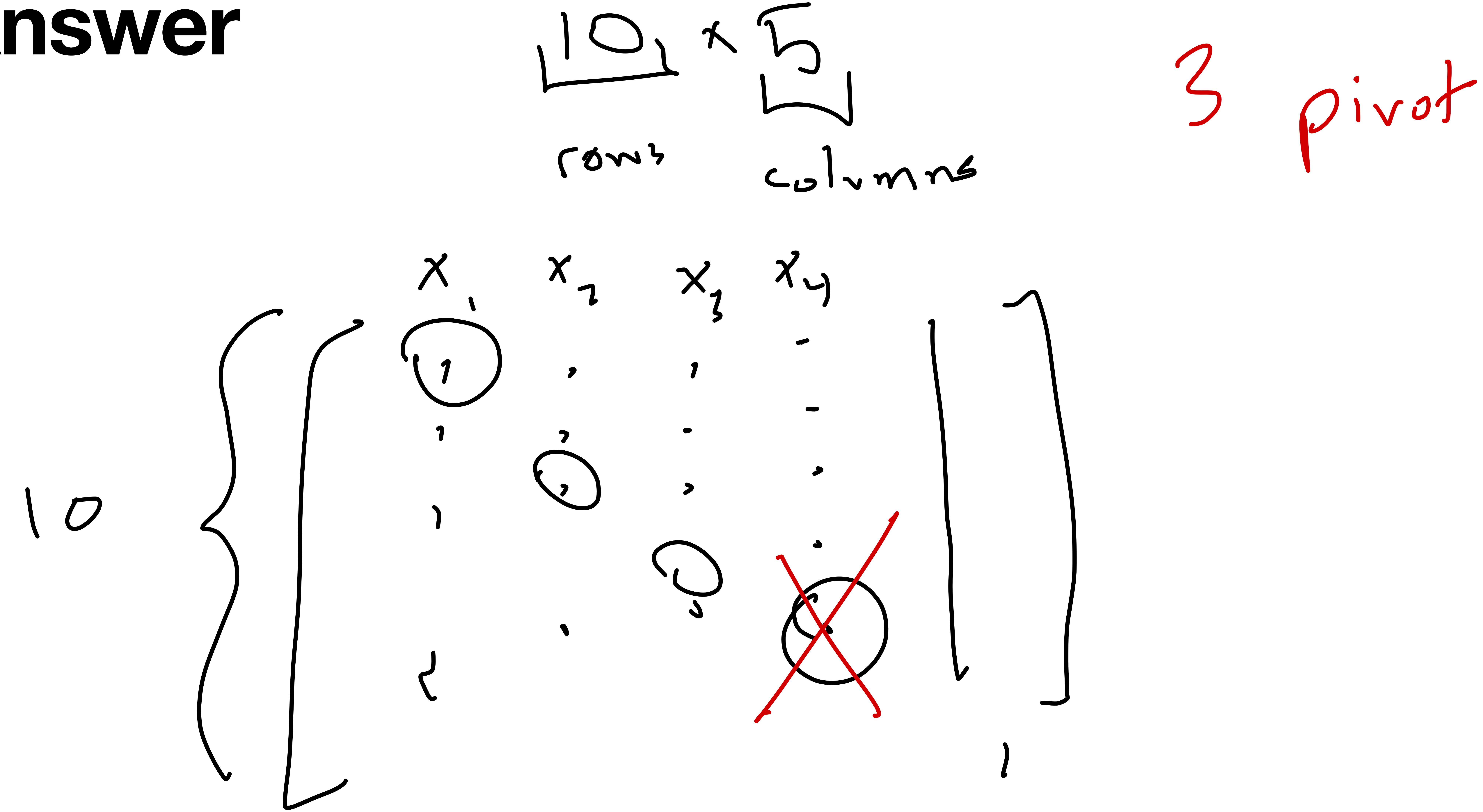
CAS CS 132

Practice Problem

Suppose that A is a 322×245 augmented matrix for a system with infinitely many solutions. What is the maximum number of pivot positions that A can have?

what about 245×322 ?

Answer



5×10

5

pivots

max

$$x_1 \quad \ldots \quad x_9 =$$

$$\begin{bmatrix} 0 & & & & & & & & & \\ & 0 & & & & & & & & \\ & & 0 & & & & & & & \\ & & & 0 & & & & & & \\ & & & & 0 & & & & & \end{bmatrix}$$

Objectives

1. Define vectors
2. Discuss vector operations and vector algebra
3. Draw the connection between vectors and systems of linear equations

Keywords

vector

vector addition

vector scaling/multiplication

the zero vector

vector equations

linear combinations

span

Motivation (An Aside)

Changing Perspective

$$\sum_{i=0}^{n-1} 2^i = 2^n - 1$$

Show that this holds for all n

Changing Perspective

$$\sum_{i=0}^{n-1} 2^i = 2^n - 1$$

$$100...000 - 000...001 = 011...111$$

show that this holds for all n

Changing Perspective

$$\sum_{i=0}^{n-1} 2^i = 2^n - 1$$

$$100...000 - 000...001 = 011...111$$

show that this holds for all n

much easier in binary

Motivation?

vectors will be one of the most important shifts of perspective in this course

the insight is simple yet elegant

Motivation?

vectors will be one of the most important shifts of perspective in this course

the insight is simple yet elegant

maybe I'm reaching...

Big Data

a piece of data is a bunch of distinct values (numbers)

How can we tell if two piece of data are similar?

maybe if they are **close together** in a geometric sense

A Note on Algebra

A Note on Algebra

in programming an "interface" is an abstract collection of related functions (e.g., a printing interface, or a comparison interface)

A Note on Algebra

```
in programming an "interface" is an abstract collection
of related functions (e.g., a printing interface, or a
comparison interface)

and object then "implements" an interface
```

A Note on Algebra

in programming an "interface" is an abstract collection of related functions (e.g., a printing interface, or a comparison interface)

and object then "implements" an interface

doing abstract algebra is like implementing an interface

A Note on Algebra

in programming an "interface" is an abstract collection of related functions (e.g., a printing interface, or a comparison interface)

and object then "implements" an interface

doing abstract algebra is like implementing an interface

we're defining an new thing called a "column vector"

A Note on Algebra

in programming an "interface" is an abstract collection of related functions (e.g., a printing interface, or a comparison interface)

and object then "implements" an interface

doing abstract algebra is like implementing an interface

we're defining an new thing called a "column vector"

we need to define what "equality" and "adding" and "multiplying by a number" means for column vectors

Vectors

What is a vector (in $\mathbb{R}^n$)?

A. an n–tuple of real numbers

B. a point in $\mathbb{R}^n$

C. a 1–column matrix with real values

D. all of the above

E. none of the above?

What is a vector (in $\mathbb{R}^n$)?

A. an n–tuple of real numbers

B. a point in $\mathbb{R}^n$

C. a 1–column matrix with real values

D. all of the above

E. none of the above?

What is a vector (in $\mathbb{R}^n$)?

A. an n−tuple of real numbers

B. a point in $\mathbb{R}^n$

C. a 1−column matrix with real values

D. all of the above

E. none of the above?

it's common to conflate points and vectors

Column Vectors

$$\mathbb{R}^{n \times 1} = \mathbb{R}^{n}$$

Definition. a ***column vector*** is a matrix with a single column, e.g.,

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

A Note on Matrix Size

an $(m \times n)$ matrix is a matrix with m rows and n columns

$$m \overbrace{\begin{bmatrix} * & * & \cdots & * & * \\ * & * & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & * & * \\ * & * & \cdots & * & * \end{bmatrix}}^{n}$$

$$4 \overbrace{\begin{bmatrix} 2 \\ 3 \\ 0.1 \\ -2 \end{bmatrix}}^{1}$$

A Note on Matrix Size

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \in \mathbb{R}^{2\times 2}$$

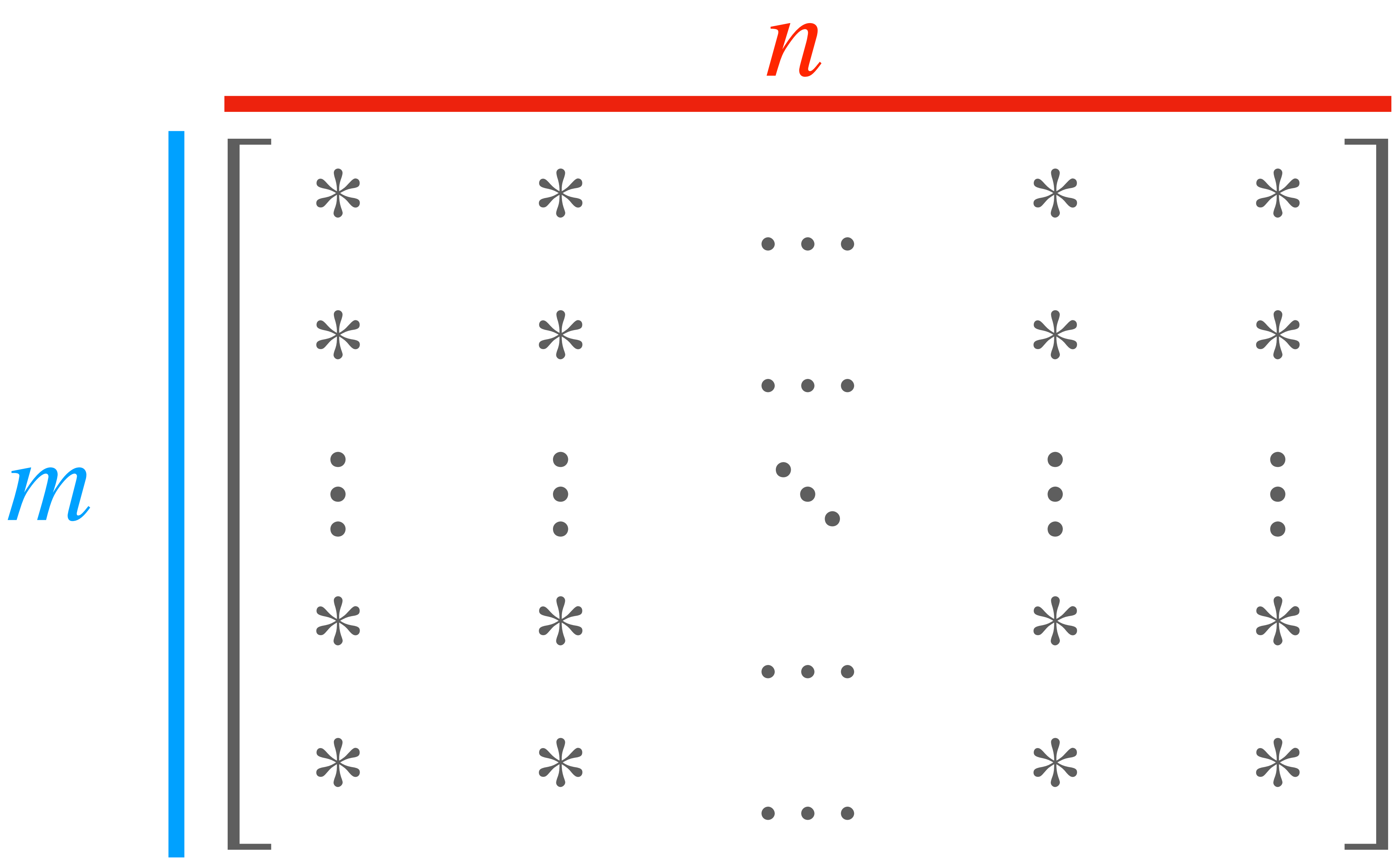
an $(m \times n)$ matrix is a matrix with m rows and n columns

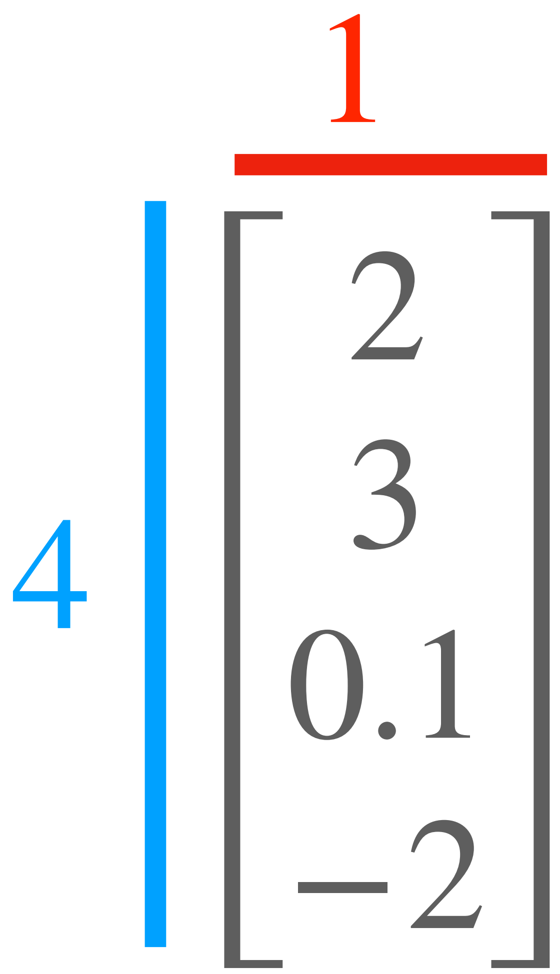
$$m \left\{ \overbrace{\begin{bmatrix} * & * & \cdots & * & * \\ * & * & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & * & * \\ * & * & \cdots & * & * \end{bmatrix}}^{n} \right. \qquad 4 \left\{ \overbrace{\begin{bmatrix} 2 \\ 3 \\ 0.1 \\ -2 \end{bmatrix}}^{1} \right.$$

$\mathbb{R}^{m\times n}$ is set of matrices with $\mathbb{R}$ entries

A Note on Matrix Size

an $(m \times n)$ matrix is a matrix with m rows and n columns





the number of rows of a vectors is called its **dimension**

$\mathbb{R}^{m \times n}$ is set of matrices with $\mathbb{R}$ entries

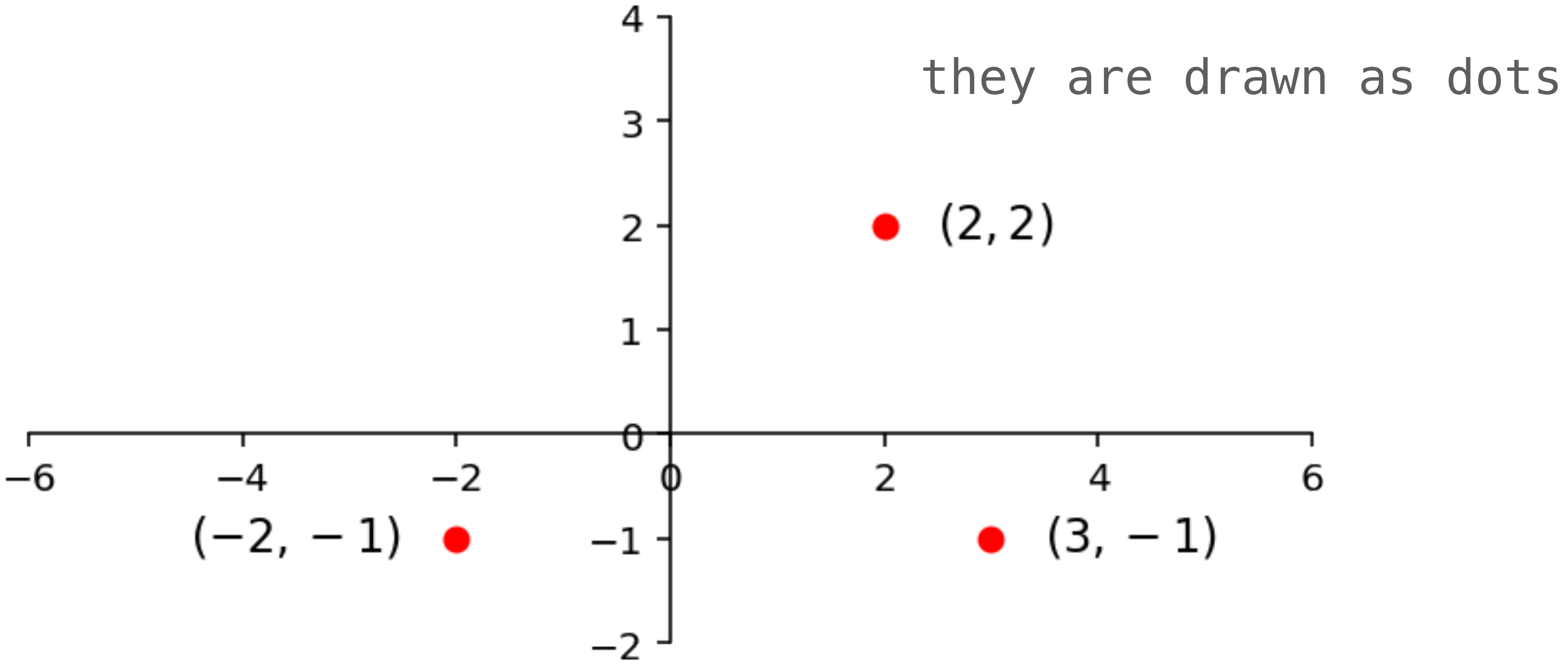
Examples

$$\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^5 \in \mathbb{R}^5 \equiv \mathbb{R}^{5 \times 1}$$

$$\vec{w} = \begin{bmatrix} \pi \\ 0.2 \\ e \\ -1 \end{bmatrix}^4 \in \mathbb{R}^4$$

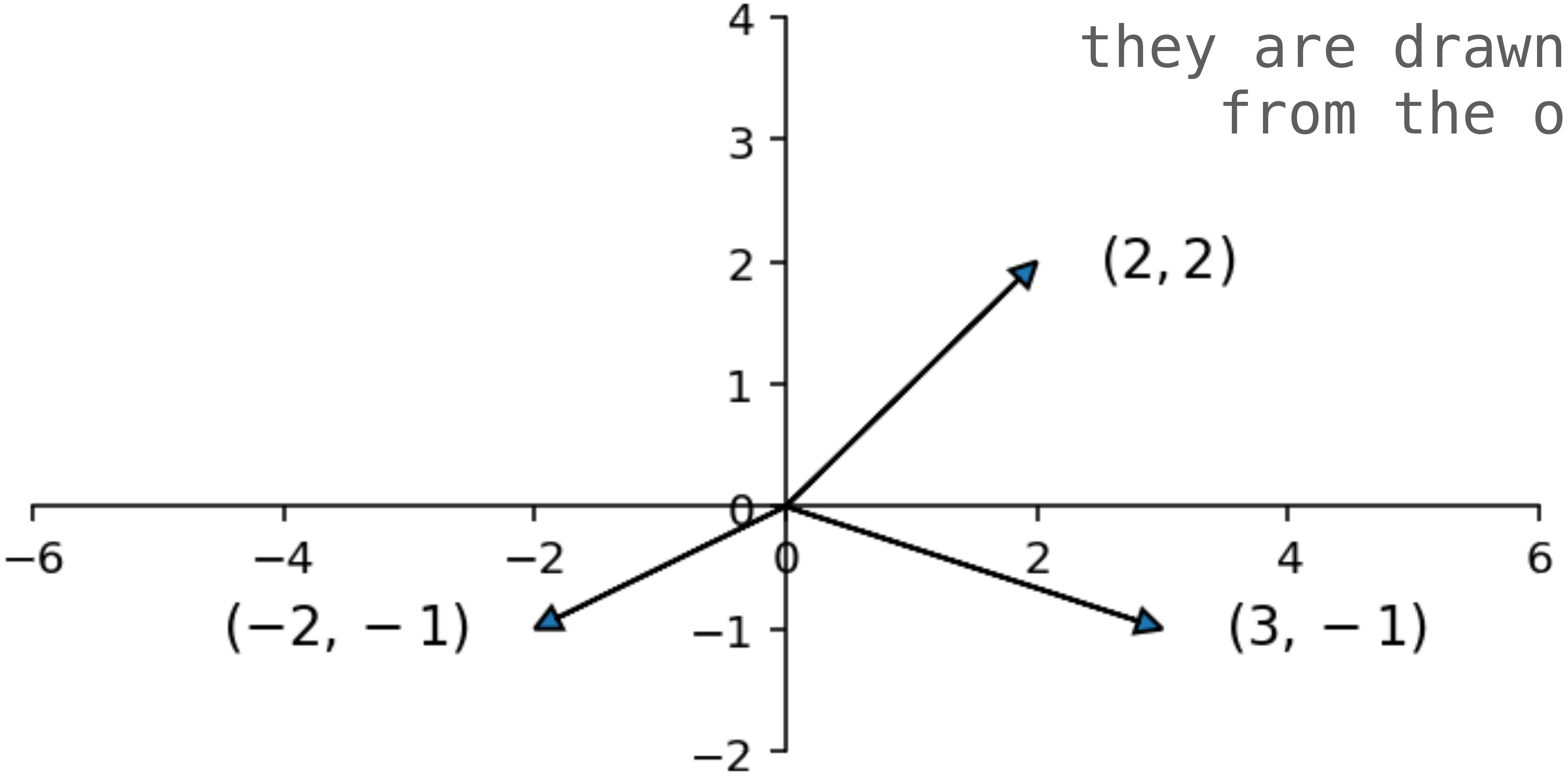
$$\vec{u} = [1]^1 \in \mathbb{R}^1 \equiv \mathbb{R}$$

Notation (Points)



points in $\mathbb{R}^2$ are notated as (a, b)

Notation (Vectors)



vectors in $\mathbb{R}^2$ are notated as $\begin{bmatrix} a \\ b \end{bmatrix}$

Notation (Looking ahead)

we will often write $[a_1 \ a_2 \ \dots \ a_n]^T$ for the vector

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

!!IMPORTANT!!

$(a_1, a_2, \dots, a_n)$ is not the same as $[a_1 \ \ a_2 \ \ \dots \ \ a_n]$

Vector Operations

Vector "Interface"

Vector "Interface"

`equality`	what does it mean for two vectors to be equal?

Vector "Interface"

equality	what does it mean for two vectors to be equal?
addition	what does $\mathbf{u} + \mathbf{v}$ (adding two vectors mean?

Vector "Interface"

equality	what does it mean for two vectors to be equal?
addition	what does $\mathbf{u} + \mathbf{v}$ (adding two vectors mean?
scaling	what does $a\mathbf{v}$ (multiplying a vector by a real number) mean?

Vector "Interface"

equality	what does it mean for two vectors to be equal?
addition	what does $\mathbf{u} + \mathbf{v}$ (adding two vectors mean?
scaling	what does $a\mathbf{v}$ (multiplying a vector by a real number) mean?

What properties do they need to satisfy?

Vector "Interface"

equality	what does it mean for two vectors to be equal?
addition	what does $\mathbf{u} + \mathbf{v}$ (adding two vectors mean?
scaling	what does $a\mathbf{v}$ (multiplying a vector by a real number) mean?

What properties do they need to satisfy?

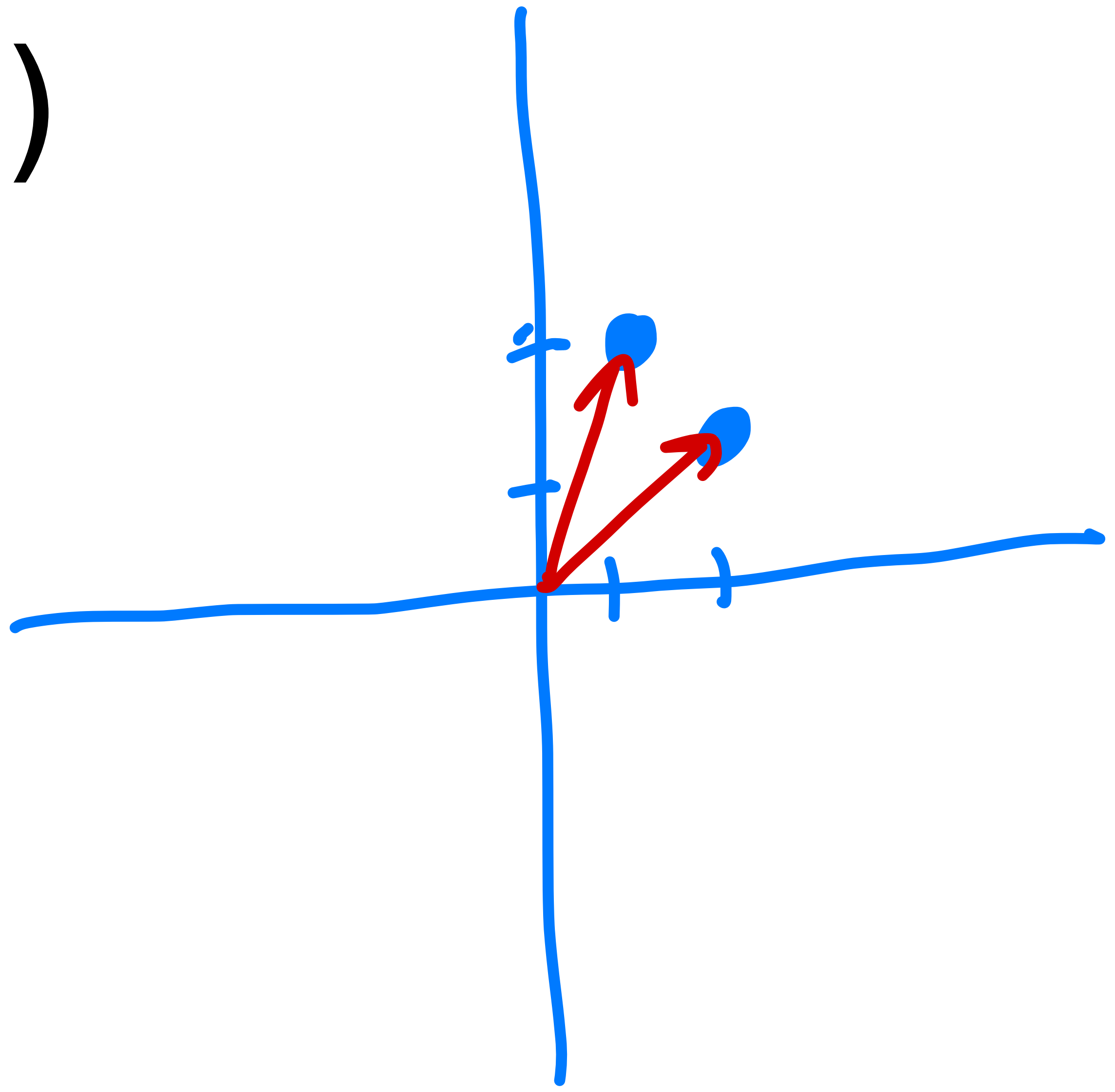
Vector Equality

two vectors are equal if their entries at each position are equal

(this is also the case for matrices)

!!IMPORTANT!!
ORDER MATTERS

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$



Vector Equality

$$\begin{bmatrix} 2+3 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

`is the same as`

$$\begin{gathered} a_1 = b_1 \\ a_2 = b_2 \\ \vdots \\ a_n = b_n \end{gathered}$$

Examples

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 1+0 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Vector "Interface"

equality	what does it mean for two vectors to be equal?
addition	what does $\mathbf{u} + \mathbf{v}$ (adding two vectors mean?
scaling	what does $a\mathbf{v}$ (multiplying a vector by a real number) mean?

Vector "Interface"

equality	what does it mean for two vectors to be equal?
addition	what does $\mathbf{u} + \mathbf{v}$ (adding two vectors mean?
scaling	what does $a\mathbf{v}$ (multiplying a vector by a real number) mean?

What properties do they need to satisfy?

Vector Addition

adding two vectors means adding their entries

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix}$$

Vector Addition

adding two vectors means adding their entries

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix}$$

!!IMPORTANT!!

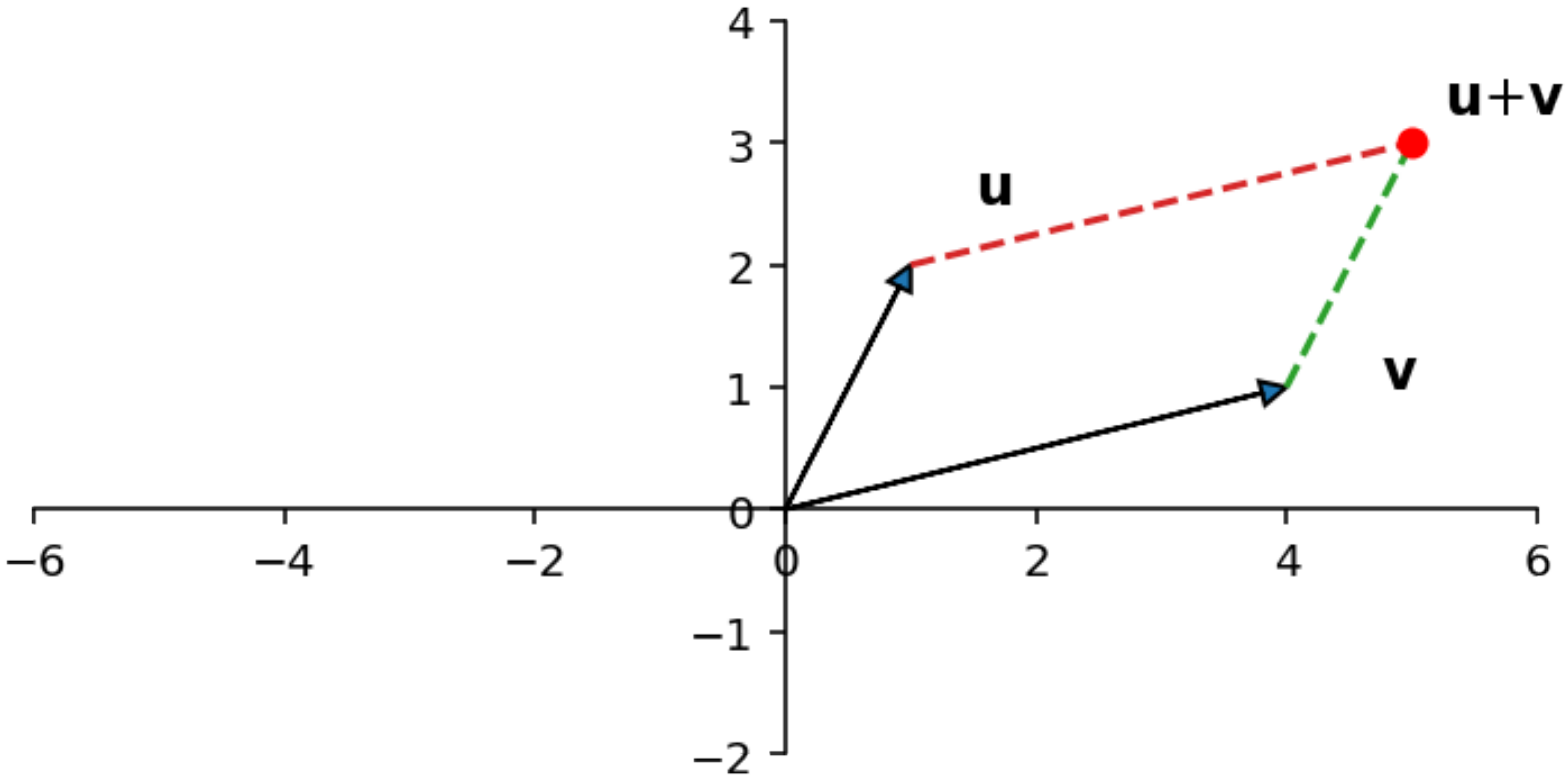
WE CAN ONLY ADD VECTORS OF THE SAME SIZE

Examples

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1+-1 \\ 2+0 \\ 3+2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}$$

Vector Addition (Geometrically)

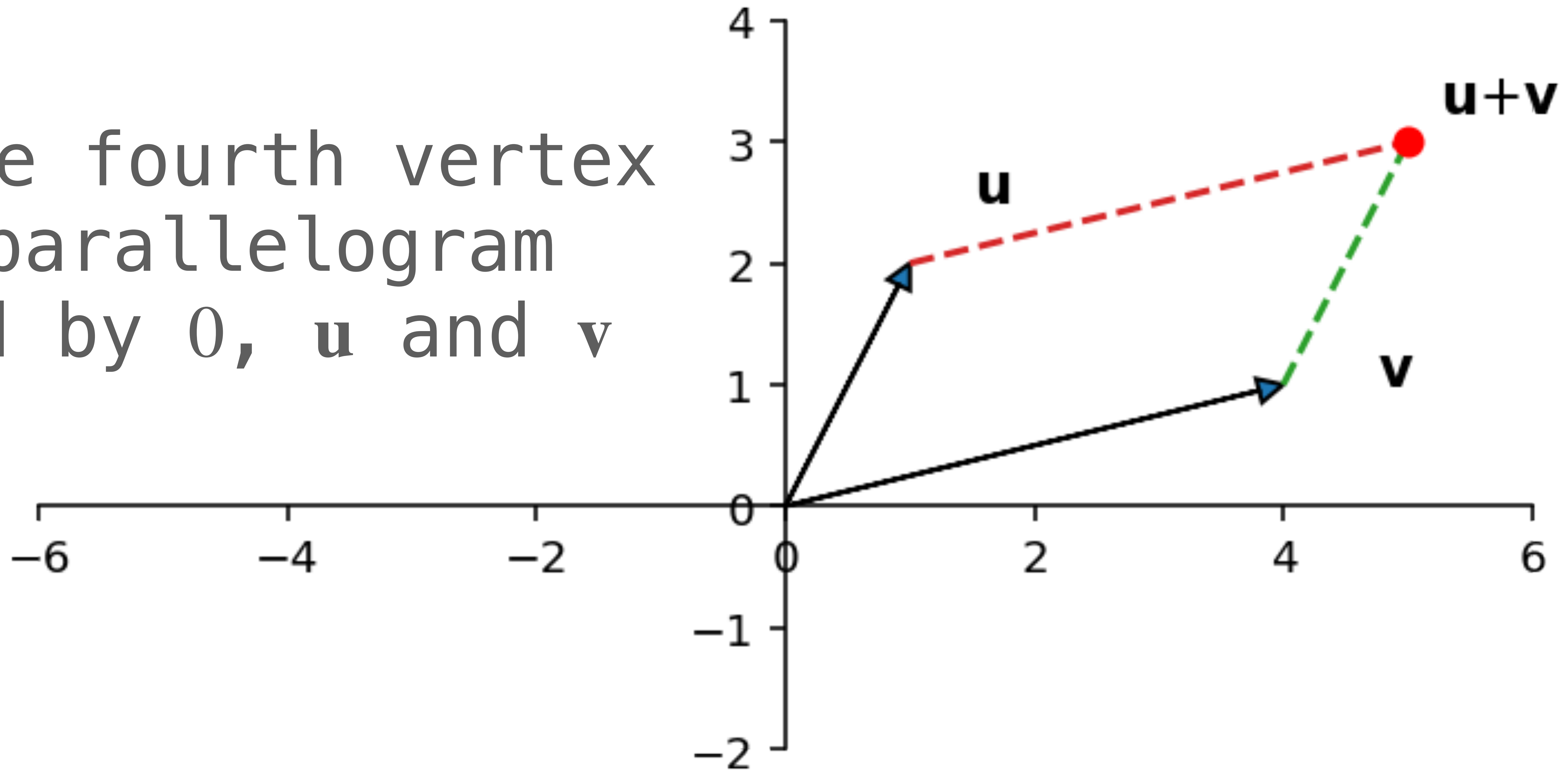
in $\mathbb{R}^2$ it's called the *parallelogram rule*



Vector Addition (Geometrically)

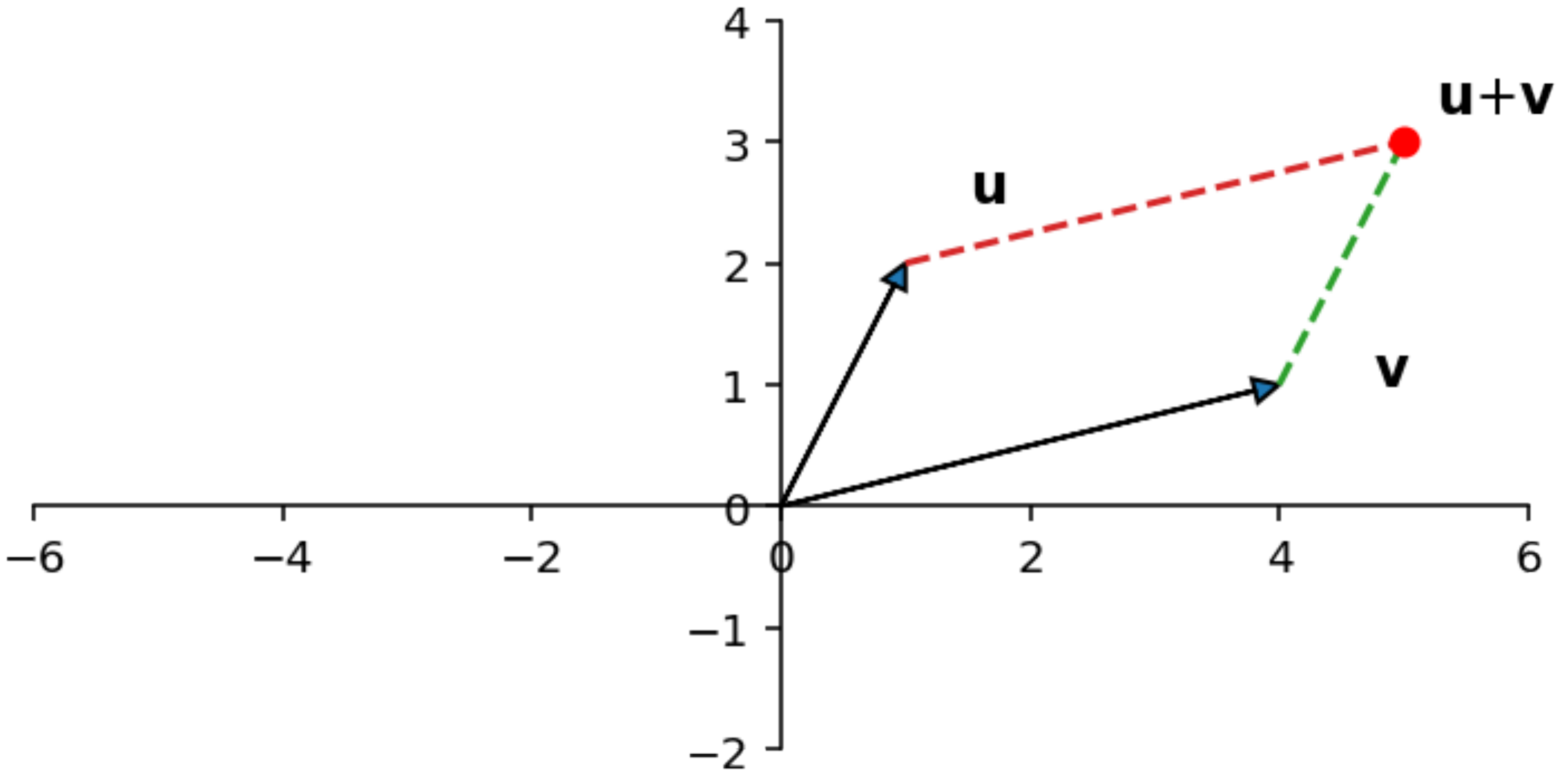
in $\mathbb{R}^2$ it's called the *parallelogram rule*

$\mathbf{u} + \mathbf{v}$ is the fourth vertex of the parallelogram generated by 0, $\mathbf{u}$ and $\mathbf{v}$



Vector Addition (Geometrically)

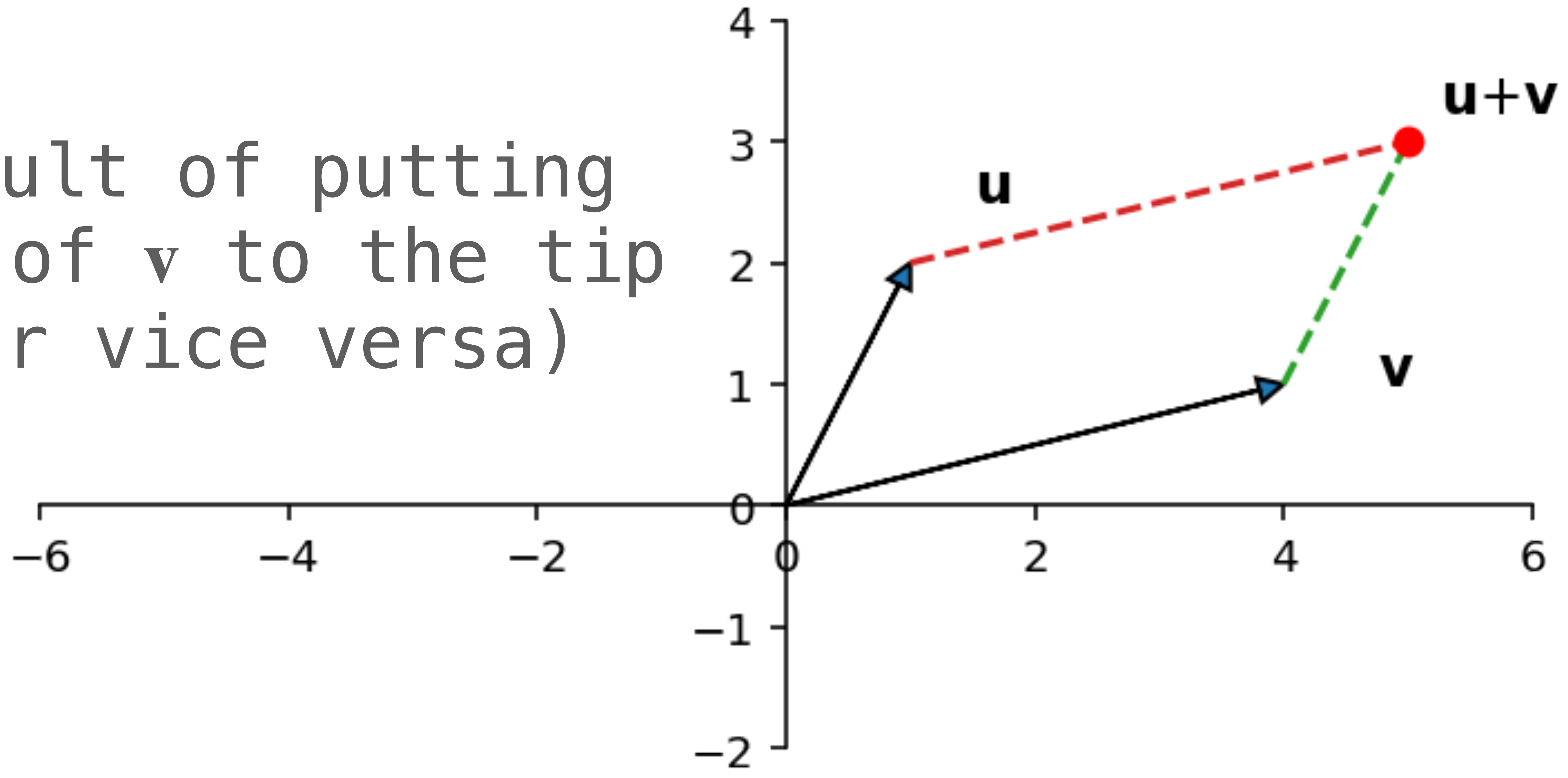
`or the` *`tip-to-tail rule`*



Vector Addition (Geometrically)

or the *tip–to–tail rule*

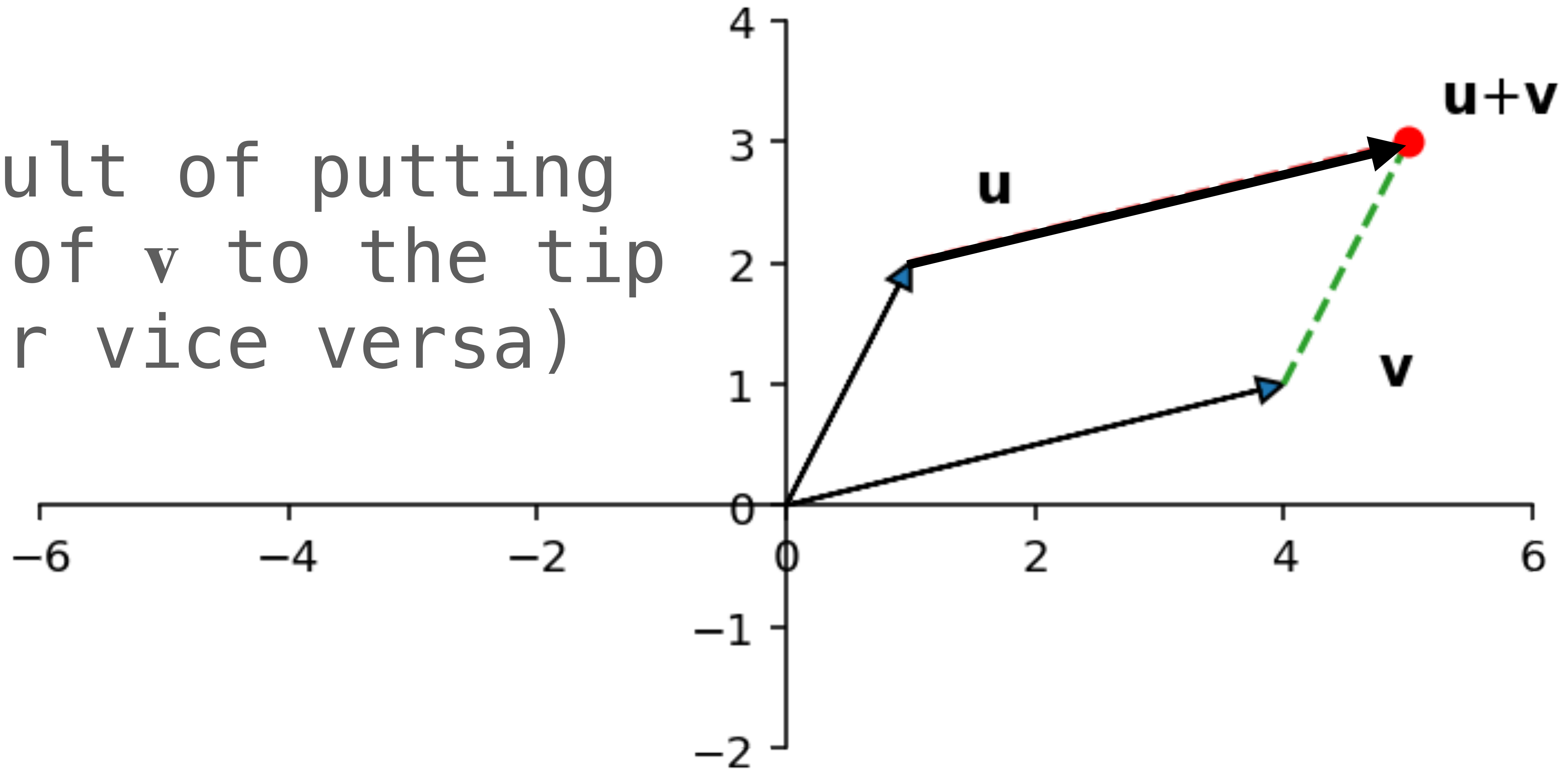
$\mathbf{u} + \mathbf{v}$ result of putting the tail of $\mathbf{v}$ to the tip of $\mathbf{u}$ (or vice versa)



Vector Addition (Geometrically)

or the *tip-to-tail rule*

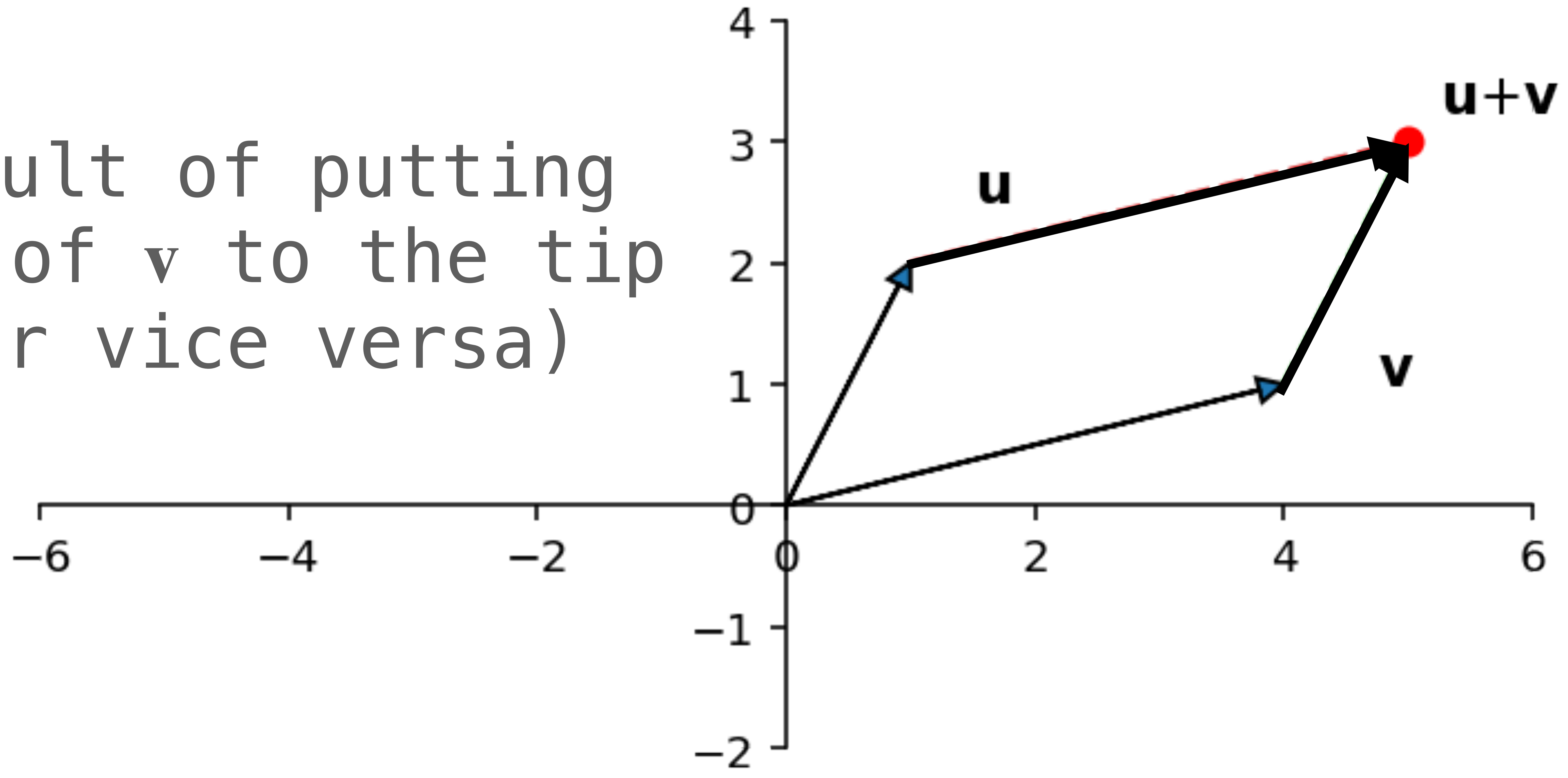
$\mathbf{u} + \mathbf{v}$ result of putting the tail of $\mathbf{v}$ to the tip of $\mathbf{u}$ (or vice versa)



Vector Addition (Geometrically)

or the *tip-to-tail rule*

$\mathbf{u} + \mathbf{v}$ result of putting the tail of $\mathbf{v}$ to the tip of $\mathbf{u}$ (or vice versa)



demo
(from ILA)

Vector "Interface"

equality	what does it mean for two vectors to be equal?
addition	what does $\mathbf{u} + \mathbf{v}$ (adding two vectors mean?
scaling	what does $a\mathbf{v}$ (multiplying a vector by a real number) mean?

What properties do they need to satisfy?

Vector Scaling/Multiplication

```
scaling/multiplying a vector by a number means
multiplying each of it's elements
```

$$a \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} ab_1 \\ ab_2 \\ \vdots \\ ab_n \end{bmatrix}$$

Vector Scaling/Multiplication (Example)

$$3 \begin{bmatrix} 2 \\ 1 \\ 3.5 \\ 4 \end{bmatrix}$$

Vector Scaling/Multiplication (Example)

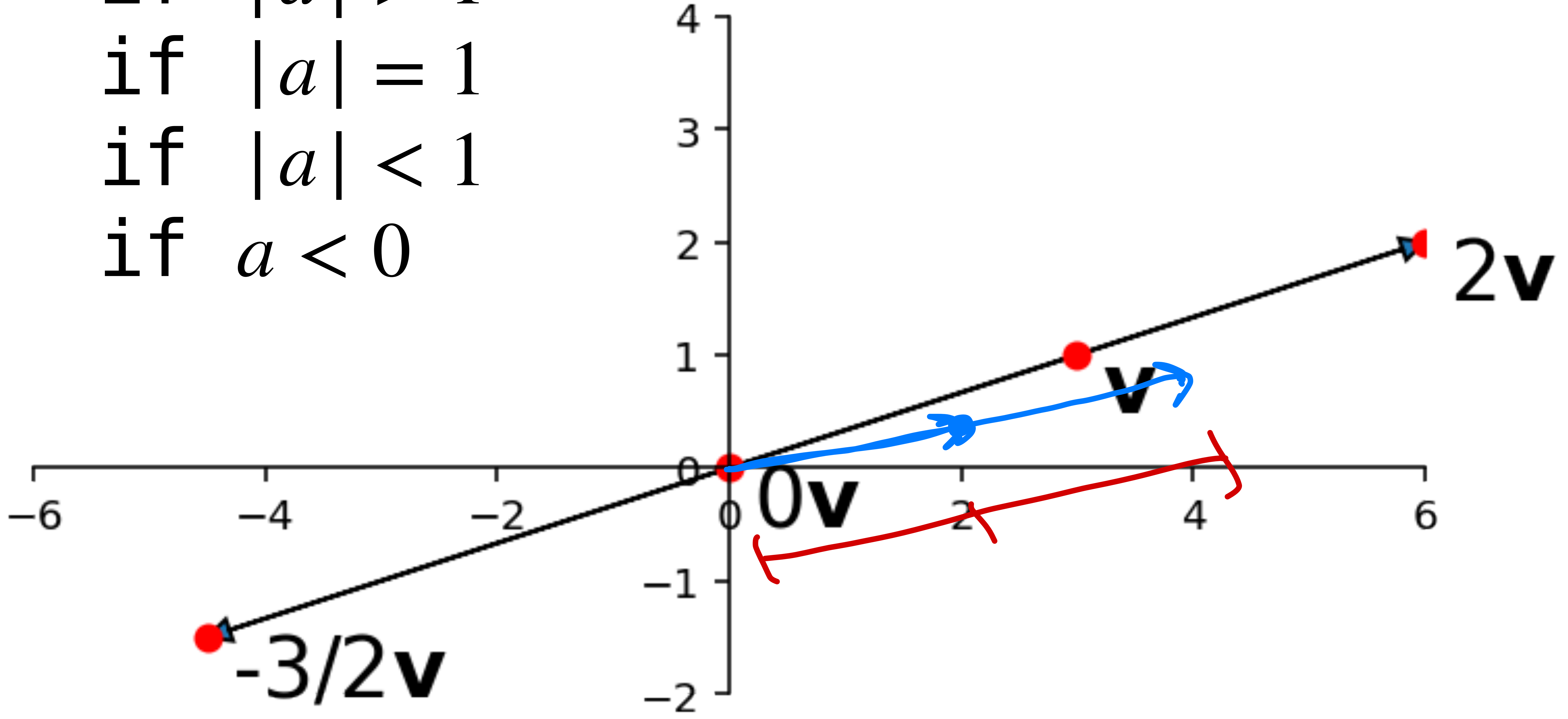
$$3 \begin{bmatrix} 2 \\ 1 \\ 3.5 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \cdot 2 \\ 3 \cdot 1 \\ 3 \cdot 3.5 \\ 3 \cdot 4 \end{bmatrix}$$

Vector Scaling/Multiplication (Example)

$$3 \begin{bmatrix} 2 \\ 1 \\ 3.5 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \cdot 2 \\ 3 \cdot 1 \\ 3 \cdot 3.5 \\ 3 \cdot 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 10.5 \\ 12 \end{bmatrix}$$

Vector Scaling (Geometrically)

longer	if $|a| > 1$
the same length	if $|a| = 1$
shorter	if $|a| < 1$
reversed	if $a < 0$



$\vec{u} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$\frac{1}{2}\vec{u} = \begin{bmatrix} 2 \\ 1/2 \end{bmatrix}$

demo

(from ILA)

Algebraic Properties

For any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ and any real numbers c, d:

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$$

$$\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$$

$$\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$$

$$c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$$

$$(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$$

$$c(d\mathbf{u}) = (cd)\mathbf{u}$$

$$1\mathbf{u} = \mathbf{u}$$

Algebraic Properties

For any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ and any real numbers c, d:

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$$

$$\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$$

$$\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$$

$$c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$$

$$(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$$

$$c(d\mathbf{u}) = (cd)\mathbf{u}$$

$$1\mathbf{u} = \mathbf{u}$$

these are requirements for any **vector space**
they matter more for *bizarre* vector spaces

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

Example "Proof"

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix} = \begin{bmatrix} v_1 + u_1 \\ v_2 + u_2 \\ v_3 + u_3 \end{bmatrix}$$

$$= \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Question (Practice)

$$3\begin{bmatrix}1\\1\\1\\1\end{bmatrix}+2\begin{bmatrix}2\\0\\3\\-1\end{bmatrix}-\begin{bmatrix}-3\\4\\2\\0\end{bmatrix}$$

Compute the value of the above vector.

Answer

$$3\begin{bmatrix}1\\1\\1\\1\end{bmatrix}+2\begin{bmatrix}2\\0\\3\\-1\end{bmatrix}-\begin{bmatrix}-3\\4\\2\\0\end{bmatrix}$$

$$\begin{bmatrix}3\\3\\3\\3\end{bmatrix}+\begin{bmatrix}4\\0\\6\\-2\end{bmatrix}-\begin{bmatrix}-3\\4\\2\\0\end{bmatrix}=\begin{bmatrix}10\\-1\\7\\1\end{bmatrix}$$

In Sum

In Sum

`we can add vectors`

In Sum

we can add vectors

we can scale vectors

In Sum

we can add vectors

we can scale vectors

this gives us a way of generating new vectors from old ones

In Sum

we can add vectors

we can scale vectors

this gives us a way of generating new vectors from old ones

What vectors can we make in this way?

Linear Combinations

Linear Combinations

Definition. a ***linear combination*** of vectors

$$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$$

is a vector of the form

$$\alpha_1 \mathbf{v}_1 + \alpha_1 \mathbf{v}_2 + \ldots + \alpha_n \mathbf{v}_n$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are in $\mathbb{R}$

Linear Combinations

Definition. a ***linear combination*** of vectors

$$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$$

is a vector of the form

$$\alpha_1 \mathbf{v}_1 + \alpha_1 \mathbf{v}_2 + \ldots + \alpha_n \mathbf{v}_n$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are in $\mathbb{R}$

weights

Linear Combinations

Definition. a ***linear combination*** of vectors

$$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$$

is a vector of the form

$$\alpha_1 \mathbf{v}_1 + \alpha_1 \mathbf{v}_2 + \ldots + \alpha_n \mathbf{v}_n$$

Looks suspiciously like a linear equation...

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are in $\mathbb{R}$

weights

Linear Combinations (Example)

$$3\begin{bmatrix}1\\1\\1\\1\end{bmatrix}+2\begin{bmatrix}2\\0\\3\\-1\end{bmatrix}-\begin{bmatrix}-3\\4\\2\\0\end{bmatrix}$$

Linear Combinations (Geometrically)

demo
(from ILA)

The Fundamental Concern

Can $\mathbf{u}$ be written as a linear combination of

$$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n?$$

That is, are there weights $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots \alpha_n \mathbf{v}_n = \mathbf{u}?$$

Why is this fundamental?

I'm going to ask that you suspend your disbelief...

Why is this fundamental?

I'm going to ask that you suspend your disbelief...

For now, how do we solve this problem?

Vector Equations and Linear Systems

The Fundamental Connection

The Fundamental Connection

```
we don't know the weights, that's want we want
to find
```

The Fundamental Connection

we don't know the weights, that's want we want to find

what if we write them as *unknowns*?

The Fundamental Connection

`we don't know the weights, that's want we want to find`

`what if we write them as` *`unknowns`*`?`

$$x_1 \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

Some Symbol Pushing...

$$x_1 \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

$$x_1 + 2x_2 = 7$$
$$-2x_1 + 5_2 = 4$$
$$-5x_1 + 6x_2 = 3$$

$$\begin{bmatrix} x_1 \\ -2x_1 \\ -5x_1 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ 5x_2 \\ 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 + 2x_2 \\ -2x_1 + 5x_2 \\ -5x_1 + 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 3 \end{bmatrix}$$

Some Symbol Pushing...

$$x_1 \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

The Fundamental Connection

The Fundamental Connection

```
we don't know the weights, that's want we want
to find
```

The Fundamental Connection

we don't know the weights, that's want we want to find

what if we write them as *unknowns*?

The Fundamental Connection

we don't know the weights, that's want we want to find

what if we write them as *unknowns*?

$$x_1 \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

The Fundamental Connection

we don't know the weights, that's want we want to find

what if we write them as *unknowns*?

$$\begin{bmatrix} x_1 \\ (-2)x_1 \\ (-5)x_1 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ 5x_2 \\ 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

The Fundamental Connection

we don't know the weights, that's want we want to find

what if we write them as *unknowns*?

$$\begin{bmatrix} x_1 + 2x_2 \\ (-2)x_1 + 5x_2 \\ -5x_1 + 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

The Fundamental Connection

```
we don't know the weights, that's want we want
to find
```

what if we write them as *unknowns*?

$$x_1 + 2x_2 = 7$$

$$(-2)x_1 + 5x_2 = 4$$

$$-5x_1 + 6x_2 = -3$$

The Fundamental Connection

we don't know the weights, that's want we want to find

what if we write them as *unknowns*?

$$x_1 + 2x_2 = 7$$
$$(-2)x_1 + 5x_2 = 4$$
$$-5x_1 + 6x_2 = -3$$

we get a system of linear equations we know how to solve

The Fundamental Connection

More generally:

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{1m} \end{bmatrix} + x_2 \begin{bmatrix} a_{21} \\ a_{21} \\ \vdots \\ a_{2m} \end{bmatrix} + \ldots + x_n \begin{bmatrix} a_{n1} \\ a_{n2} \\ \vdots \\ a_{nm} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

The Fundamental Connection

$$\begin{bmatrix} a_{11}x_1 \\ a_{21}x_1 \\ \vdots \\ a_{1m}x_1 \end{bmatrix} + \begin{bmatrix} a_{21}x_2 \\ a_{21}x_2 \\ \vdots \\ a_{2m}x_1 \end{bmatrix} + \ldots + \begin{bmatrix} a_{n1}x_n \\ a_{n2}x_n \\ \vdots \\ a_{nm}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

`by vector scaling`

The Fundamental Connection

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

`by vector addition`

The Fundamental Connection

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}
$$

`by vector equality`

The Fundamental Connection

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$$

augmented matrix

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

system of linear equations

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{1m} \end{bmatrix} + x_2 \begin{bmatrix} a_{21} \\ a_{21} \\ \vdots \\ a_{2m} \end{bmatrix} + \dots + x_n \begin{bmatrix} a_{n1} \\ a_{n2} \\ \vdots \\ a_{nm} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

vector equation

The Fundamental Connection

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$$

augmented matrix

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m$$

system of linear equations

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{1m} \end{bmatrix} + x_2 \begin{bmatrix} a_{21} \\ a_{21} \\ \vdots \\ a_{2m} \end{bmatrix} + \ldots + x_n \begin{bmatrix} a_{n1} \\ a_{n2} \\ \vdots \\ a_{nm} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

vector equation

The Fundamental Connection

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

augmented matrix

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

system of linear equations

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{1m} \end{bmatrix} + x_2 \begin{bmatrix} a_{21} \\ a_{21} \\ \vdots \\ a_{2m} \end{bmatrix} + \ldots + x_n \begin{bmatrix} a_{n1} \\ a_{n2} \\ \vdots \\ a_{nm} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

vector equation

The Fundamental Connection

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

augmented matrix

this is our big shift in perspective

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

system of linear equations

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{1m} \end{bmatrix} + x_2 \begin{bmatrix} a_{21} \\ a_{21} \\ \vdots \\ a_{2m} \end{bmatrix} + \ldots + x_n \begin{bmatrix} a_{n1} \\ a_{n2} \\ \vdots \\ a_{nm} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

vector equation

HOW TO: Linear Combination Problems

HOW TO: Linear Combination Problems

Question. Can $\mathbf{b}$ be written as a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_n$?

HOW TO: Linear Combination Problems

Question. Can $\mathbf{b}$ be written as a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_n$?

Solution. Solve the system of linear equations with the augmented matrix

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}$$

HOW TO: Linear Combination Problems

Question. Can $\mathbf{b}$ be written as a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_n$?

Solution. Solve the system of linear equations with the augmented matrix

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}$$

A solution to this system is a set of weights to define $\mathbf{b}$ as a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$

HOW TO: Linear Combination Problems

Question. Can $\mathbf{b}$ be written as a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_n$?

Solution. Solve the system of linear equations with the augmented matrix

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}$$

this is notation for building a matrix out of column vectors

A solution to this system is a set of weights to define $\mathbf{b}$ as a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$

Question

Can $\begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$ *be written as a linear combination of*

$\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$ *and* $\begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$*?*

Answer

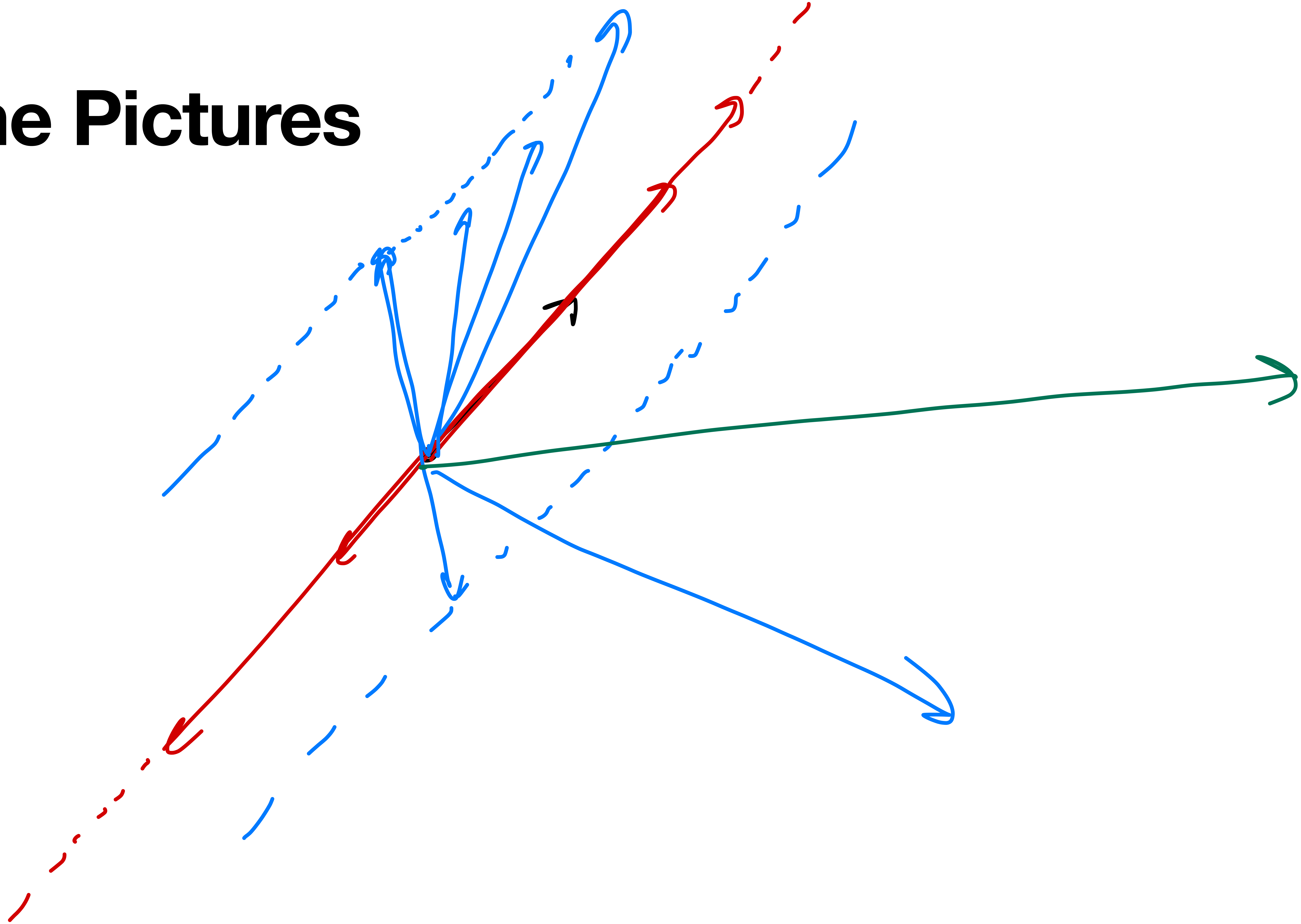
$$3\begin{bmatrix}1\\-2\\-5\end{bmatrix} + 2\begin{bmatrix}2\\5\\6\end{bmatrix} = \begin{bmatrix}7\\4\\-3\end{bmatrix}$$

$$\begin{bmatrix}1 & 2 & 7\\-2 & 5 & 4\\-5 & 6 & -3\end{bmatrix} \sim$$

$$\begin{bmatrix}1 & 0 & 3\\0 & 1 & 2\\0 & 0 & 0\end{bmatrix}$$

Spans

Some Pictures



Spans

Spans

Definition. the ***span*** of a set of vectors is the set of all possible linear combinations of them

Spans

Definition. the ***span*** of a set of vectors is the set of all possible linear combinations of them

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = \{\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots \alpha_n\mathbf{v}_n : \alpha_1, \alpha_2, \ldots, \alpha_n \text{ are in } \mathbb{R}\}$$

Spans

Definition. the ***span*** of a set of vectors is the set of all possible linear combinations of them

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = \{\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots\alpha_n\mathbf{v}_n : \alpha_1, \alpha_2, \ldots, \alpha_n \text{ are in } \mathbb{R}\}$$

$\mathbf{u} \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ **exactly when u can be expressed as a linear combination of those vectors**

Spans

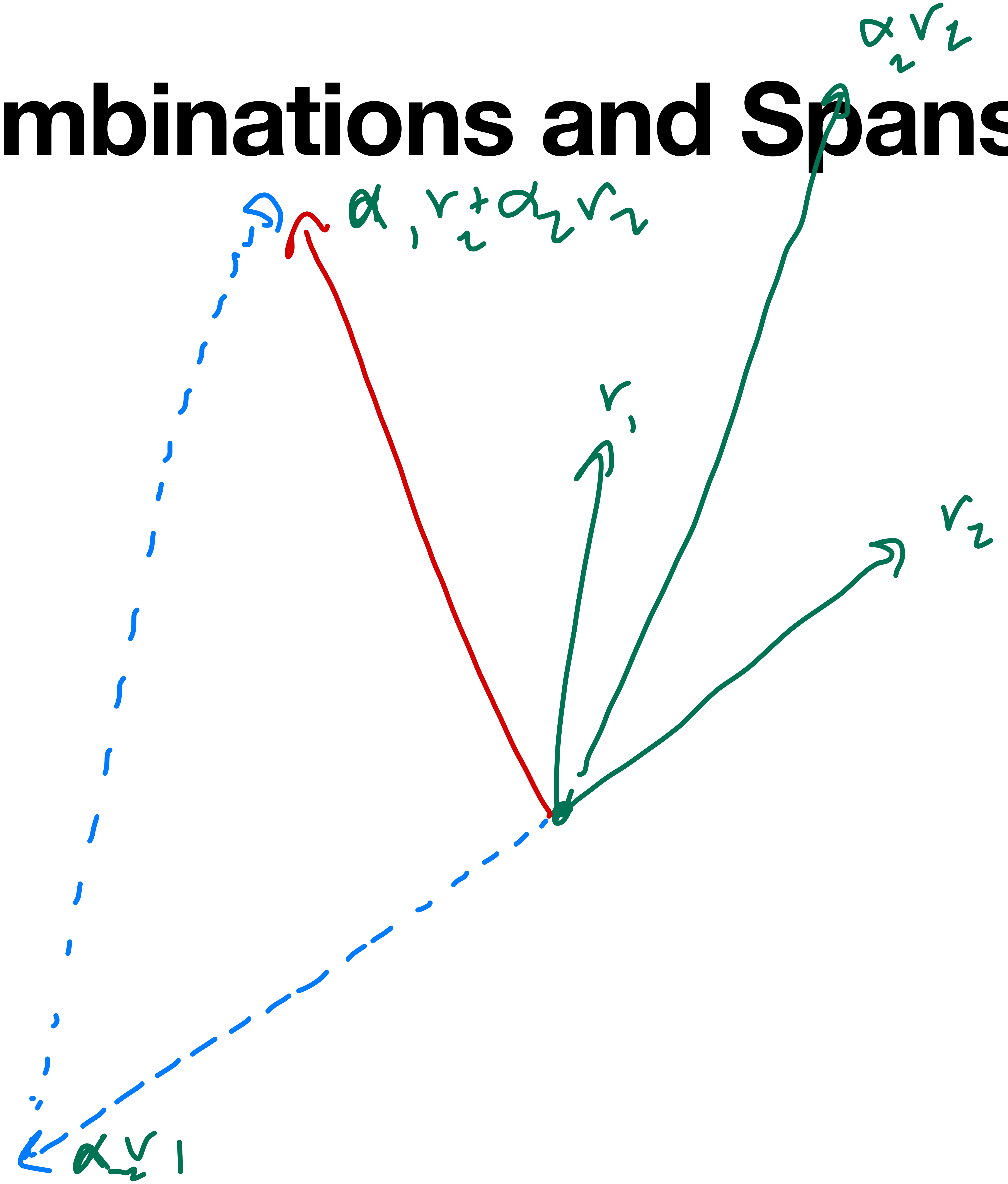
Definition. the ***span*** of a set of vectors is the set of all possible linear combinations of them

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = \{\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots\alpha_n\mathbf{v}_n : \alpha_1, \alpha_2, \ldots, \alpha_n \text{ are in } \mathbb{R}\}$$

read: $\mathbf{u}$ is an element of $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$

$\mathbf{u} \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ **exactly when $\mathbf{u}$ can be expressed as a linear combination of those vectors**

Linear Combinations and Spans (A Picture)



Spans (Geometrically)

Spans (Geometrically)

```
for one vector
```

Spans (Geometrically)

`for one vector`

$$\text{span}\{\mathbf{v}\} = \{\alpha\mathbf{v} : \alpha \in \mathbb{R}\}$$

Spans (Geometrically)

for one vector

$$\text{span}\{\mathbf{v}\} = \{\alpha\mathbf{v} : \alpha \in \mathbb{R}\}$$

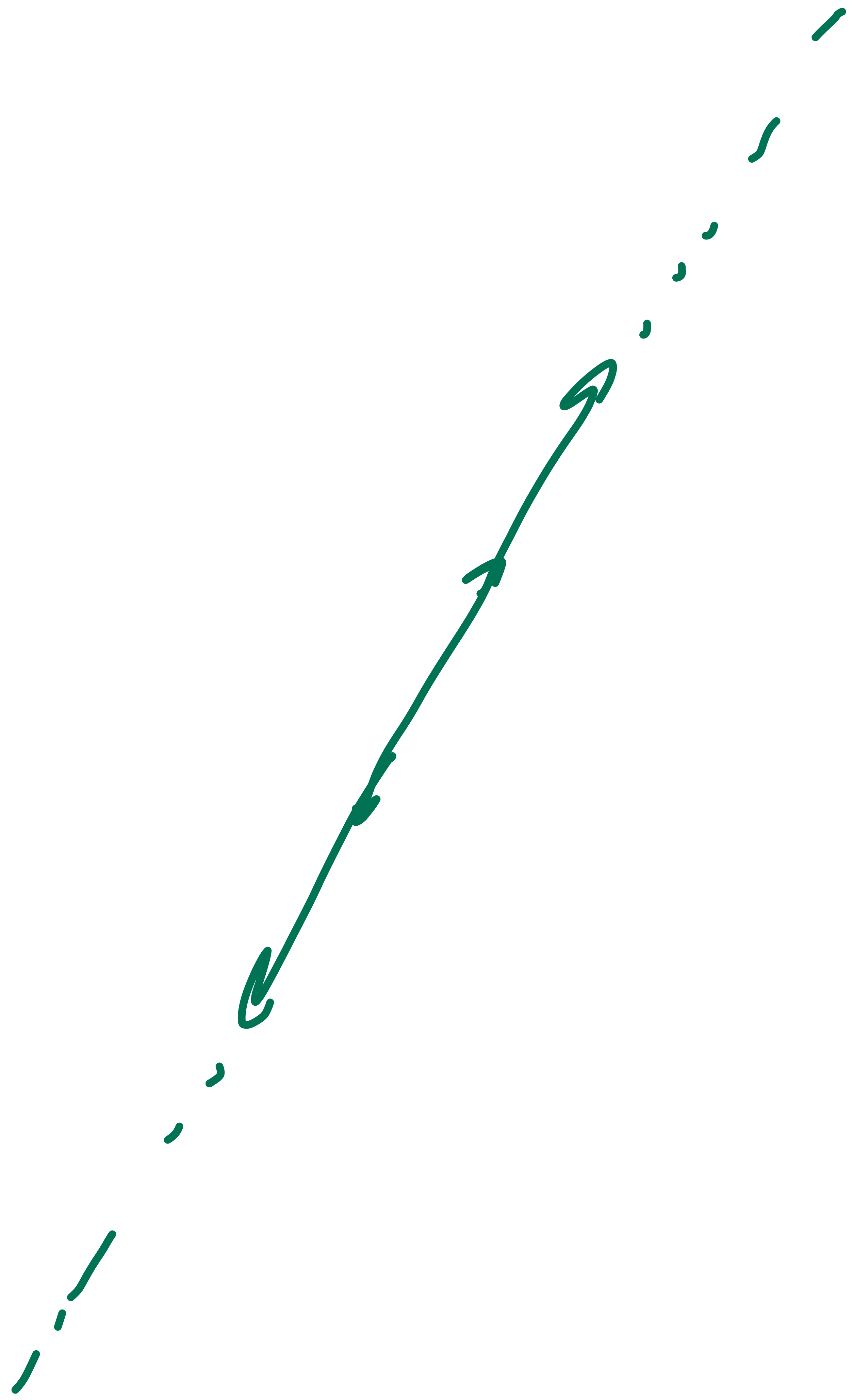
this is **all scalar multiple of** $\mathbf{v}$

Spans (Geometrically)

for one vector

$$\text{span}\{\mathbf{v}\} = \{\alpha\mathbf{v} : \alpha \in \mathbb{R}\}$$

this is **all scalar multiple of** $\mathbf{v}$



the span of one vector is a **line**

Spans (Geometrically)

Spans (Geometrically)

`the span of **two** vectors can be a **plane**`

Spans (Geometrically)

the span of **two** vectors can be a **plane**

the span of **three** vectors can be a **hyperplane**

Spans (Geometrically)

the span of **two** vectors can be a **plane**

the span of **three** vectors can be a **hyperplane**

!!IMPORTANT!!
In all cases they pass through the origin

Spans (Geometrically)

demo
(from ILA)

HOW TO: Span Problems

HOW TO: Span Problems

Question. Is $\mathbf{b} \in \mathrm{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$?

HOW TO: Span Problems

Question. Is $\mathbf{b} \in \text{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$?

Solution. Determine if $\mathbf{b}$ can be written as a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$

HOW TO: Span Problems

Question. Is $\mathbf{b} \in \mathrm{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$?

Solution. Determine if $\mathbf{b}$ can be written as a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$

you know how to do this now

Example

$$\textit{Is} \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix} \textit{in} \ \text{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} \right\}?$$

Question (Conceptual)

What does it mean geometrically if $\mathbf{b} \notin \text{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$*?*

demo
(from ILA)

HOW TO: Inconsistency and Spans

HOW TO: Inconsistency and Spans

Question. find a vector $\mathbf{b}$ which *does not* appear in $\text{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$

HOW TO: Inconsistency and Spans

Question. find a vector $\mathbf{b}$ which *does not* appear in $\text{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$

Solution. Choose $\mathbf{b}$ so that

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}$$

is the augmented matrix of an *inconsistent* system

HOW TO: Inconsistency and Spans

Question. find a vector $\mathbf{b}$ which *does not* appear in $\text{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$

Solution. Choose $\mathbf{b}$ so that

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}$$

is the augmented matrix of an *inconsistent* system

There is **no way** to write $\mathbf{b}$ as a linear combination

Example

Find a vector ***not*** *in* $\text{span}\left\{\begin{bmatrix}2\\1\\1\end{bmatrix}, \begin{bmatrix}5\\3\\3\end{bmatrix}\right\}$**.**

Summary

vectors are fundamental objects

we can think of them as the columns of a linear system

we can scale them and add them together

they can span spaces which represent hyperplanes