Matrix-Vector Equations

Geometric Algorithms
Lecture 6

CAS CS 132

Practice Problem

Is the vector $\begin{bmatrix} 9 \\ 3 \\ -14 \end{bmatrix}$ *in* $\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix} \right\}$*?*

Is the vector $\begin{bmatrix} 9 \\ 3 \\ -14 \end{bmatrix}$ *in* span $\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix} \right\}$?

Answer

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ -14 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 3 & 9 \\ 0 & 1 & 2 & 3 \\ -2 & -1 & -4 & -14 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 + 2R_1}$$

$+2 \quad +2 \quad +6 \quad +18$

$$\begin{bmatrix} 1 & 1 & 3 & 9 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 4 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - R_2} \begin{bmatrix} 1 & 1 & 3 & 9 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$-0 \quad -1 \quad -2 \quad -3$

$\boxed{\text{NO}}$

Answer

solve the system of linear equations with the augmented matrix

$$\begin{bmatrix} 1 & 1 & 3 & 9 \\ 0 & 1 & 2 & 3 \\ -2 & -1 & -4 & -14 \end{bmatrix}$$

Answer

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$$\begin{bmatrix} 1 & 1 & 3 & 9 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_3 \leftarrow R_3 - R_1$$

Answer

$$\begin{bmatrix} 1 & 1 & 3 & 9 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

`no solution ≡ not in the span`

Objectives

1. motivate the study of matrix–vector equations
2. define matrix–vector multiplication
3. Revisit span
4. take stock of our perspectives on systems of linear equations

Keywords

matrix–vector multiplication

the matrix equation

inner–product

row–column rule

Recap

Recall: Vector "Interface"

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`equality`	`what does it mean for two vectors to be equal?`

Recall: Vector "Interface"

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addition what does $\mathbf{u} + \mathbf{v}$ (adding two vectors mean?

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Recall: Vector "Interface"

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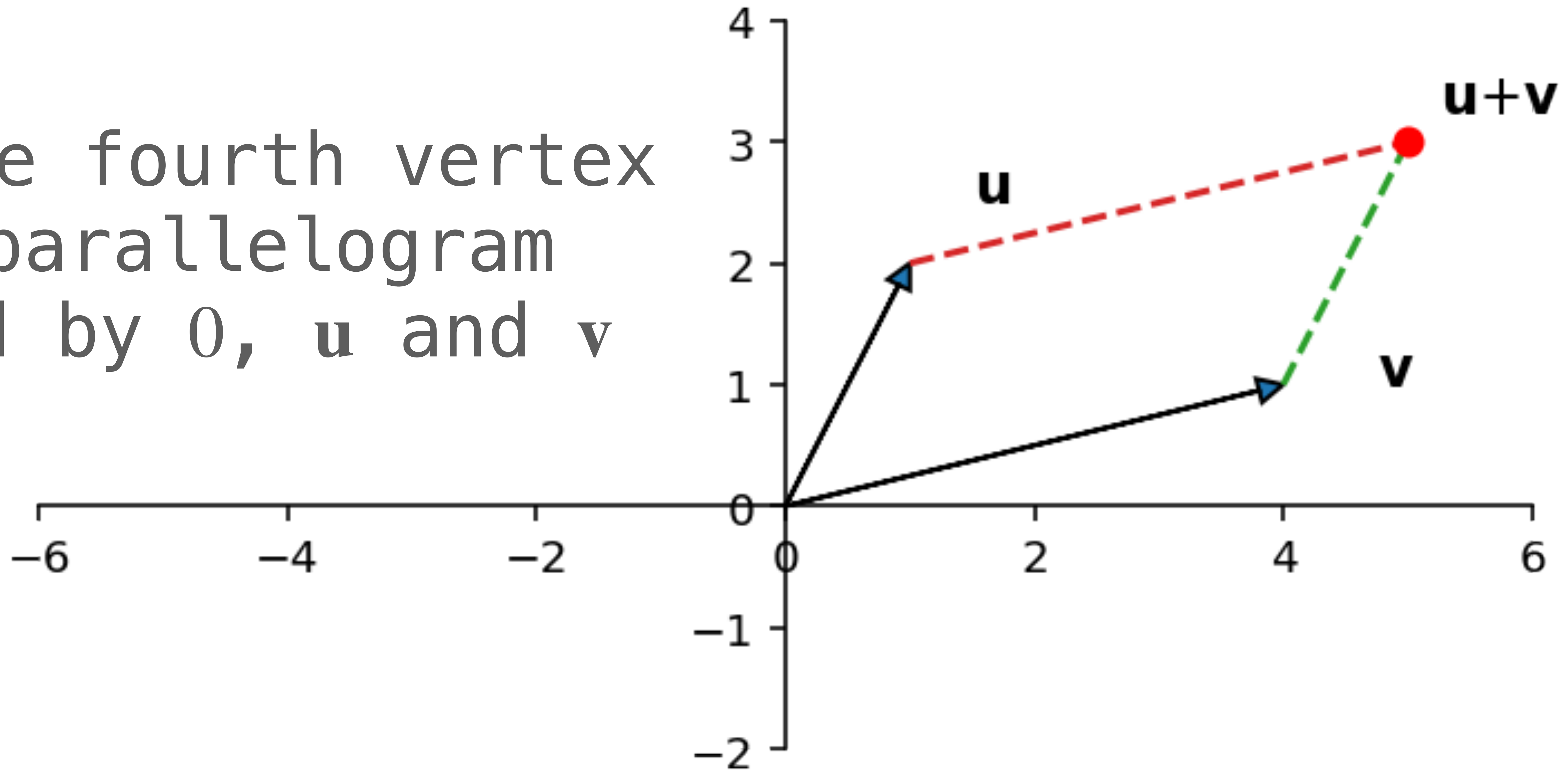
What properties do they need to satisfy?

Recall: Vector Addition (Geometrically)

in $\mathbb{R}^2$ it's called the *parallelogram rule*

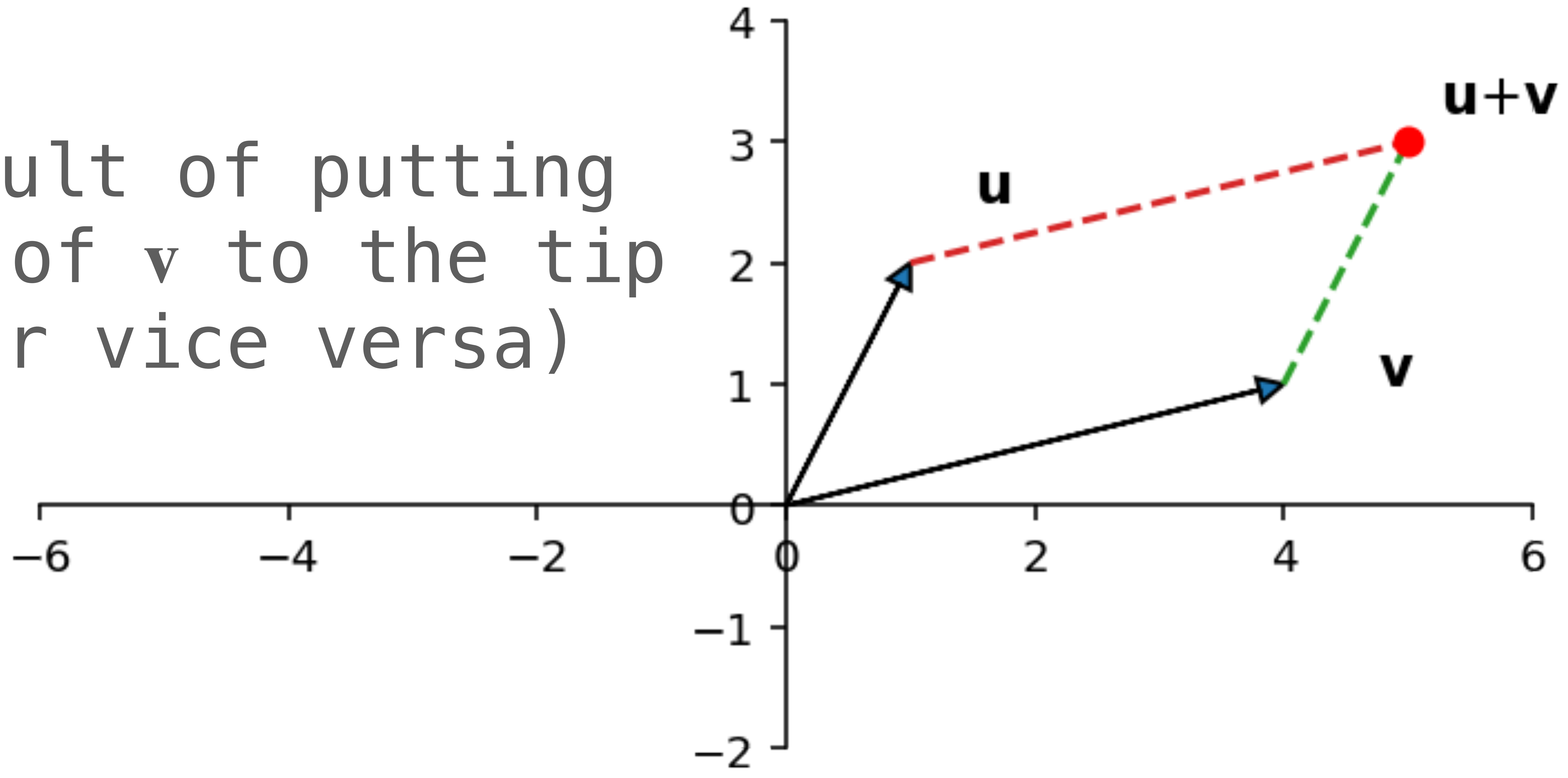
$\mathbf{u} + \mathbf{v}$ is the fourth vertex of the parallelogram generated by 0, $\mathbf{u}$ and $\mathbf{v}$



Vector Addition (Geometrically)

or the *tip-to-tail rule*

$\mathbf{u} + \mathbf{v}$ result of putting the tail of $\mathbf{v}$ to the tip of $\mathbf{u}$ (or vice versa)



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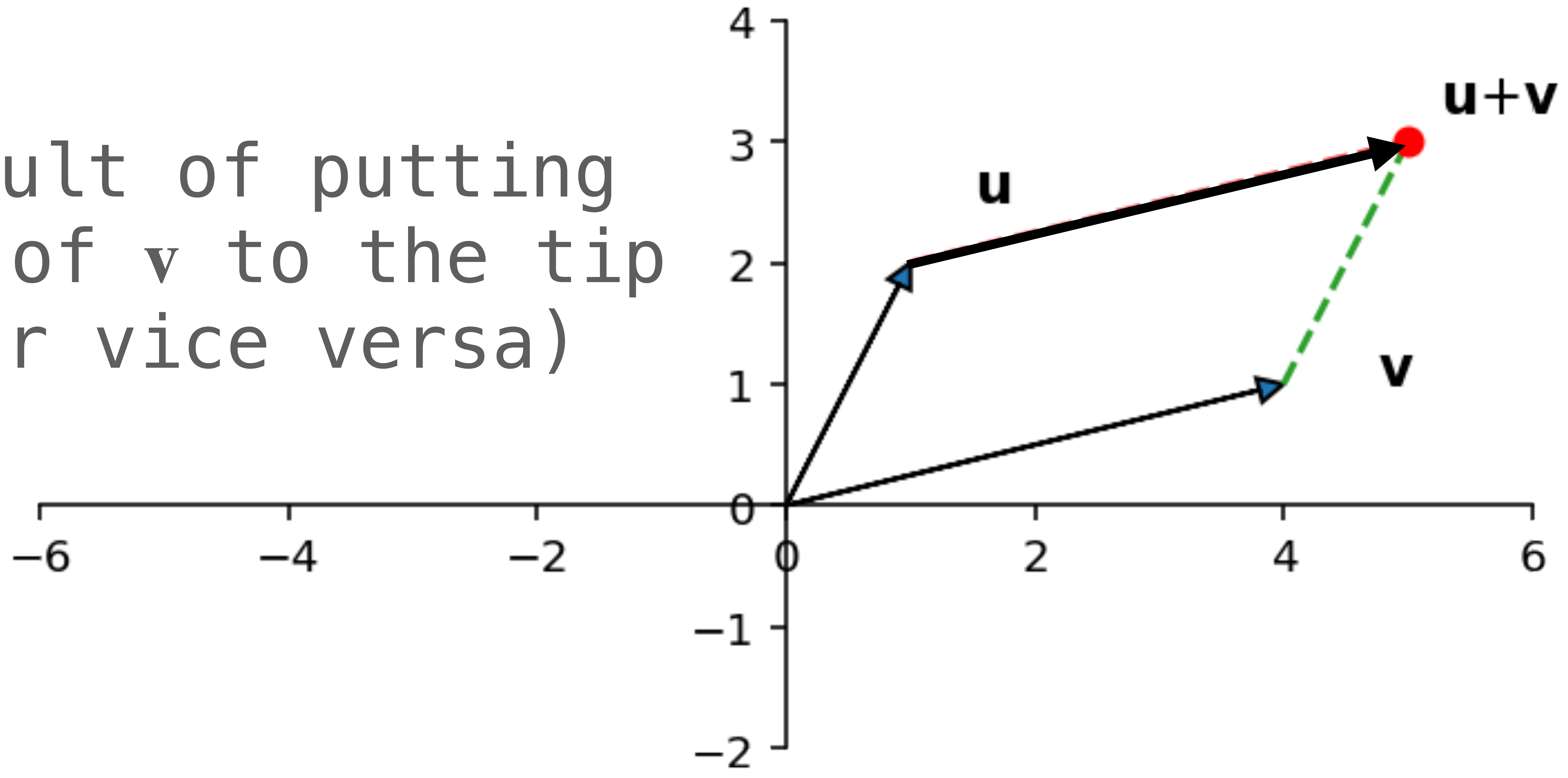

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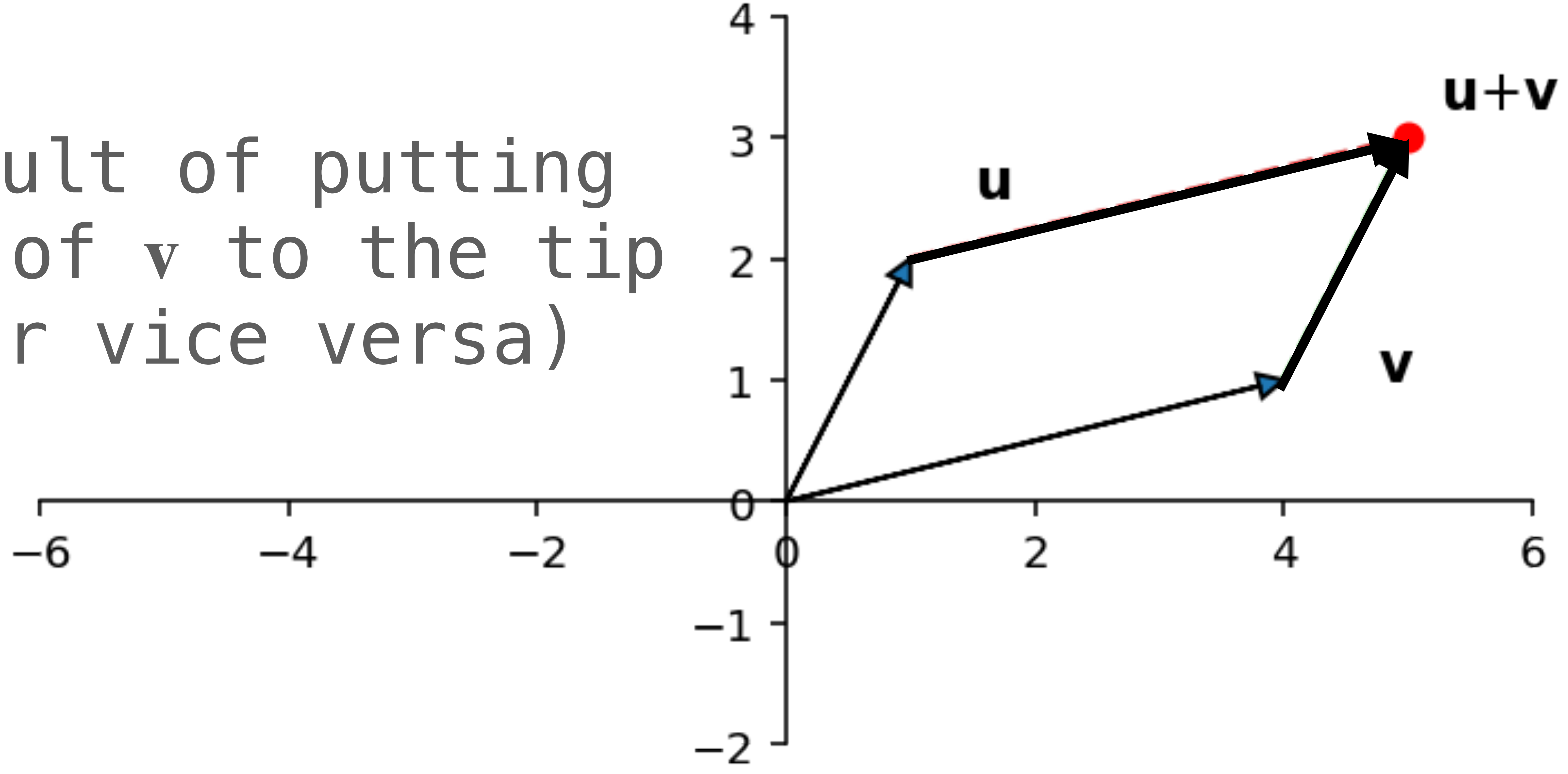
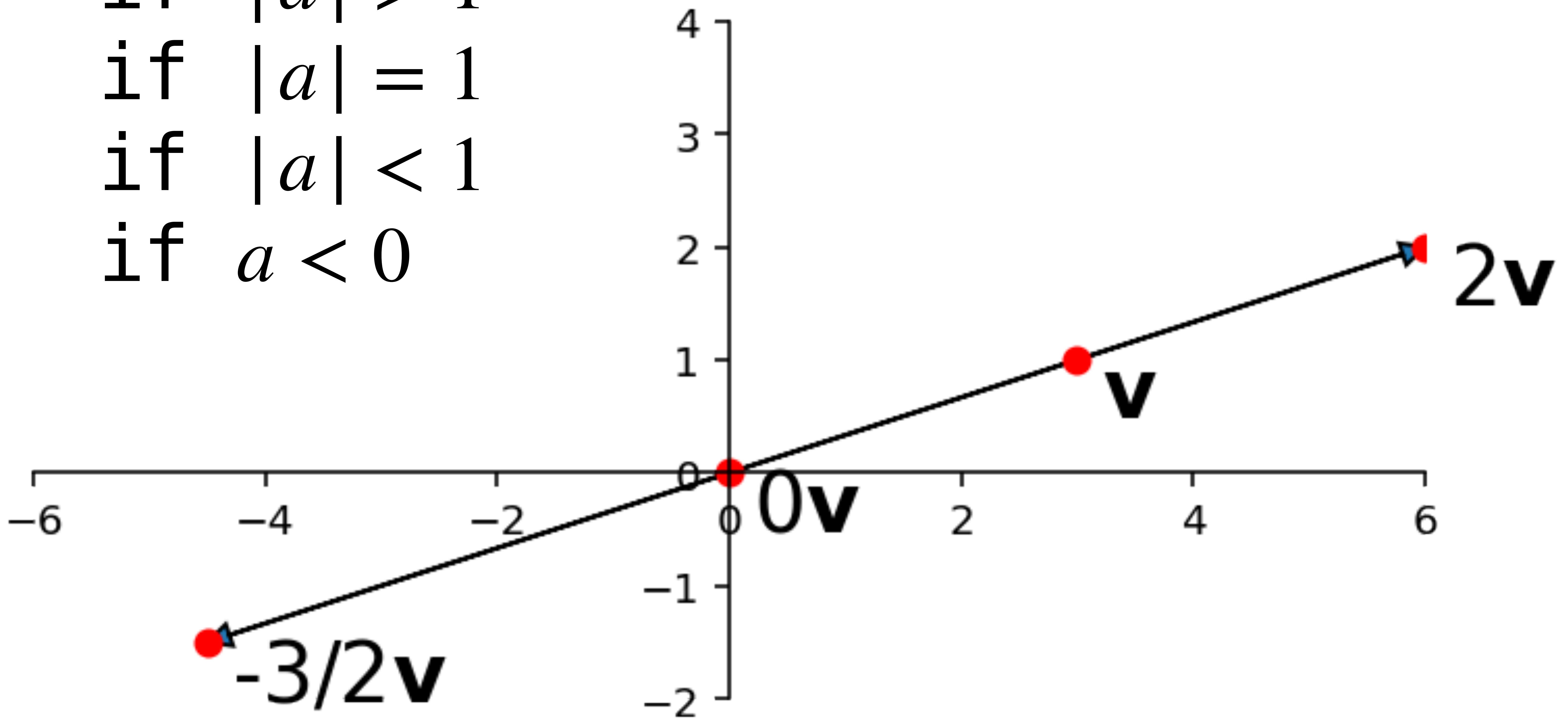


Recall Vector Scaling (Geometrically)

longer	if $\|a\| > 1$
the same length	if $\|a\| = 1$
shorter	if $\|a\| < 1$
reversed	if $a < 0$



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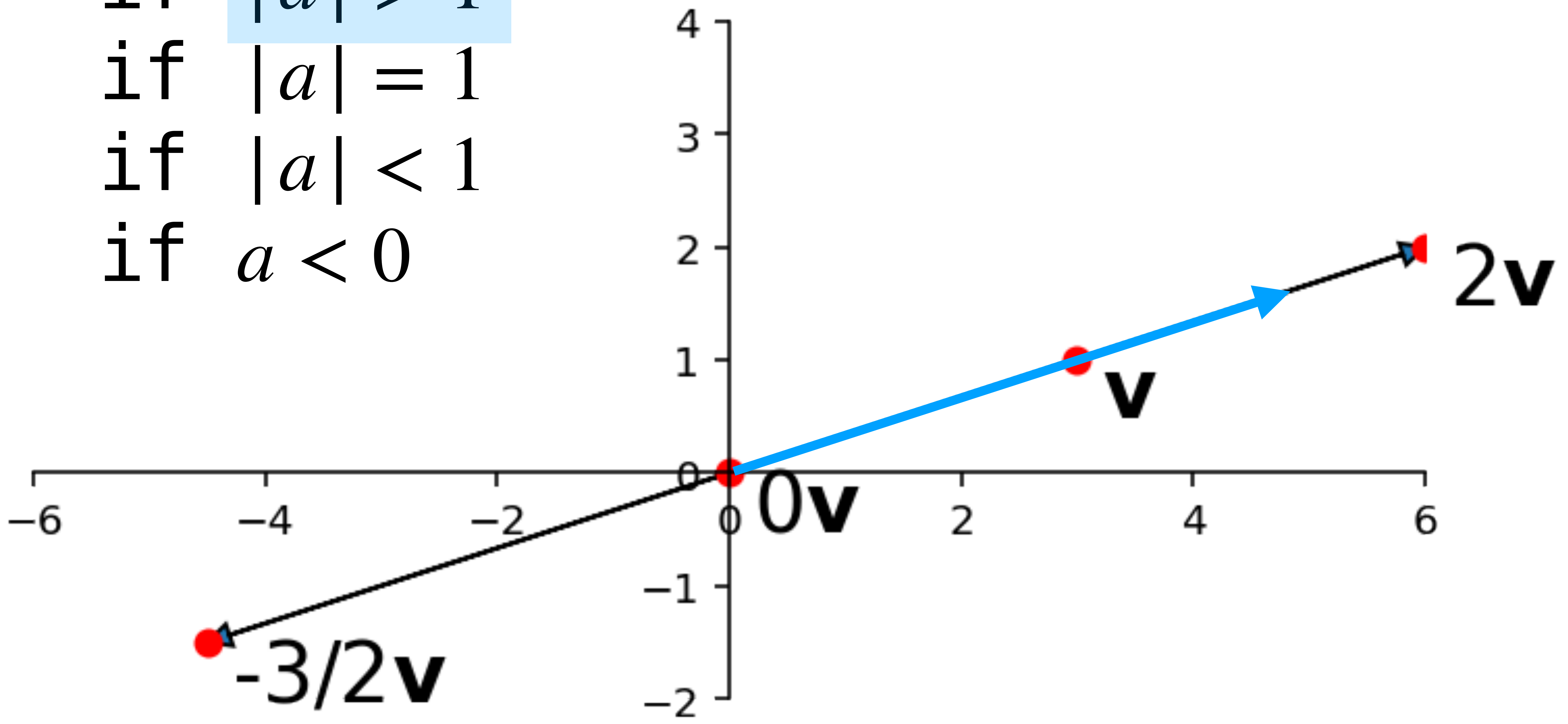

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```
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```

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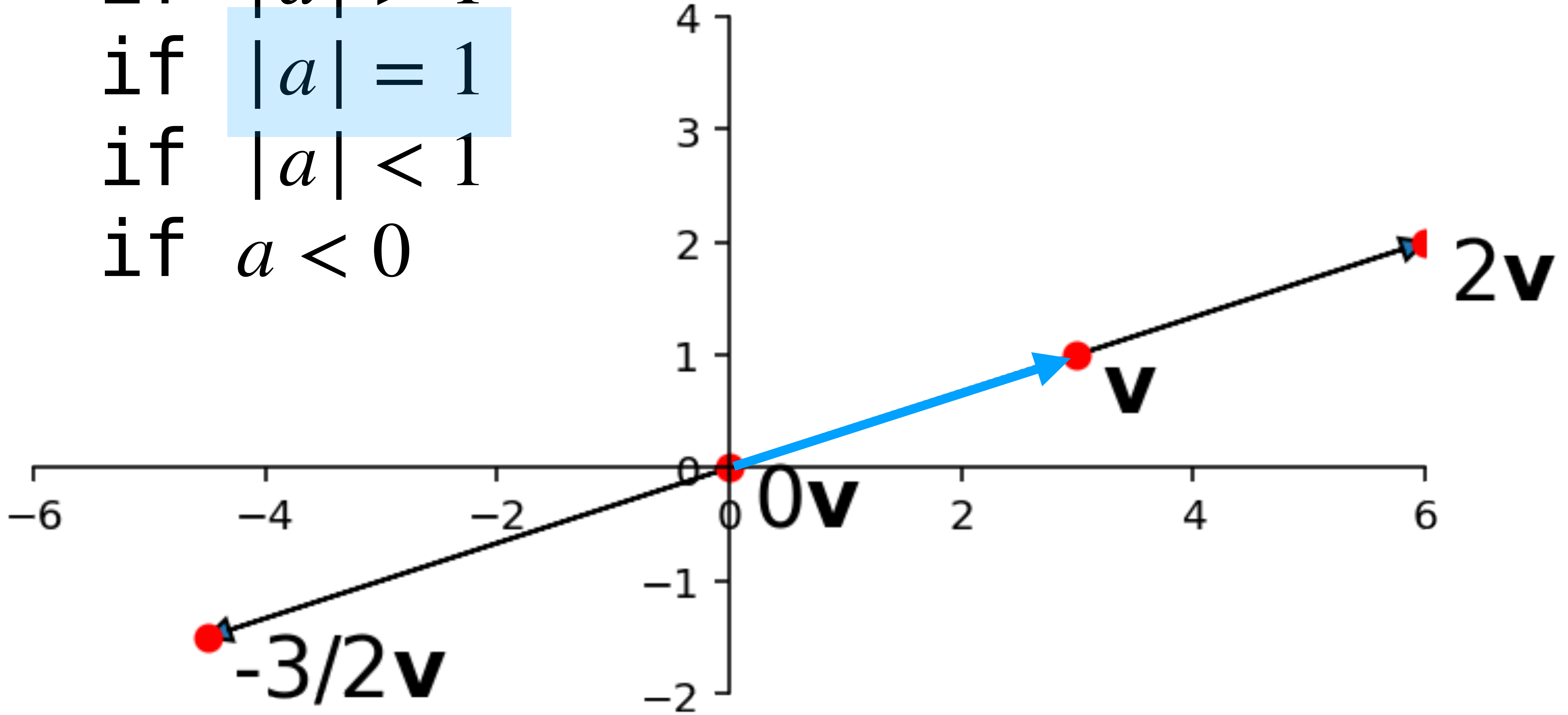

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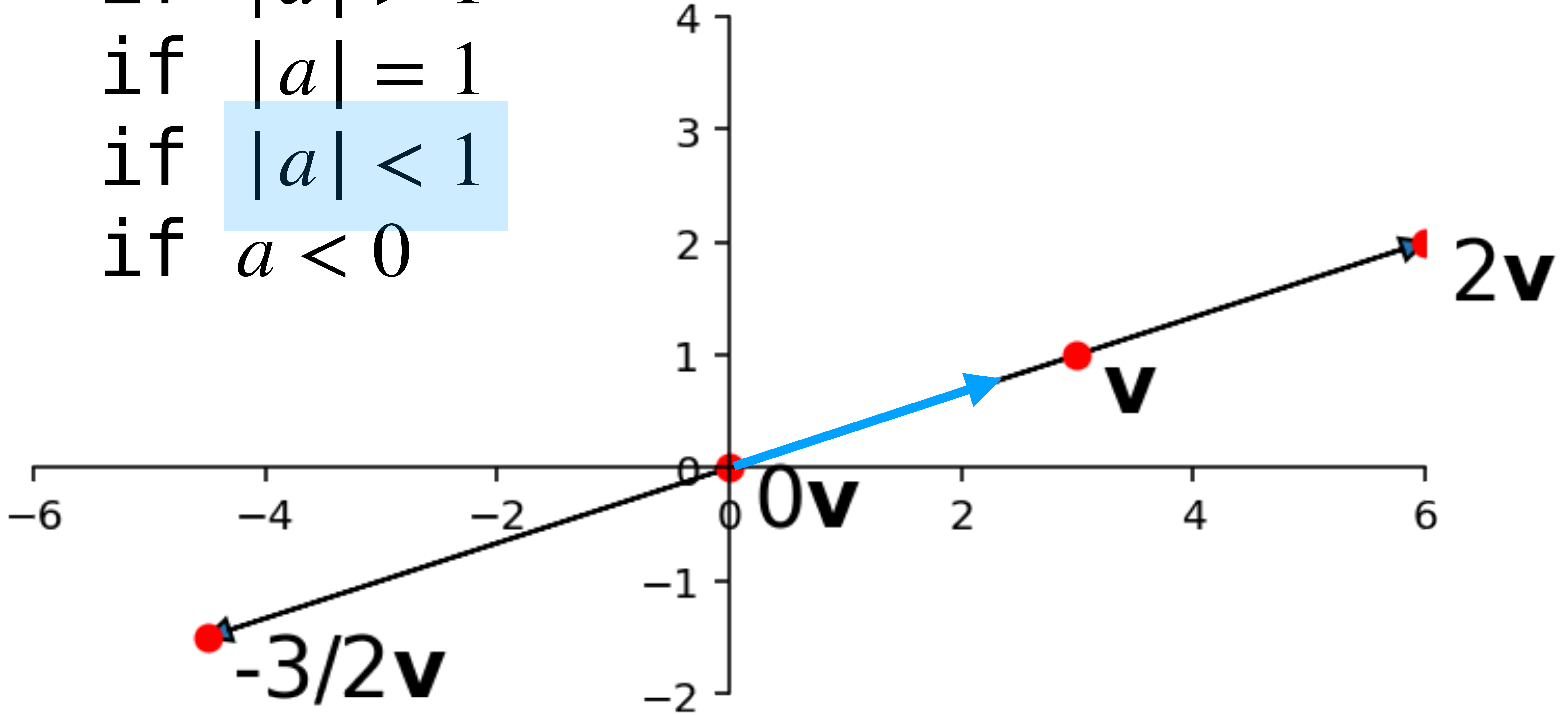

Recall Vector Scaling (Geometrically)

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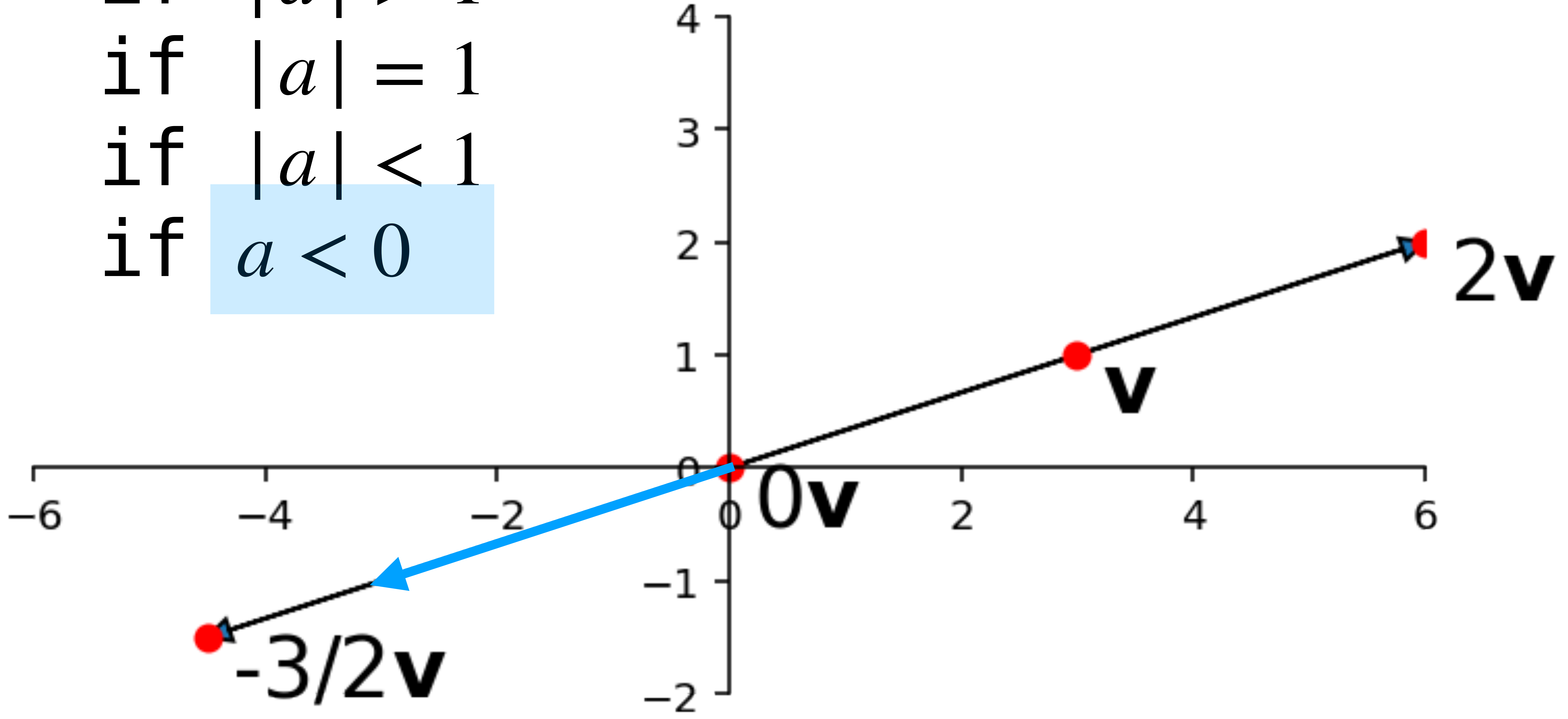

Recall: Linear Combinations

Definition. a ***linear combination*** of vectors

$$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$$

is a vector of the form

$$\alpha_1 \mathbf{v}_1 + \alpha_1 \mathbf{v}_2 + \ldots + \alpha_n \mathbf{v}_n$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are in $\mathbb{R}$

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where $\underbrace{\alpha_1, \alpha_2, \ldots, \alpha_n}_{\text{weights}}$ are in $\mathbb{R}$

Recall: Linear Combinations (Example)

$$3\begin{bmatrix}1\\1\\1\\1\end{bmatrix}+2\begin{bmatrix}2\\0\\3\\-1\end{bmatrix}-\begin{bmatrix}-3\\4\\2\\0\end{bmatrix}$$

Recall: Linear Combinations (Geometrically)

demo

(from ILA)

Recall: The Fundamental Concern

Can $\mathbf{u}$ be written as a linear combination of

$$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n?$$

That is, are there weights $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots \alpha_n \mathbf{v}_n = \mathbf{u}?$$

Recall: The Fundamental Connection

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$$

augmented matrix

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

system of linear equations

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{1m} \end{bmatrix} + x_2 \begin{bmatrix} a_{21} \\ a_{21} \\ \vdots \\ a_{2m} \end{bmatrix} + \ldots + x_n \begin{bmatrix} a_{n1} \\ a_{n2} \\ \vdots \\ a_{nm} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

vector equation

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vector equation

where we left off...

Question (Conceptual)

What does it mean geometrically if $\mathbf{b} \notin \text{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$*?*

demo

(from ILA)

HOW TO: Inconsistency and Spans

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$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}$$

is the augmented matrix of an *inconsistent* system

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There is **no way** to write $\mathbf{b}$ as a linear combination

Example

Find a vector ***not*** *in* span $\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 3 \end{bmatrix} \right\}$.

$$\begin{bmatrix} 2 & 5 & b_1 \\ 1 & 3 & b_2 \\ 1 & 3 & b_3 \end{bmatrix} \qquad \boxed{\begin{bmatrix} 2 & 5 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 1 \end{bmatrix}}$$

Motivation (Very Short)

Recall: The Fundamental Connection

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$$

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vector equation

Recall: The Fundamental Connection

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system of linear equations

Why not view these as a vector too?

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{1m} \end{bmatrix} + x_2 \begin{bmatrix} a_{21} \\ a_{21} \\ \vdots \\ a_{2m} \end{bmatrix} + \ldots + x_n \begin{bmatrix} a_{n1} \\ a_{n2} \\ \vdots \\ a_{nm} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

vector equation

Solutions as Vectors

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Can we view a linear system as a single equation with matrices and vectors?

How do matrices and vectors "interface"?

Matrix-Vector Multiplication

Matrix-Vector "Interface"

`multiplication`   what does $A\mathbf{v}$ mean when A is a matrix and $\mathbf{v}$ is a vector?

Matrix-Vector Multiplication (Pictorially)

$$As$$

Matrix-Vector Multiplication (Pictorially)

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

Matrix-Vector Multiplication (Pictorially)

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

Matrix-Vector Multiplication (Pictorially)

$$s_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + s_2 \begin{bmatrix} a_{21} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \ldots + s_n \begin{bmatrix} a_{m1} \\ a_{m2} \\ \vdots \\ a_{mn} \end{bmatrix}$$

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a linear combination of the columns where $\mathbf{s}$ defines the weights

Why keeping track of matrix size is important

`this only works if the number of` *`columns`* `of the matrix matches the number of` *`rows`* `of the vector`

$$\begin{bmatrix} * & \cdots & * \\ * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \\ * & \cdots & * \end{bmatrix} \begin{bmatrix} * \\ \vdots \\ * \end{bmatrix} = \begin{bmatrix} * \\ * \\ \vdots \\ * \\ * \end{bmatrix}$$

$$(m \times n) \qquad (n \times 1) \qquad (m \times 1)$$

Why keeping track of matrix size is important

`this only works if the number of` *`columns`* `of the matrix matches the number of` *`rows`* `of the vector`

$$
\begin{array}{cc} & n \\ m & \begin{bmatrix} * & \cdots & * \\ * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \\ * & \cdots & * \end{bmatrix} \end{array}
\begin{array}{cc} & 1 \\ n & \begin{bmatrix} * \\ \vdots \\ * \end{bmatrix} \end{array}
=
\begin{array}{cc} & 1 \\ m & \begin{bmatrix} * \\ * \\ \vdots \\ * \\ * \end{bmatrix} \end{array}
$$

$(m \times n)$ $\quad$ $(n \times 1)$ $\quad$ $(m \times 1)$

Non-Example

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} + 3???$$

Non-Example

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} + 3???$$

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$(2 \times 2) \quad (3 \times 1)$

Non-Example

$$\underset{2\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} 7 \\ 8 \\ 0 \end{bmatrix}} = 7\begin{bmatrix} 1 \\ 4 \end{bmatrix} + 8\begin{bmatrix} 2 \\ 5 \end{bmatrix} + ??\begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 4 \end{bmatrix} + 3???$$

THESE DON'T MATCH

(2×2) (3×1)

Example

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Example

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$(2 \times 2) \quad (2 \times 1)$

Example

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

THESE MATCH

$$(2 \times 2) \quad (2 \times 1)$$

$$= \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

Matrix-Vector Multiplication

Definition. Given a $(m \times n)$ matrix A with columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$, and a vector $\mathbf{v}$ in $\mathbb{R}^n$, we define

$\in \mathbb{R}^m$

$$A\mathbf{v} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \ldots \quad \mathbf{a}_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + \ldots v_n\mathbf{a}_n$$

Matrix-Vector Multiplication

Definition. Given a $(m \times n)$ matrix A with columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$, and a vector $\mathbf{v}$ in $\mathbb{R}^n$, we define

$$A\mathbf{v} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + \ldots v_n\mathbf{a}_n$$

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$A\mathbf{v}$ is a linear combination of the columns of A with weights given by $\mathbf{v}$

Example

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

$$3 \times 2$$

$$\vec{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \vec{a}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = (1)\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (-1)\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ -3 \\ -3 \end{bmatrix}$$

Algebraic Properties

The algebraic properties of matrix-vector multiplication are **very important.**

1. $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v}$
2. $A(c\mathbf{v}) = c(A\mathbf{v})$

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The algebraic properties of matrix-vector multiplication are **very important.**

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2. $A(c\mathbf{v}) = c(A\mathbf{v})$

There are only two, please memorize them...

Derivation of (1) for A in $\mathbb{R}^{n\times 3}$

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3]\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right)$$

Derivation of (1) for A in $\mathbb{R}^{n\times 3}$

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3]\left(\begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix}\right)$$

by vector addition

Derivation of (1) for A in $\mathbb{R}^{n\times 3}$

$$(u_1 + v_1)\mathbf{a}_1 + (u_2 + v_2)\mathbf{a}_2 + (u_3 + v_3)\mathbf{a}_3$$

`by matrix vector multiplication`

Derivation of (1) for A in $\mathbb{R}^{n \times 3}$

$$(2+3)5 = 2(5) + 3(5)$$

$$u_1\mathbf{a}_1 + v_1\mathbf{a}_1 + u_2\mathbf{a}_2 + v_2\mathbf{a}_2 + u_3\mathbf{a}_3 + v_3\mathbf{a}_3$$

`by vector scaling (distribution)`

Derivation of (1) for A in $\mathbb{R}^{n\times 3}$

$$(u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + u_3\mathbf{a}_3) + (v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + v_3\mathbf{a}_3)$$

`by rearranging`

Derivation of (1) for A in $\mathbb{R}^{n\times 3}$

$$\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix} + \begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}$$

`by matrix vector multiplication`

Derivation of (1) for A in $\mathbb{R}^{n\times 3}$

$$\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3\end{bmatrix}\left(\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}+\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}\right)$$

equals

$$\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}+\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}$$

fin

A Common Error

$$A\mathbf{v} \neq \mathbf{v}A$$

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it is **important** that we write our matrix-vectors multiplications with the matrix on the <u>left</u>

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it is **important** that we write our matrix-vectors multiplications with the matrix on the <u>left</u>

this may feel artificial now, since the RHS is meaningless to us now, but it won't be for long

Looking forward a bit

Remember. column vectors are matrices with 1 column

Eventually we'll be able to view all of these as <u>matrix operations</u>

Question

Compute the following matrix-vector multiplication

$$\begin{bmatrix} 2 & -3 & 4 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix}$$

Answer

$$\begin{bmatrix} 2 & -3 & 4 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix}$$

$$5\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 5\begin{bmatrix} -3 \\ 1 \end{bmatrix} + 4\begin{bmatrix} 4 \\ 0 \end{bmatrix} =$$

$$\begin{bmatrix} 10 \\ -5 \end{bmatrix} + \begin{bmatrix} -15 \\ 5 \end{bmatrix} + \begin{bmatrix} 16 \\ 0 \end{bmatrix} = \begin{bmatrix} 11 \\ 0 \end{bmatrix}$$

Answer

$$\begin{bmatrix} 2 & -3 & 4 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix}$$

Answer

$$5\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 5\begin{bmatrix} -3 \\ 1 \end{bmatrix} + 4\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

Answer

$$\begin{bmatrix} 10 \\ -5 \end{bmatrix} + \begin{bmatrix} -15 \\ 5 \end{bmatrix} + \begin{bmatrix} 16 \\ 0 \end{bmatrix}$$

Answer

$$\begin{bmatrix} 11 \\ 0 \end{bmatrix}$$

A Tip

$$\begin{bmatrix} 2 & -3 & 4 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} ? \\ ? \end{bmatrix}$$

A Tip

$$\begin{bmatrix} \boxed{2 \quad -3 \quad 4} \\ -1 \quad\; 1 \quad\; 0 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} \boxed{?} \\ ? \end{bmatrix}$$

$$5(2) + 5(-3) + 4(4) = 11$$

A Tip

$$\begin{bmatrix} 2 & -3 & 4 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 11 \\ ? \end{bmatrix}$$

$$5(-1) + 5(1) + 4(0) = 0$$

A Tip

$$\begin{bmatrix} 2 & -3 & 4 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 11 \\ 0 \end{bmatrix}$$

Calculating $A\mathbf{v}$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ \vdots \\ ? \end{bmatrix}$$

$$v_1 = a_{11}s_1 + a_{12}s_2 + \ldots + a_{1n}s_n = \sum_{i=1}^{n} a_{1i}s_i$$

Calculating $A\mathbf{v}$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} v_1 \\ ? \\ \vdots \\ ? \end{bmatrix}$$

$$v_2 = a_{21}s_1 + a_{22}s_2 + \ldots + a_{2n}s_n = \sum_{i=1}^{n} a_{2i}s_i$$

Calculating $A\mathbf{v}$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ ? \end{bmatrix}$$

$$v_m = a_{m1}s_1 + a_{m2}s_2 + \ldots + a_{mn}s_n = \sum_{i=1}^{n} a_{mi}s_i$$

Calculating $A\mathbf{v}$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$$

Row-Column Rule

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} a_{1i}s_i \\ \sum_{i=1}^{n} a_{2i}s_i \\ \vdots \\ \sum_{i=1}^{n} a_{mi}s_i \end{bmatrix}$$

Inner product: $\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \sum_{i=1}^{n} a_i s_i$

Inner Product

Definition. The **inner product** of vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ is defined the

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} u_i v_i$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 1(4) + 2(5) + 3(6)$$

$$4 + 10 + 18$$

$$\boxed{32}$$

Row-Column Rule

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} a_{1i}s_i \\ \sum_{i=1}^{n} a_{2i}s_i \\ \vdots \\ \sum_{i=1}^{n} a_{mi}s_i \end{bmatrix}$$

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The ith entry of the $A\mathbf{s}$ is the inner product of the ith row of A and $\mathbf{s}$

Example

The Matrix Equation

Recall: Vector Equations

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{b}$$

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Question. Can $\mathbf{b}$ be written as a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$?

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The Idea. think of the weights as *unknowns*

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we can use the same idea for matrix-vector multiplication

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$$A\mathbf{x} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n]\,\mathbf{x} = \mathbf{b}$$

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Can $\mathbf{b}$ be written as a linear combination of **the columns of A**?

The Matrix Equation

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Can $\mathbf{b}$ be written as a linear combination of **the columns of A**?

The Idea. write the "vector part" of our matrix-vector multiplication as an *unknown*

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

HOW TO: The Matrix Equation

Question. Does $A\mathbf{x} = \mathbf{b}$ have a solution?

Question. Is $A\mathbf{x} = \mathbf{b}$ consistent?

Question. write down a solution to the equation $A\mathbf{x} = \mathbf{b}$

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(augmented matrix) $\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n & \mathbf{b}\end{bmatrix}$

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(augmented matrix) $\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n & \mathbf{b}\end{bmatrix}$

!!they all have the same solution set!!

HOW TO: The Matrix Equation

Question. write down a solution to the equation $A\mathbf{x} = \mathbf{b}$

Solution.

use Gaussian elimination (or other means) to convert $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}$ to reduced echelon form

then read off a solution from the reduced echelon form

Full Span

Recall: Span

Recall: Span

Definition. the ***span*** of a set of vectors is the set of all possible linear combinations of them

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = \{\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots\alpha_n\mathbf{v}_n : \alpha_1, \alpha_2, \ldots, \alpha_n \text{ are in } \mathbb{R}\}$$

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$\mathbf{u} \in \mathsf{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ exactly when $\mathbf{u}$ can be expressed as a linear combination of those vectors

Spans (with Matrices)

Definition. the ***span*** of the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ is:

$$\text{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\} = \{[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n]\,\mathbf{v} : \mathbf{v} \in \mathbb{R}^n\}$$

$$\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$v_1 \vec{a}_1 + \cdots + v_n \vec{a}_n$$

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the span of the columns of a matrix A
is the set of of vectors resulting
from multiplying A by any vector

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the span of the columns of a matrix A
is the set of of vectors resulting
from multiplying A by any vector

(we will soon start thinking of A as a way of *transforming* vectors)

Spanning all of $\mathbb{R}^2$

Spanning all of $\mathbb{R}^2$

if two (or more) vectors in $\mathbb{R}^2$ span a plane, they must span all of $\mathbb{R}^2$. They "fill up" $\mathbb{R}^2$

Spanning all of $\mathbb{R}^2$

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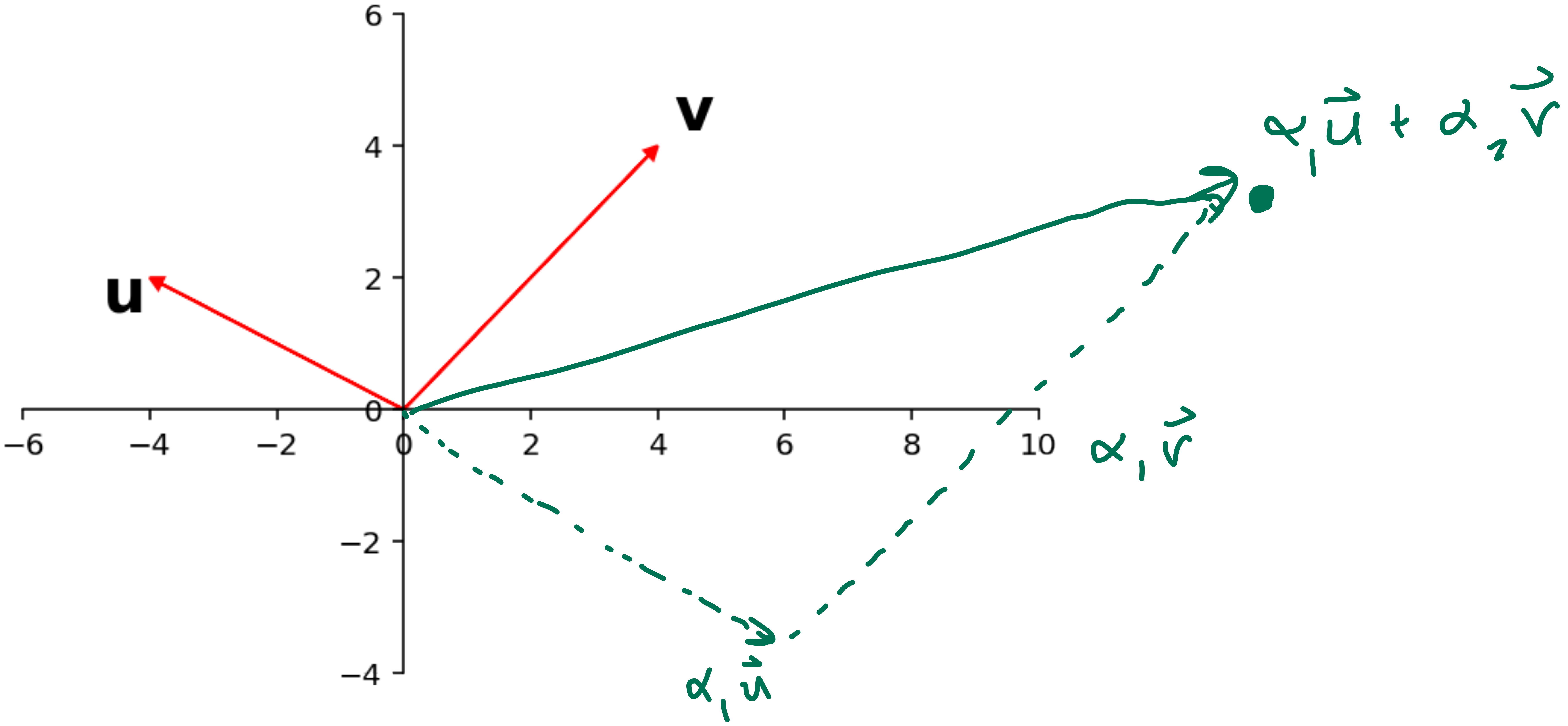
`they must span all of` $\mathbb{R}^2$`. They "fill up"` $\mathbb{R}^2$



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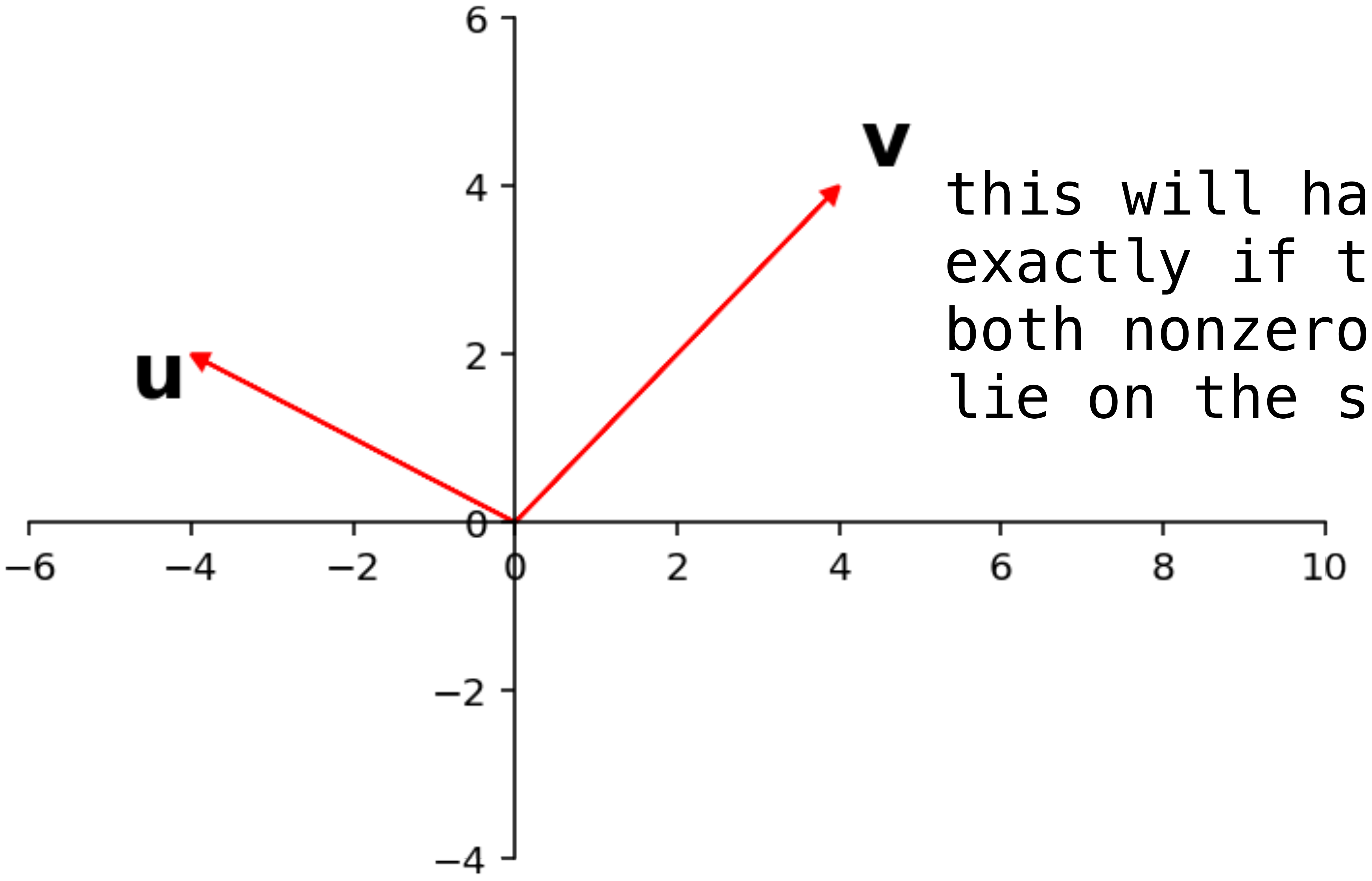

this will happen exactly if they are both nonzero and don't lie on the same line

What about $\mathbb{R}^n$?

When do a set of vectors span all of $\mathbb{R}^n$?
When do a set of vectors "fill up" $\mathbb{R}^n$?

A Few Questions

Can two vectors in $\mathbb{R}^3$ span all of $\mathbb{R}^3$?

Is it required that five vectors $\mathbb{R}^3$ span all of $\mathbb{R}^3$?

A Thought Experiment

suppose I give you the augmented matrix of a linear system but I cover up the last column

$$\left[\begin{array}{cccc} 1 & 2 & 3 & \text{[redacted]} \\ 2 & 1 & 0 & \end{array}\right]$$

A Thought Experiment

```
then we reduce it to echelon form
```

$$\begin{bmatrix} 1 & 2 & 3 & \text{[redacted]} \\ 2 & 1 & 0 & \end{bmatrix}$$

A Thought Experiment

`then we reduce it to echelon form`

$$\begin{bmatrix} 1 & 2 & 3 & \text{[redacted]} \\ 2 & 1 & 0 & \end{bmatrix}$$

$$R_2 \leftarrow R_2 - 2R_1$$

A Thought Experiment

then we reduce it to echelon form

$$\begin{bmatrix} 1 & 2 & 3 & \text{[redacted]} \\ 0 & -3 & -6 & \end{bmatrix}$$

A Thought Experiment

then we reduce it to echelon form

$$\begin{bmatrix} 1 & 2 & 3 & \text{[redacted]} \\ 0 & -3 & -6 & \end{bmatrix}$$

Does it have a solution?

A Thought Experiment

then we reduce it to echelon form

$$\begin{bmatrix} 1 & 2 & 3 & \text{[redacted]} \\ 0 & -3 & -6 & \end{bmatrix}$$

Yes. It doesn't have an inconsistent row

A Thought Experiment

`what about this system?`

$$\begin{bmatrix} 1 & 1 & 2 & \text{[redacted]} \\ 2 & 2 & 4 & \end{bmatrix}$$

A Thought Experiment

`what about this system?`

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$$\begin{bmatrix} 1 & 1 & 2 & \text{[redacted]} \\ 0 & 0 & 0 & \end{bmatrix}$$

A Thought Experiment

`what about this system?`

$$\begin{bmatrix} 1 & 1 & 2 & \text{[redacted]} \\ 0 & 0 & 0 & \end{bmatrix}$$

`it depends...`

Pivots and Spanning $\mathbb{R}^m$

$$\begin{bmatrix} 1 & 2 & 3 & \text{[redacted]} \\ 2 & 1 & 0 & \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & \text{[redacted]} \\ 0 & -3 & -6 & \end{bmatrix}$$

Pivots and Spanning $\mathbb{R}^m$

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`if it doesn't matter what the last column is, then` **`every choice must be possible`**

Pivots and Spanning $\mathbb{R}^m$

$$\begin{bmatrix} 1 & 2 & 3 & \text{[redacted]} \\ 2 & 1 & 0 & \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & \text{[redacted]} \\ 0 & -3 & -6 & \end{bmatrix}$$

`if it doesn't matter what the last column is, then` **every choice must be possible**

every vector in $\mathbb{R}^2$ can be written as a linear combination of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$

Spanning $\mathbb{R}^m$

Theorem. For any $m \times n$ matrix, the following are logically equivalent

1. For every $\mathbf{b}$ in $\mathbb{R}^m$, $A\mathbf{x} = \mathbf{b}$ has a solution

2. The columns of A span $\mathbb{R}^m$

3. A has a pivot position in every row

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1. For every $\mathbf{b}$ in $\mathbb{R}^m$, $A\mathbf{x} = \mathbf{b}$ has a solution
2. The columns of A span $\mathbb{R}^m$
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HOW TO: Spanning $\mathbb{R}^m$

Question. Does the set of vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ from $\mathbb{R}^m$ span all if $\mathbb{R}^m$?

Solution. Reduce $[\mathbf{a}_1 \ \ \mathbf{a}_2 \ \ \ldots \ \ \mathbf{a}_n]$ to echelon form and check if every row has a pivot

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Question. Does the set of vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ from $\mathbb{R}^m$ span all if $\mathbb{R}^m$?

Solution. Reduce $[\mathbf{a}_1 \;\; \mathbf{a}_2 \;\; \ldots \;\; \mathbf{a}_n]$ to echelon form and check if every row has a pivot

!! We only need the echelon form !!

Question

Do $\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 2023 \end{bmatrix}$ span all of $\mathbb{R}^3$?

Answer: No

the matrix

$$\begin{bmatrix} 2 & 0 \\ 2 & 1 \\ 3 & 2023 \end{bmatrix}$$

cannot have more than 2 pivot positions

Not spanning $\mathbb{R}^m$

$$\begin{bmatrix} 1 & 1 & 2 & \text{[redacted]} \\ 2 & 2 & 4 & \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & \text{[redacted]} \\ 0 & 0 & 0 & \end{bmatrix}$$

Not spanning $\mathbb{R}^m$

$$\begin{bmatrix} 1 & 1 & 2 & \text{[redacted]} \\ 2 & 2 & 4 & \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & \text{[redacted]} \\ 0 & 0 & 0 & \end{bmatrix}$$

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in this case the choice matters
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Not spanning $\mathbb{R}^m$

$$\begin{bmatrix} 1 & 1 & 2 & \blacksquare \\ 2 & 2 & 4 & \blacksquare \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & \blacksquare \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}$$

in this case the choice matters

we can't make the last column $[0 \;\; 0 \;\; 0 \;\; \blacksquare]$ for nonzero $\blacksquare$

Not spanning $\mathbb{R}^m$

$$\begin{bmatrix} 1 & 1 & 2 & \blacksquare \\ 2 & 2 & 4 & \blacksquare \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & \blacksquare \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}$$

in this case the choice matters

we can't make the last column $\begin{bmatrix} 0 & 0 & 0 & \blacksquare \end{bmatrix}$ for nonzero $\blacksquare$

but we can make the last column <u>parameters</u> to find equations that must hold

Not spanning $\mathbb{R}^m$

$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 2 & 2 & 4 & b_2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \end{bmatrix}$$

Not spanning $\mathbb{R}^m$

$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 2 & 2 & 4 & b_2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \end{bmatrix}$$

as long as $(-2)b_1 + b_2 = 0$, the system is consistent

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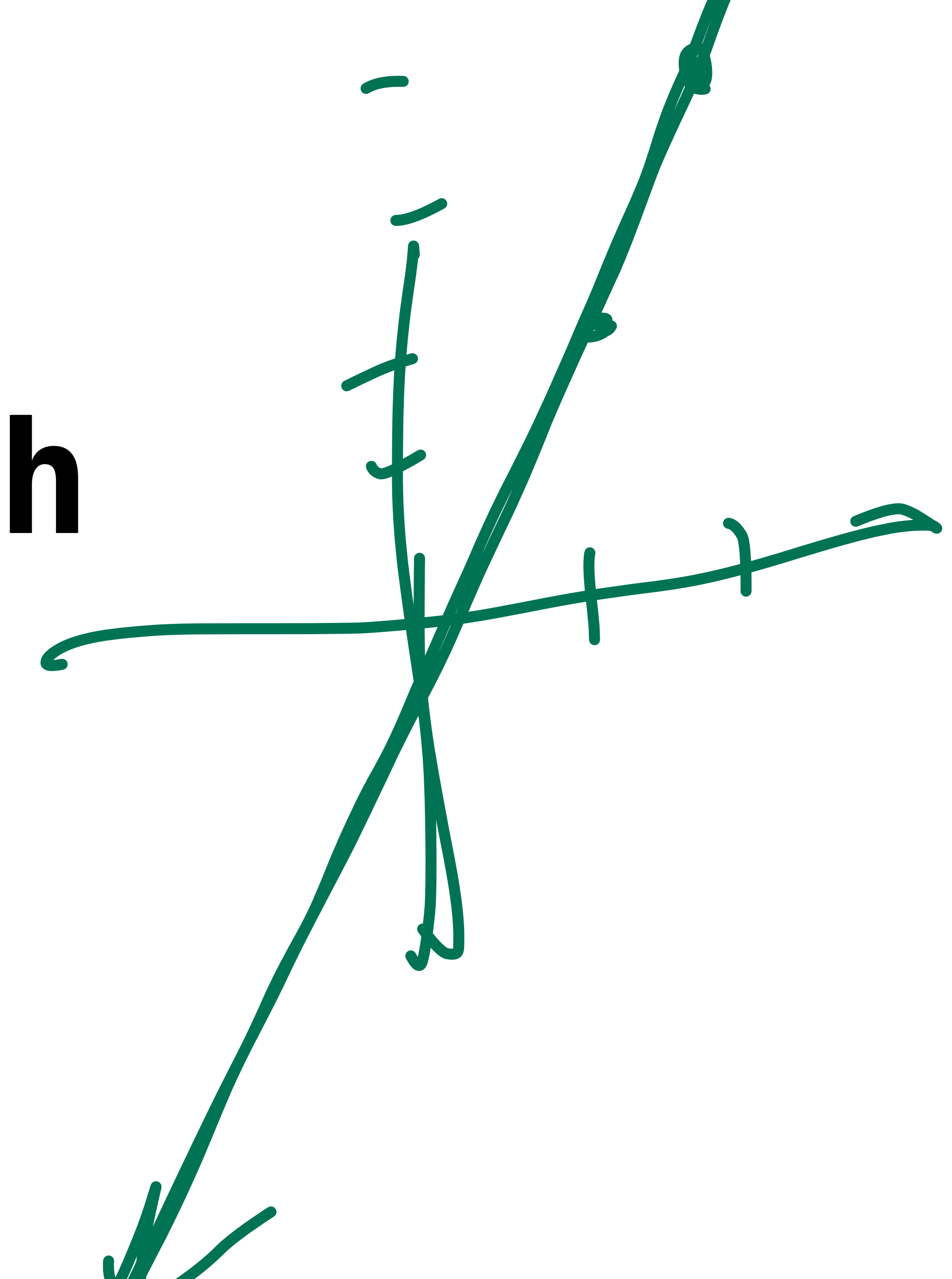
as long as $(-2)b_1 + b_2 = 0$, the system is consistent

$-2x + y = 0$ $\quad y = 2x$



this gives use a <u>linear equation</u> which describes the span of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$

Question (Understanding Check)

True *or* ***False,*** *the echelon form of any matrix has at most one row of the form* $[0 \;\; 0 \;\; \ldots \;\; 0 \;\; \blacksquare]$ *where* $\blacksquare$ *is nonzero.*

Answer: True

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

leading entry not to the right

this is not in echelon form

Question (More Challenging)

Give a linear equation for the span of the vectors $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ *and* $\begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$.

Answer

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

Answer

$$\begin{bmatrix} 1 & -1 & b_1 \\ 2 & -1 & b_2 \\ 0 & -1 & b_3 \end{bmatrix}$$

Answer

$$\begin{bmatrix} 1 & -1 & b_1 \\ 0 & 2 & b_2 - 2b_1 \\ 0 & -1 & b_3 \end{bmatrix}$$

$$R_2 \leftarrow R_2 - 2R_1$$

Answer

$$\begin{bmatrix} 1 & -1 & b_1 \\ 0 & 2 & b_2 - 2b_1 \\ 0 & 0 & b_3 + (1/2)(b_2 - 2b_1) \end{bmatrix}$$

$$R_3 \leftarrow R_3 - (1/2)R_2$$

Answer

$$\begin{bmatrix} 1 & -1 & b_1 \\ 0 & 2 & b_2 - 2b_1 \\ 0 & 0 & b_3 + (1/2)(b_2 - 2b_1) \end{bmatrix}$$

$$R_3 \leftarrow R_3 - (1/2)R_2$$

Answer

$$0 = b_3 + (1/2)(b_2 - 2b_1)$$

Answer

$$b_1 - (1/2)b_2 - b_3 = 0$$

Answer

$$x_1 - (1/2)x_2 - x_3 = 0$$

Taking Stock

Four Representations

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$$

augmented matrix

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

matrix equation

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

system of linear equations

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{1m} \end{bmatrix} + x_2 \begin{bmatrix} a_{21} \\ a_{21} \\ \vdots \\ a_{2m} \end{bmatrix} + \dots + x_n \begin{bmatrix} a_{n1} \\ a_{n2} \\ \vdots \\ a_{nm} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

vector equation

Four Representations

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$$

augmented matrix

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

matrix equation

they all have the same solution sets

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

system of linear equations

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{1m} \end{bmatrix} + x_2 \begin{bmatrix} a_{21} \\ a_{21} \\ \vdots \\ a_{2m} \end{bmatrix} + \dots + x_n \begin{bmatrix} a_{n1} \\ a_{n2} \\ \vdots \\ a_{nm} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

vector equation

Summary

Matrix and vectors can be multiplied together to get new vectors

The matrix equation is another representation of systems of linear equations

Looking forward: Matrices *transform* vectors