Vector Equations

Geometric Algorithms
Lecture 5

CAS CS 132

Practice Problem

Suppose that A is a 322×245 augmented matrix for a system with infinitely many solutions. What is the maximum number of pivot positions that A can have?

Answer

Objectives

1. Define vectors
2. Discuss vector operations and vector algebra
3. Draw the connection between vectors and systems of linear equations

Keywords

vector

vector addition

vector scaling/multiplication

the zero vector

vector equations

linear combinations

span

Motivation (An Aside)

Changing Perspective

$$\sum_{i=0}^{n-1} 2^i = 2^n - 1$$

Show that this holds for all n

Changing Perspective

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$$100...000 - 000...001 = 011...111$$

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Changing Perspective

$$\sum_{i=0}^{n-1} 2^i = 2^n - 1$$

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show that this holds for all n

much easier in binary

Motivation?

vectors will be one of the most important shifts of perspective in this course

the insight is simple yet elegant

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maybe I'm reaching...

Big Data

a piece of data is a bunch of distinct values (numbers)

How can we tell if two piece of data are similar?

maybe if they are **close together** in a geometric sense

A Note on Algebra

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we're defining an new thing called a "column vector"

we need to define what "equality" and "adding" and "multiplying by a number" means for column vectors

Vectors

What is a vector (in $\mathbb{R}^n$)?

A. an n–tuple of real numbers

B. a point in $\mathbb{R}^n$

C. a 1–column matrix with real values

D. all of the above

E. none of the above?

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it's common to conflate points and vectors

Column Vectors

Definition. a ***column vector*** is a matrix with a single column, e.g.,

A Note on Matrix Size

an $(m \times n)$ matrix is a matrix with m rows and n columns

$$m \underbrace{\overbrace{\begin{bmatrix} * & * & \cdots & * & * \\ * & * & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & * & * \\ * & * & \cdots & * & * \end{bmatrix}}^{n}}_{}$$

$$4 \overbrace{\begin{bmatrix} 2 \\ 3 \\ 0.1 \\ -2 \end{bmatrix}}^{1}$$

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$$m \left\{ \overbrace{\begin{bmatrix} * & * & \cdots & * & * \\ * & * & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & * & * \\ * & * & \cdots & * & * \end{bmatrix}}^{n} \right. \qquad 4 \left\{ \overbrace{\begin{bmatrix} 2 \\ 3 \\ 0.1 \\ -2 \end{bmatrix}}^{1} \right.$$

$\mathbb{R}^{m \times n}$ is set of matrices with $\mathbb{R}$ entries

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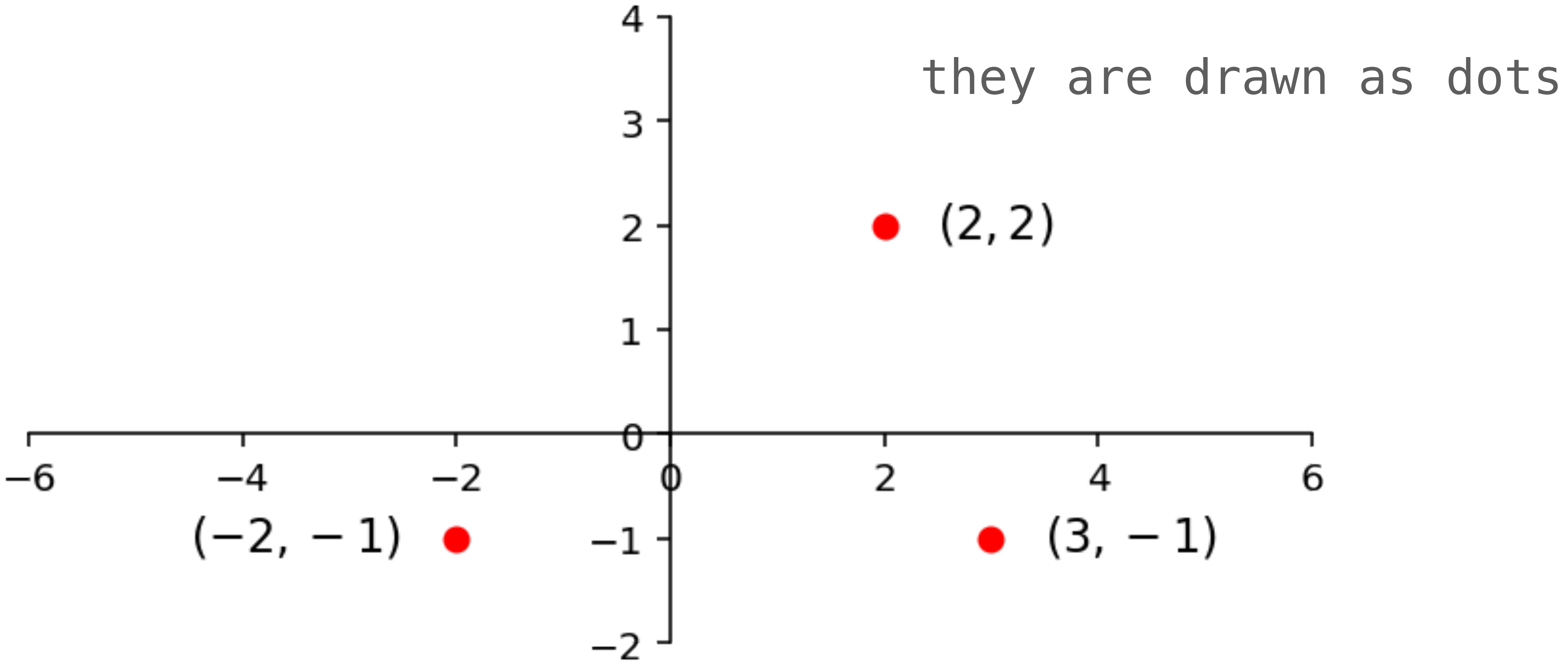
$$4 \left\{ \overbrace{\begin{bmatrix} 2 \\ 3 \\ 0.1 \\ -2 \end{bmatrix}}^{1} \right.$$

the number of rows of a vectors is called its **dimension**

$\mathbb{R}^{m \times n}$ is set of matrices with $\mathbb{R}$ entries

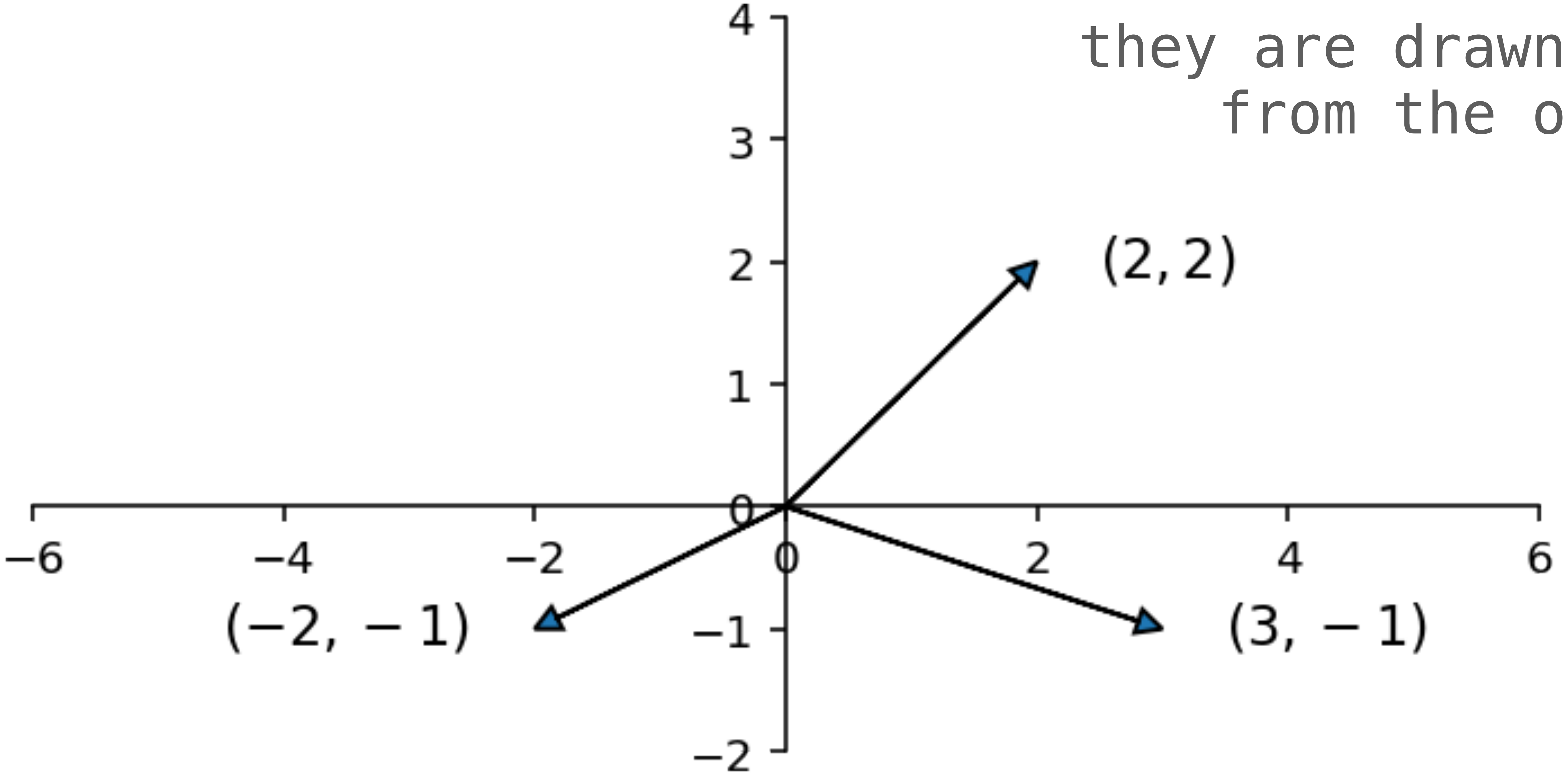
Examples

Notation (Points)



points in $\mathbb{R}^2$ are notated as (a, b)

Notation (Vectors)



they are drawn as arrow from the origin

vectors in $\mathbb{R}^2$ are notated as $\begin{bmatrix} a \\ b \end{bmatrix}$

Notation (Looking ahead)

we will often write $[a_1\ a_2\ \dots\ a_n]^T$ for the vector

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

!!IMPORTANT!!

$(a_1, a_2, \dots, a_n)$ is not the same as $[a_1\ \ a_2\ \ \dots\ \ a_n]$

Vector Operations

Vector "Interface"

Vector "Interface"

equality what does it mean for two vectors to be equal?

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equality — what does it mean for two vectors to be equal?

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Vector Equality

two vectors are equal if their entries at each position are equal

(this is also the case for matrices)

!!IMPORTANT!!
ORDER MATTERS

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Vector Equality

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

`is the same as`

$$\begin{gathered} a_1 = b_1 \\ a_2 = b_2 \\ \vdots \\ a_n = b_n \end{gathered}$$

Examples

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Vector Addition

`adding two vectors means adding their entries`

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix}$$

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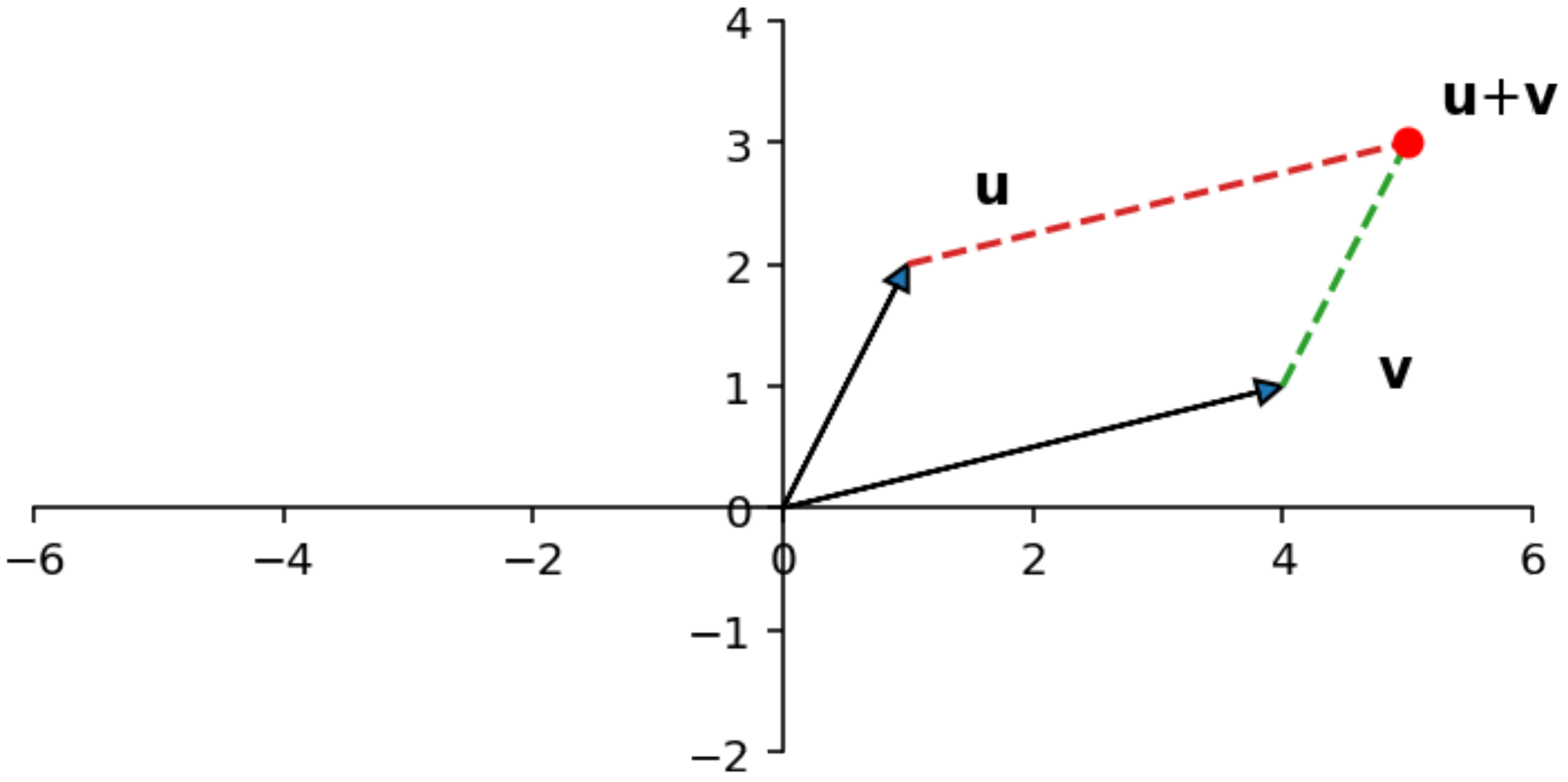
!!IMPORTANT!!

WE CAN ONLY ADD VECTORS OF THE SAME SIZE

Examples

Vector Addition (Geometrically)

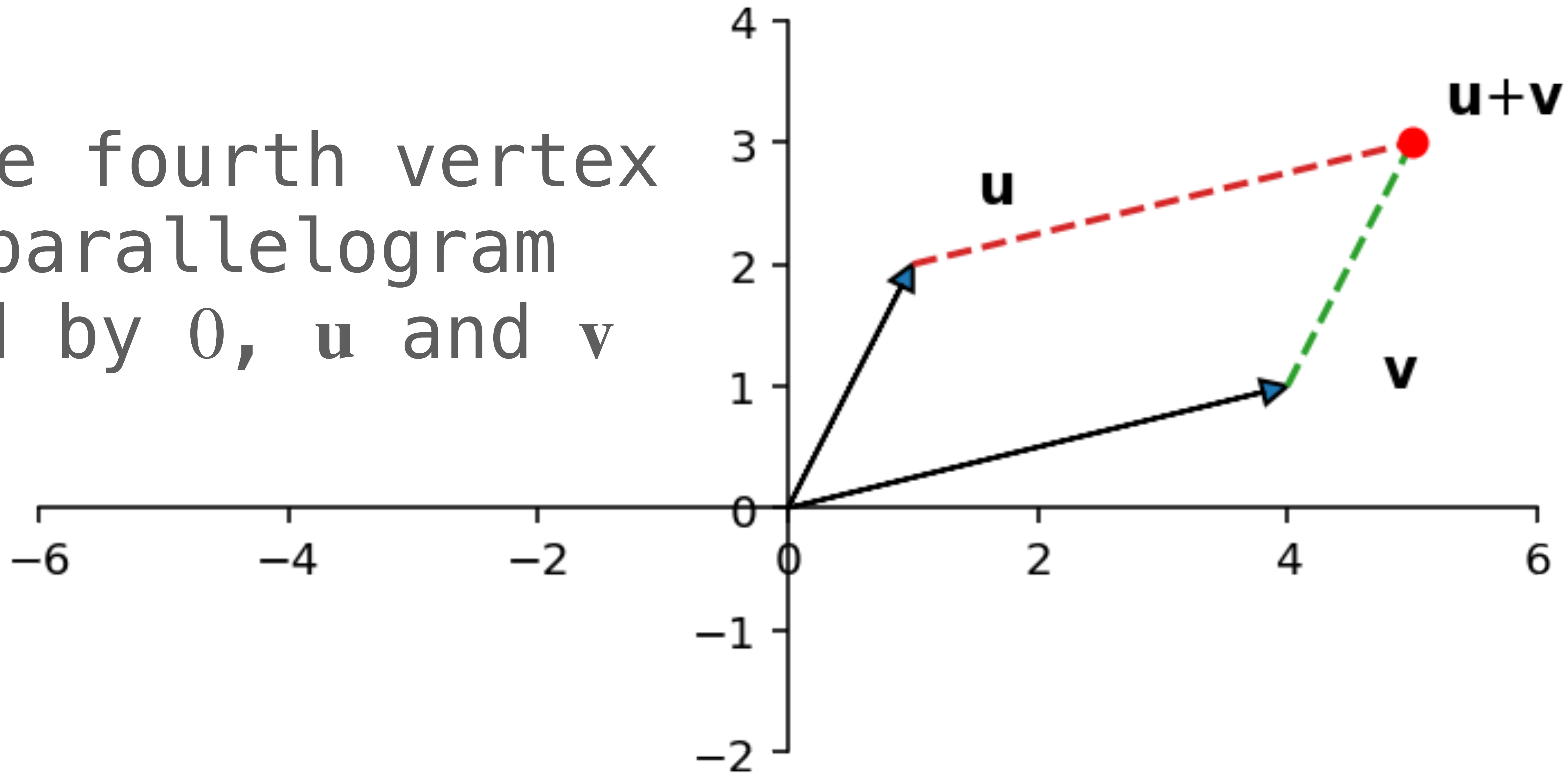
`in` $\mathbb{R}^2$ `it's called the` *`parallelogram rule`*



Vector Addition (Geometrically)

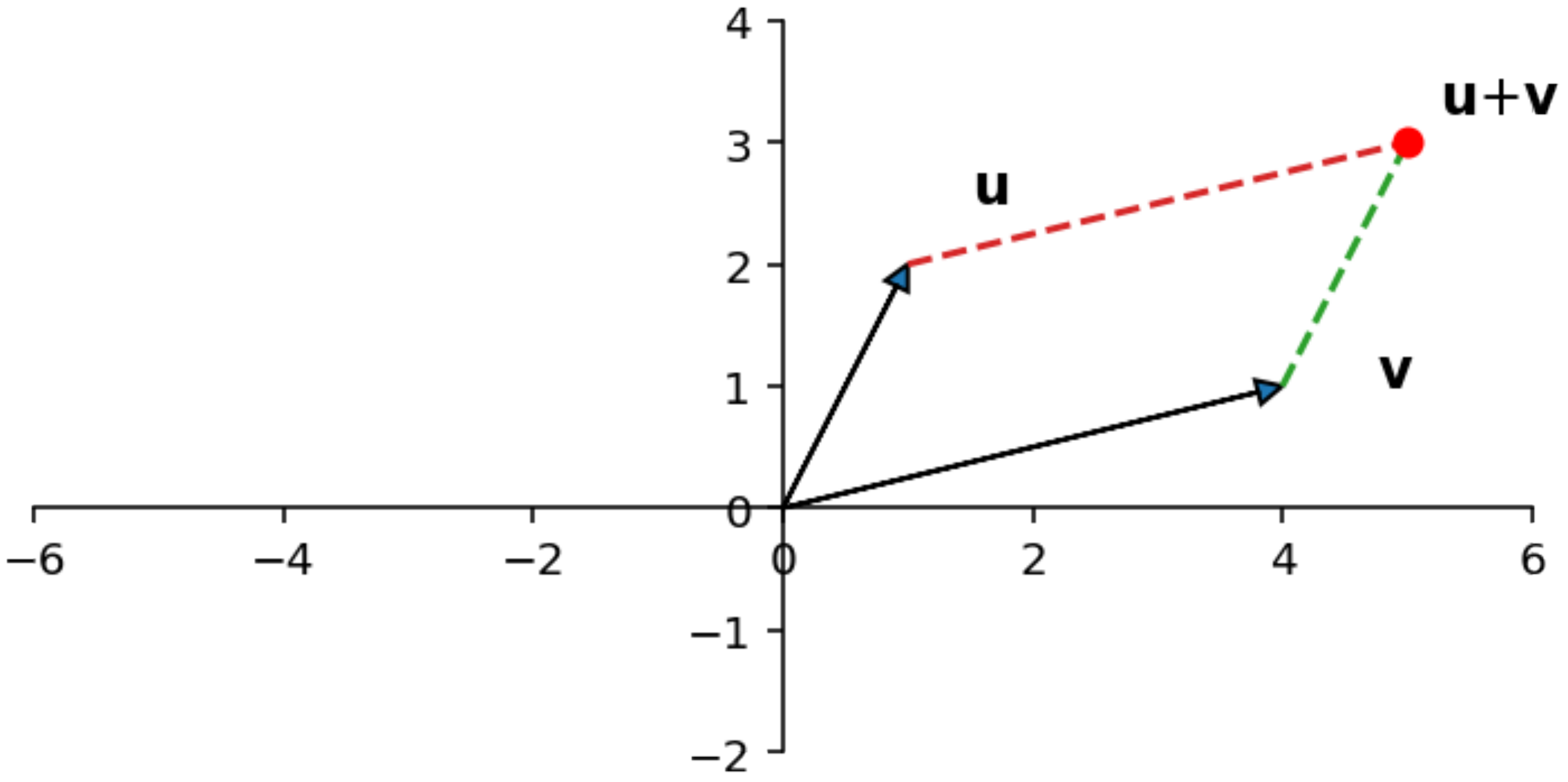
in $\mathbb{R}^2$ it's called the *parallelogram rule*

$\mathbf{u} + \mathbf{v}$ is the fourth vertex of the parallelogram generated by 0, $\mathbf{u}$ and $\mathbf{v}$



Vector Addition (Geometrically)

or the *tip-to-tail rule*



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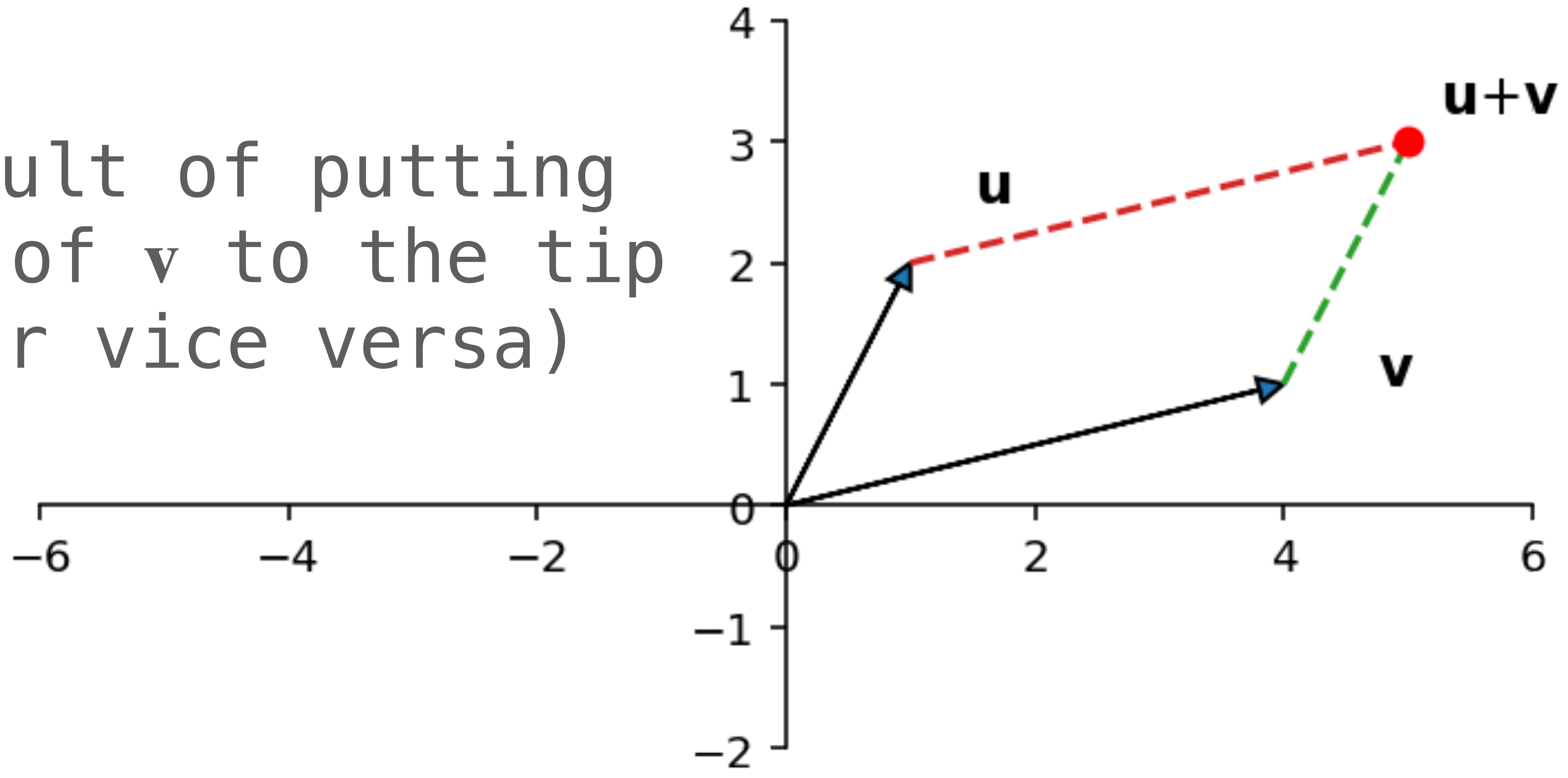
$\mathbf{u}+\mathbf{v}$ result of putting the tail of $\mathbf{v}$ to the tip of $\mathbf{u}$ (or vice versa)



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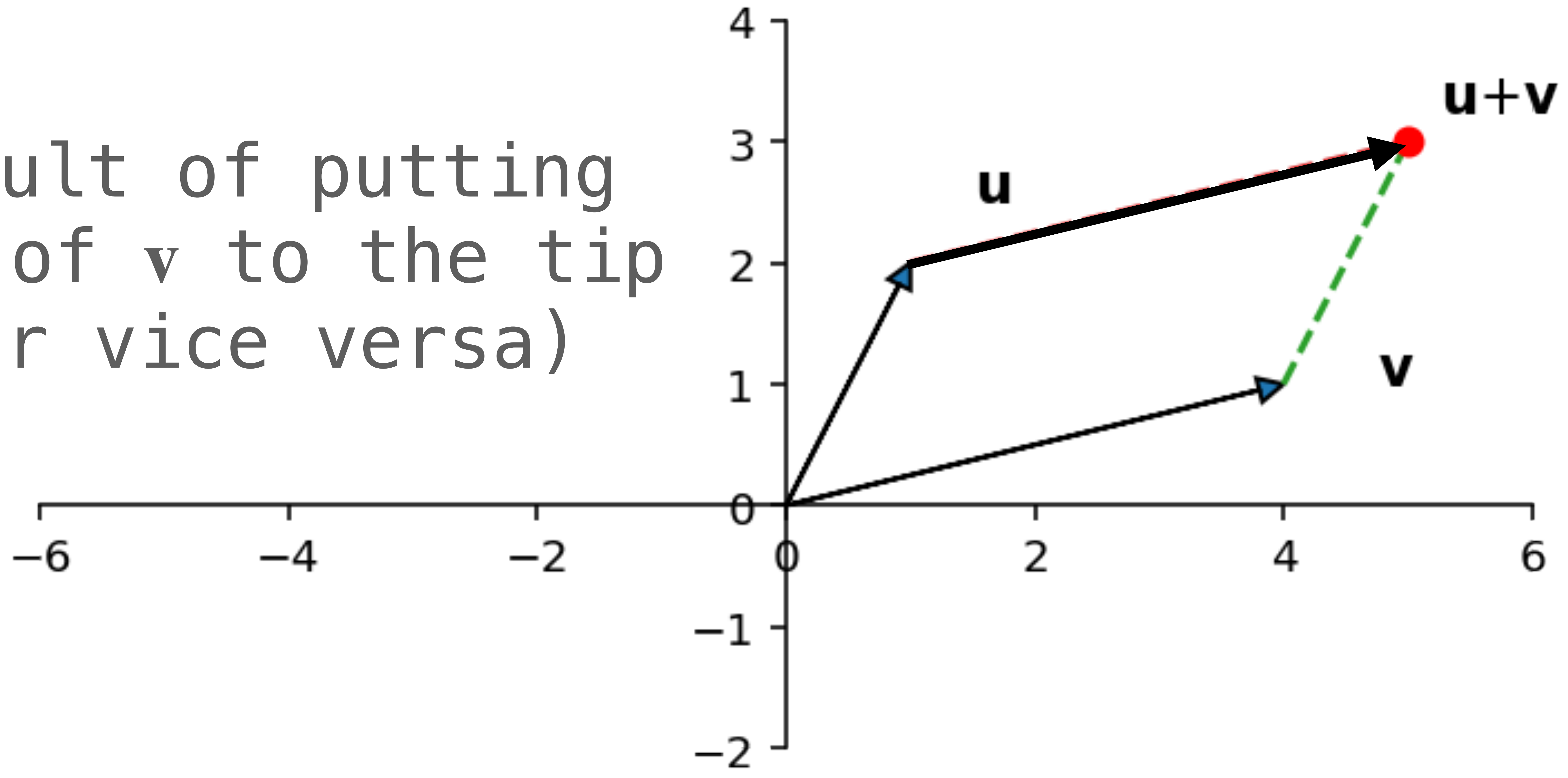

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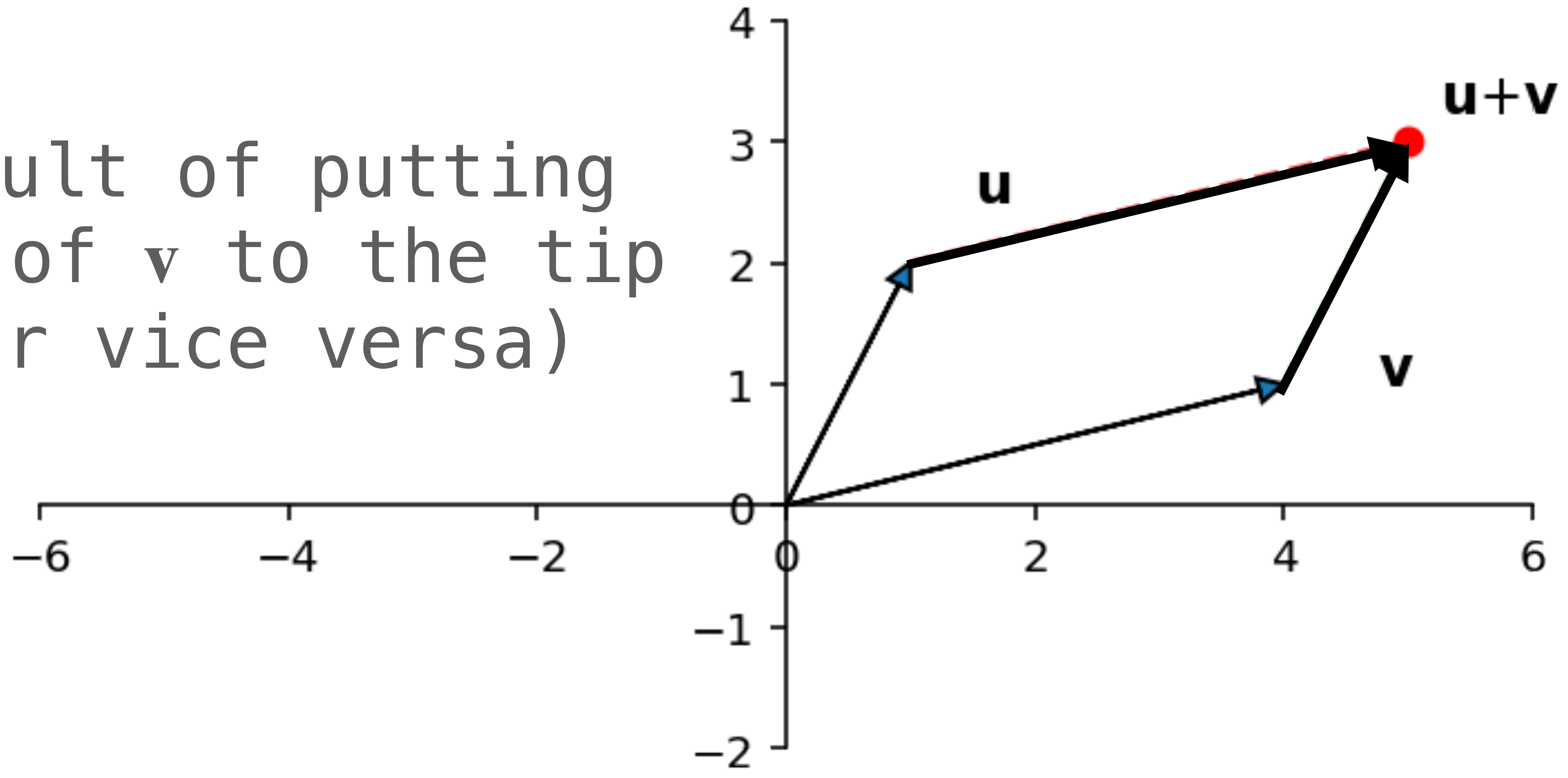

demo
(from ILA)

Vector "Interface"

equality	what does it mean for two vectors to be equal?
addition	what does $\mathbf{u} + \mathbf{v}$ (adding two vectors mean?
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What properties do they need to satisfy?

Vector Scaling/Multiplication

scaling/multiplying a vector by a number means multiplying each of it's elements

$$a \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} ab_1 \\ ab_2 \\ \vdots \\ ab_n \end{bmatrix}$$

Vector Scaling/Multiplication (Example)

$$3 \begin{bmatrix} 2 \\ 1 \\ 3.5 \\ 4 \end{bmatrix}$$

Vector Scaling/Multiplication (Example)

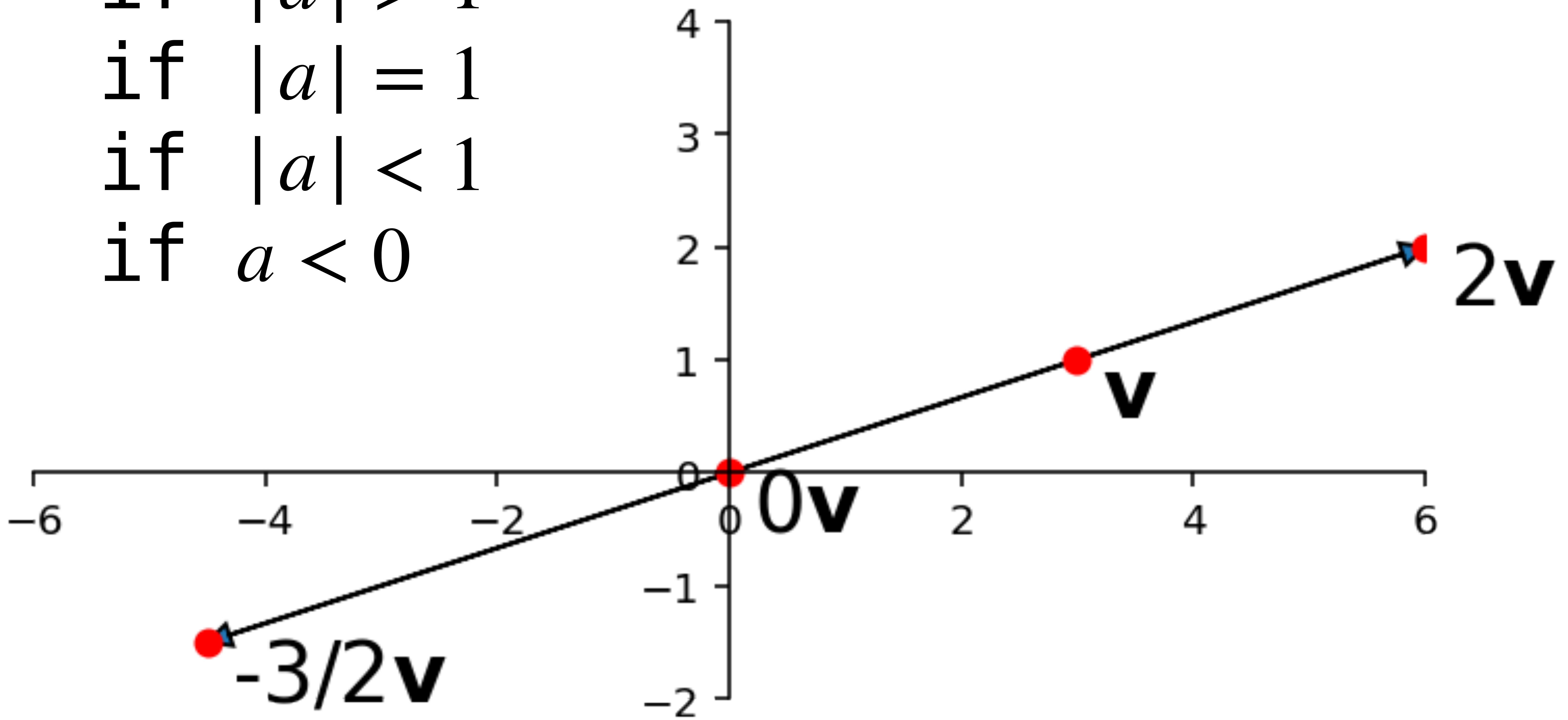
$$3 \begin{bmatrix} 2 \\ 1 \\ 3.5 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \cdot 2 \\ 3 \cdot 1 \\ 3 \cdot 3.5 \\ 3 \cdot 4 \end{bmatrix}$$

Vector Scaling/Multiplication (Example)

$$3 \begin{bmatrix} 2 \\ 1 \\ 3.5 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \cdot 2 \\ 3 \cdot 1 \\ 3 \cdot 3.5 \\ 3 \cdot 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 10.5 \\ 12 \end{bmatrix}$$

Vector Scaling (Geometrically)

longer	if $\|a\| > 1$
the same length	if $\|a\| = 1$
shorter	if $\|a\| < 1$
reversed	if $a < 0$



demo
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Algebraic Properties

For any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ and any real numbers c, d:

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$$

$$\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$$

$$\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$$

$$c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$$

$$(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$$

$$c(d\mathbf{u}) = (cd)\mathbf{u}$$

$$1\mathbf{u} = \mathbf{u}$$

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these are requirements for any **vector space**
they matter more for *bizarre* vector spaces

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

Example "Proof"

Question (Practice)

$$3\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + 2\begin{bmatrix}2\\0\\3\\-1\end{bmatrix} - \begin{bmatrix}-3\\4\\2\\0\end{bmatrix}$$

Compute the value of the above vector.

Answer

$$3\begin{bmatrix}1\\1\\1\\1\end{bmatrix}+2\begin{bmatrix}2\\0\\3\\-1\end{bmatrix}-\begin{bmatrix}-3\\4\\2\\0\end{bmatrix}$$

In Sum

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`we can add vectors`

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```
we can add vectors

we can scale vectors
```

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this gives us a way of generating new vectors from old ones

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What vectors can we make in this way?

Linear Combinations

Linear Combinations

Definition. a ***linear combination*** of vectors

$$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$$

is a vector of the form

$$\alpha_1 \mathbf{v}_1 + \alpha_1 \mathbf{v}_2 + \ldots + \alpha_n \mathbf{v}_n$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are in $\mathbb{R}$

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weights

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Looks suspiciously like a linear equation...

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are in $\mathbb{R}$

weights

Linear Combinations (Example)

$$3\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + 2\begin{bmatrix}2\\0\\3\\-1\end{bmatrix} - \begin{bmatrix}-3\\4\\2\\0\end{bmatrix}$$

Linear Combinations (Geometrically)

demo

(from ILA)

The Fundamental Concern

Can $\mathbf{u}$ be written as a linear combination of

$$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n?$$

That is, are there weights $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots \alpha_n \mathbf{v}_n = \mathbf{u}?$$

Why is this fundamental?

I'm going to ask that you suspend your disbelief...

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For now, how do we solve this problem?

Vector Equations and Linear Systems

The Fundamental Connection

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we don't know the weights, that's want we want to find

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what if we write them as *unknowns*?

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$$x_1 \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

Some Symbol Pushing...

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The Fundamental Connection

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what if we write them as *unknowns*?

$$x_1 + 2x_2 = 7$$
$$(-2)x_1 + 5x_2 = 4$$
$$-5x_1 + 6x_2 = -3$$

we get a system of linear equations we know how to solve

The Fundamental Connection

```
More generally:
```

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{1m} \end{bmatrix} + x_2 \begin{bmatrix} a_{21} \\ a_{21} \\ \vdots \\ a_{2m} \end{bmatrix} + \ldots + x_n \begin{bmatrix} a_{n1} \\ a_{n2} \\ \vdots \\ a_{nm} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

The Fundamental Connection

$$\begin{bmatrix} a_{11}x_1 \\ a_{21}x_1 \\ \vdots \\ a_{1m}x_1 \end{bmatrix} + \begin{bmatrix} a_{21}x_2 \\ a_{21}x_2 \\ \vdots \\ a_{2m}x_1 \end{bmatrix} + \ldots + \begin{bmatrix} a_{n1}x_n \\ a_{n2}x_n \\ \vdots \\ a_{nm}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

`by vector scaling`

The Fundamental Connection

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

`by vector addition`

The Fundamental Connection

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}
$$

`by vector equality`

The Fundamental Connection

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$$

augmented matrix

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

system of linear equations

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{1m} \end{bmatrix} + x_2 \begin{bmatrix} a_{21} \\ a_{21} \\ \vdots \\ a_{2m} \end{bmatrix} + \ldots + x_n \begin{bmatrix} a_{n1} \\ a_{n2} \\ \vdots \\ a_{nm} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

vector equation

The Fundamental Connection

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augmented matrix

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m$$

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vector equation

The Fundamental Connection

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

augmented matrix

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

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augmented matrix

this is our big shift in perspective

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

system of linear equations

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{1m} \end{bmatrix} + x_2 \begin{bmatrix} a_{21} \\ a_{21} \\ \vdots \\ a_{2m} \end{bmatrix} + \ldots + x_n \begin{bmatrix} a_{n1} \\ a_{n2} \\ \vdots \\ a_{nm} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

vector equation

HOW TO: Linear Combination Problems

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this is notation for building a matrix out of column vectors

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Question

Can $\begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$ *be written as a linear combination of* $\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$ *and* $\begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$*?*

Answer

$$3\begin{bmatrix}1\\-2\\-5\end{bmatrix}+2\begin{bmatrix}2\\5\\6\end{bmatrix}=\begin{bmatrix}7\\4\\-3\end{bmatrix}$$

Spans

Some Pictures

Spans

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read: $\mathbf{u}$ is an element of $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$

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Linear Combinations and Spans (A Picture)

Spans (Geometrically)

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```
for one vector
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`for one vector`

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the span of one vector is a **line**

Spans (Geometrically)

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`the span of` **`two`** `vectors can be a` **`plane`**

Spans (Geometrically)

the span of **two** vectors can be a **plane**

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!!IMPORTANT!!
In all cases they pass through the origin

Spans (Geometrically)

demo

(from ILA)

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you know how to do this now

Example

Is $\begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$ *in* $\text{span}\left\{\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}\right\}$*?*

Question (Conceptual)

What does it mean geometrically if $\mathbf{b} \notin \text{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$*?*

demo
(from ILA)

HOW TO: Inconsistency and Spans

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There is **no way** to write $\mathbf{b}$ as a linear combination

Example

*Find a vector **not** in* $\text{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 3 \end{bmatrix} \right\}$*.*

Summary

vectors are fundamental objects

we can think of them as the columns of a linear system

we can scale them and add them together

they can span spaces which represent hyperplanes